Partial Differential Equations (PDE)
Chapter 2: Waves and Diffusions

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- TextBooks:
- Partial Differential Equations: An Introduction(second edition) Walter A. Strauss John Wiley & Sons, 2007
- Applied Partial Differential Equations, Paul Duchateau and David Zachmann

October 29, 2021

Chapter Two

The general solution

- Consider the wave equation on the whole real axis:

$$u_{tt} = c^2 u_{xx} \text{ for } \infty < x < \infty.$$

- Basic idea: rewrite this second order equation into two first order equations so that the method of characteristic lines/curves can be used.

The general solution

- Another method to solve wave equation $u_{tt} + cu_{xx} = 0$ is coordinate method, see page 34 of the textbook.

Initial Value Problem

- We consider the following initial value problem (IVP)

$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty$$
$$u(x,0) = \phi(x) \text{ and } u_t(x,0) = \psi(x).$$

- We know that the general solution of

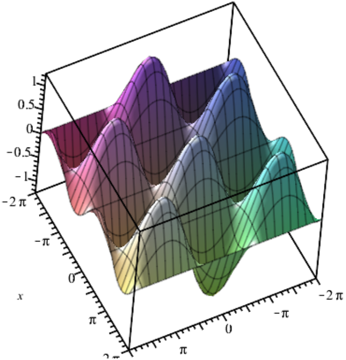
$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty,$$

is $u(x,t) = f(x+ct) + g(x-ct)$

Examples for the initial value problem

- Example (see text book page 35): solve the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty$$
$$u(x,0) = 0 \text{ and } u_t(x,0) = \cos(x).$$


1
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Examples for the initial value problem

- Example (see text book page 35): solve the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty$$

$$u(x,0) = \begin{cases} b - \dfrac{b|x|}{a} & , |x| < a \\ 0 & |x| > a \end{cases}$$

and $u_t(x,0) = 0$.

- Example: Solve the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty$$
$$u(x,0) = e^{-x^2} \text{ and } u_t(x,0) = 0.$$

Energy

- How do we calculate the total energy of a vibrating infinite string?
- Total energy(E)=kinetic energy(KE)+potential energy(PE).
- Recall the wave equation: $u_{tt} = c^2 u_{xx}$ where $\sqrt{T/\rho}$. Here ρ is the density of the string and T is the tension force of the string.
- Then $\rho u_{tt} = T u_{xx}$
- The kinetic energy is defined as: $KE = \frac{1}{2} m v^2 = \frac{1}{2} \int_{-\infty}^{\infty} \rho u_t^2 dx$
- The potential energy is defined as: $PE = \frac{1}{2} \int_{-\infty}^{\infty} T u_x^2 dx$
- Therefor, $E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx$.

Energy

- The law of conservation of energy: The energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + T u_x^2)dx$ is a constant independent of t! why?
- The idea to prove this is $\frac{dE}{dt} = 0$!
- So, $\frac{dE(t)}{dt} = \int_{-\infty}^{\infty}(\rho u_t u_{tt} + T u_x u_{xt})dx$.
- Integration by parts yields:

  $\frac{dE(t)}{dt} = \int_{-\infty}^{\infty}(\rho u_t u_{tt} - T u_t u_{xx})dx = 0!$ why?

  we consider the term $u_t u_x$ is evaluated at $x = \pm\infty$ and so it vanishes while using integration by parts.

- Example: Compute the energy of the plucked string

$$u_{tt} = c^2 u_{xx}, \quad \text{for} \quad -\infty < x < \infty$$
$$u(x,0) = \phi(x) \ \text{and} \ u_t(x,0) = \psi(x),$$

where $\psi(x) = 0$ and

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & , |x| < a \\ 0 & , |x| > a \end{cases}$$

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