Partial Differential Equations (PDE)

Chapter 5: Fourier Series

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Partial Differential Equations

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Outline

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Results from Chapter 4

- Consider the wave equation with homogeneous Dirichlet boundary condition on a bounded domain:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases}$$

- Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)\right] sin\left(\frac{n\pi}{l}x\right).$$

Results from Chapter 4

- What if we also have the initial conditions: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$?

- $\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{l}x\right).$

- $u_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \frac{n\pi c}{l} sin\left(\frac{n\pi ct}{l}\right) + B_n \frac{n\pi c}{l} cos\left(\frac{n\pi ct}{l}\right)\right] sin\left(\frac{n\pi}{l}x\right).$

- $\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} sin\left(\frac{n\pi}{l}x\right).$

- How do we work out A_n and B_n? Chapter 5!

Results from Chapter 4

- Consider the diffusion/heat equation with homogeneous Dirichlet boundary condition on a bounded domain:

$$\begin{cases} u_t = ku_{xx}, & \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0. \end{cases}$$

- Solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} sin\left(\frac{n\pi}{l}x\right).$$

Results from Chapter 4

- What if we also have the initial conditions: $u(x, 0) = \phi(x)$?
- $\phi(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{l}x\right)$.
- How do we work out A_n? Chapter 5!

Results from Chapter 4

- Consider the wave equation with homogeneous Neumann boundary condition on a bounded domain:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad \text{for } 0 < x < l, \\ u_x(0,t) = u_x(l,t) = 0. \end{cases}$$

- Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} [A_n cos\left(\frac{n\pi ct}{l}\right) \\ &\quad + B_n sin\left(\frac{n\pi ct}{l}\right)] cos\left(\frac{n\pi}{l}x\right). \end{aligned}$$

Results from Chapter 4

- What if we also have the initial conditions: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$?

- $\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=0}^{\infty} A_n cos\left(\frac{n\pi}{l}x\right).$

- $u_t(x,t) =$
$\frac{1}{2}B_0 + \sum_{n=1}^{\infty} \left[-A_n \frac{n\pi c}{l} sin\left(\frac{n\pi ct}{l}\right) + B_n \frac{n\pi c}{l} cos\left(\frac{n\pi ct}{l}\right)\right] cos\left(\frac{n\pi}{l}x\right).$

- $\psi(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} cos\left(\frac{n\pi}{l}x\right).$

- How do we work out A_n and B_n? Chapter 5!

Results from Chapter 4

- Consider the diffusion/heat equation with homogeneous Neumann boundary condition on a bounded domain:

$$\begin{cases} u_t = k u_{xx}, & \text{for } 0 < x < l, \\ u_x(0,t) = u_x(l,t) = 0. \end{cases}$$

- Solution:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi}{l}x\right).$$

Results from Chapter 4

- What if we also have the initial conditions: $u(x,0)=\phi(x)$?
- $\phi(x)=u(x,0)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n cos\left(\frac{n\pi}{l}x\right)$.
- How do we work out A_n? Chapter 5!

Definition

Definition (Fourier sine series)

The Fourier sine series in the interval $(0, l)$ is defined by

$$\phi(x) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right),$$

Definition (Fourier cosine series)

The Fourier cosine series in the interval $(0, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{l}\right),$$

Remark

Question: What are A_n?

Definition

Definition (Full Fourier series)

The full Fourier series in the interval $(-l, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n cos(\frac{n\pi x}{l}) + B_n sin\left(\frac{n\pi x}{l}\right)\right],$$

Remark

Question: What are A_n and B_n?

Fourier sine series

Lemma (1)

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{1}{2} l, & \text{if } n = m. \end{cases}$$

Fourier sine series

Proof:

- Recall the following trigonometric identity:

$$sin(a)sin(b) = \frac{1}{2}cos(a-b) - \frac{1}{2}cos(a+b).$$

- Let $a = \frac{m\pi x}{l}$ and $\frac{n\pi x}{l}$. Then

$$\begin{aligned} & \int_0^l sin\left(\frac{n\pi x}{l}\right) sin\left(\frac{m\pi x}{l}\right) dx \\ = & \int_0^l \left[\frac{1}{2}cos\frac{(m-n)\pi x}{l} - \frac{1}{2}cos\frac{(m+n)\pi x}{l}\right] dx \\ = & \frac{1}{2}\int_0^l cos\frac{(m-n)\pi x}{l}dx - \frac{1}{2}\int_0^l cos\frac{(m+n)\pi x}{l}dx. \end{aligned}$$

Fourier sine series

- If $m \neq n$,

$$\begin{aligned}
&\int_0^l sin\left(\frac{n\pi x}{l}\right) sin\left(\frac{m\pi x}{l}\right) dx \\
= \quad & \frac{1}{2}\int_0^l cos\frac{(m-n)\pi x}{l} dx - \frac{1}{2}\int_0^l cos\frac{(m+n)\pi x}{l} dx \\
= \quad & \frac{l}{2(m-n)\pi} sin\frac{(m-n)\pi x}{l}\Big|_{x=0}^{x=l} - \frac{l}{2(m+n)\pi} sin\frac{(m+n)\pi x}{l}\Big|_{x=0}^{x=l} \\
= \quad & \frac{l}{2(m-n)\pi}[sin((m-n)\pi) - sin(0)] \\
& -\frac{l}{2(m+n)\pi}[sin((m+n)\pi) - sin(0)] \\
= \quad & 0.
\end{aligned}$$

Fourier sine series

- If $m = n$,

$$\begin{aligned}
&\int_0^l sin\left(\frac{n\pi x}{l}\right) sin\left(\frac{m\pi x}{l}\right) dx \\
= \quad & \frac{1}{2}\int_0^l cos\frac{(m-n)\pi x}{l}dx - \frac{1}{2}\int_0^l cos\frac{(m+n)\pi x}{l}dx \\
= \quad & \frac{1}{2}\int_0^l 1 \; dx - \frac{1}{2}\int_0^l cos\frac{2n\pi x}{l}dx \\
= \quad & \frac{1}{2}l - \frac{l}{4n\pi}sin\frac{2n\pi x}{l}\Big|_{x=0}^{x=l} \\
= \quad & \frac{1}{2}l - \frac{l}{4n\pi}[sin(2n\pi) - sin(0)] \\
= \quad & \frac{1}{2}l.
\end{aligned}$$

Fourier sine series

Definition (Fourier sine series)

The Fourier sine series in the interval $(0, l)$ is defined by

$$\phi(x) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{2}{l}\int_0^l \phi(x) sin\left(\frac{n\pi x}{l}\right) dx, \ \ n = 1, 2, 3, \cdots.$$

Fourier sine series

- Example 1: Find the Fourier sine series for $\phi(x) = 1$ in the interval $[0, l]$.

- For $n = 1, 2, 3, \cdots,$

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l \phi(x) sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\int_0^l sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{l}\left[-\frac{l}{n\pi} cos(\frac{n\pi x}{l})\right]\Big|_{x=0}^{x=l} \\ &= -\frac{2}{n\pi}\left[cos(\frac{n\pi x}{l})\right]\Big|_{x=0}^{x=l} \\ &= -\frac{2}{n\pi}[cos(n\pi) - cos(0)] \\ &= \frac{2}{n\pi}[1 - (-1)^n]. \end{aligned}$$

Fourier sine series

- Then

$$
\begin{aligned}
1 &= \phi(x) \\
&= \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right) \\
&= \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n] sin\left(\frac{n\pi x}{l}\right).
\end{aligned}
$$

Fourier sine series

- Example 2: Find the Fourier sine series for $\phi(x) = x$ in the interval $[0, l]$.

Fourier sine series

- Example 3: Solve the problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad \text{for } 0 < x < l, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \phi(x) = x, \ \ u_t(x,0) = \psi(x) = 1. \end{cases}$$

- General solution without the initial conditions (Section 4.1):

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)\right] sin\left(\frac{n\pi}{l}x\right).$$

- Next step?

- Apply the initial conditions to compute A_n and B_n!

Fourier cosine series

Lemma (2)

$$\int_0^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{1}{2}l, & \text{if } n = m. \end{cases}$$

Proof.

Homework! Hint:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b).$$

□

Fourier cosine series

- Now we consider

$$\int_0^l \cos\left(\frac{m\pi x}{l}\right) dx.$$

- If $m = 0$,

$$\int_0^l \cos\left(\frac{m\pi x}{l}\right) dx = \int_0^l 1 \ dx = l.$$

- If $m \neq 0$,

$$\begin{aligned}\int_0^l \cos\left(\frac{m\pi x}{l}\right) dx &= \frac{l}{m\pi} \sin\left(\frac{m\pi x}{l}\right) \Big|_{x=0}^{x=l} \\ &= \frac{l}{m\pi}[\sin(m\pi) - \sin(0)] \\ &= 0.\end{aligned}$$

Fourier cosine series

- If $m = 0$, then by Lemma 2 and the above formula for $m = 0$,

$$\begin{aligned}
&\int_0^l \phi(x) cos(\frac{m\pi x}{l}) dx \\
= \quad & \frac{1}{2} A_0 \int_0^l cos\left(\frac{m\pi x}{l}\right) dx + \sum_{n=1}^{\infty} A_n \int_0^l cos\left(\frac{n\pi x}{l}\right) cos\left(\frac{m\pi x}{l}\right) dx \\
= \quad & \frac{1}{2} A_0 l.
\end{aligned}$$

- Hence

$$A_0 = \frac{2}{l} \int_0^l \phi(x) cos\left(\frac{0\pi x}{l}\right).$$

Fourier cosine series

- If $m \neq 0$, then by Lemma 2 and the above formula for $m \neq 0$,

$$\begin{aligned}
&\int_0^l \phi(x) cos(\frac{m\pi x}{l})dx \\
= \quad & \frac{1}{2}A_0 \int_0^l cos\left(\frac{m\pi x}{l}\right) dx + \sum_{n=1}^{\infty} A_n \int_0^l cos\left(\frac{n\pi x}{l}\right) cos\left(\frac{m\pi x}{l}\right) dx \\
= \quad & A_m \frac{1}{2} l.
\end{aligned}$$

- Hence

$$A_m = \frac{2}{l}\int_0^l \phi(x) cos(\frac{m\pi x}{l}), \; m = 1, 2, 3, \cdots .$$

- Finally,

$$A_n = \frac{2}{l}\int_0^l \phi(x) cos(\frac{n\pi x}{l}), \; n = 0, 1, 2, 3, \cdots .$$

Fourier cosine series

Definition (Fourier cosine series)

The Fourier cosine series in the interval $(0, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{2}{l}\int_0^l \phi(x) cos\left(\frac{n\pi x}{l}\right) dx, \;\; n = 0, 1, 2, 3, \cdots.$$

Fourier cosine series

- Example 4: Find the Fourier cosine series for $\phi(x) = 1$ in the interval $[0, l]$.

Fourier cosine series

- Example 5: Find the Fourier cosine series for $\phi(x) = x$ in the interval $[0, l]$.
- For $n = 1, 2, 3, \cdots$,

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l \phi(x) cos\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\int_0^l x\ cos\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{l}{n\pi}\frac{2}{l}\int_0^l x\ dsin\left(\frac{n\pi x}{l}\right) = \frac{2}{n\pi}\int_0^l x\ dsin\left(\frac{n\pi x}{l}\right) \\ &= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)|_{x=0}^{x=l} - \int_0^l sin\left(\frac{n\pi x}{l}\right) dx\right] \\ &= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)|_{x=0}^{x=l} + \frac{l}{n\pi} cos\left(\frac{n\pi x}{l}\right)|_{x=0}^{x=l}\right]. \end{aligned}$$

Fourier cosine series

- Then

$$
\begin{aligned}
A_n &= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)\right]\Big|_{x=0}^{x=l} + \frac{2l}{n^2\pi^2}\left[cos\left(\frac{n\pi x}{l}\right)\right]\Big|_{x=0}^{x=l} \\
&= \frac{2}{n\pi}[l\ sin(n\pi) - 0] + \frac{2l}{n^2\pi^2}[cos(n\pi) - cos(0)] \\
&= \frac{2l}{n^2\pi^2}[(-1)^n - 1].
\end{aligned}
$$

Fourier cosine series

- For $n = 0$,

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l \phi(x)cos\left(\frac{0\pi x}{l}\right)dx = \frac{2}{l}\int_0^l x\ dx \\ &= \frac{2}{l}\frac{1}{2}x^2|_{x=0}^{x=l} \\ &= \frac{1}{l}(l^2 - 0) \\ &= l. \end{aligned}$$

- Hence

$$\begin{aligned} x &= \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{l}\right) \\ &= \frac{1}{2}l + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}[(-1)^n - 1]cos\left(\frac{n\pi x}{l}\right). \end{aligned}$$

Full Fourier series

- Now we study the coefficients of the full Fourier series on $[-l, l]$:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n cos\left(\frac{n\pi x}{l}\right) + B_n sin\left(\frac{n\pi x}{l}\right)\right]$$

Full Fourier series

Lemma (3)

$$\int_{-l}^{l} cos\left(\frac{n\pi x}{l}\right) cos\left(\frac{m\pi x}{l}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ l, & \text{if } n = m. \end{cases}$$

$$\int_{-l}^{l} sin\left(\frac{n\pi x}{l}\right) cos\left(\frac{m\pi x}{l}\right) dx = 0, \quad \text{for all } n, m.$$

Proof.

Homework! Hint:

$$sin(a)cos(b) = \frac{1}{2}sin(a-b) + \frac{1}{2}sin(a+b).$$

□

Full Fourier series

Lemma (4)

$$\int_{-l}^{l} sin\left(\frac{n\pi x}{l}\right) sin(\frac{m\pi x}{l})dx = \begin{cases} 0, & if\ n \neq m \\ l, & if\ n = m. \end{cases}$$

$$\int_{-l}^{l} cos\left(\frac{n\pi x}{l}\right) sin(\frac{m\pi x}{l})dx = 0, \quad for\ all\ n, m.$$

Proof.

Homework! □

Full Fourier series

Definition (Full Fourier series)

The full Fourier series in the interval $(-l, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n cos\left(\frac{n\pi x}{l}\right) + B_n sin\left(\frac{n\pi x}{l}\right)\right],$$

where

$$\begin{aligned} A_n &= \frac{1}{l}\int_{-l}^{l}\phi(x) cos\left(\frac{n\pi x}{l}\right) dx,\ \ n = 0, 1, 2, 3, \cdots, \\ B_n &= \frac{1}{l}\int_{-l}^{l}\phi(x) sin\left(\frac{n\pi x}{l}\right) dx,\ \ n = 1, 2, 3, \cdots. \end{aligned}$$

Full Fourier series

- Example 6: Find the Full Fourier series for $\phi(x) = x$ in the interval $[-l, l]$.

- For $n = 0$,

$$\begin{aligned} A_0 &= \frac{2}{l}\int_{-l}^{l} \phi(x) cos\left(\frac{0\pi x}{l}\right) dx = \frac{2}{l}\int_{-l}^{l} x \ dx \\ &= \frac{2}{l}\frac{1}{2}x^2|_{x=-l}^{x=l} \\ &= \frac{1}{l}(l^2 - l^2) \\ &= 0. \end{aligned}$$

Full Fourier series

- For $n = 1, 2, 3, \cdots,$

$$\begin{aligned}
A_n &= \frac{2}{l}\int_{-l}^{l} \phi(x) cos\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\int_{-l}^{l} x\ cos\left(\frac{n\pi x}{l}\right) dx \\
&= \frac{l}{n\pi}\frac{2}{l}\int_{-l}^{l} x\ dsin\left(\frac{n\pi x}{l}\right) \\
&= \frac{2}{n\pi}\int_{-l}^{l} x\ dsin\left(\frac{n\pi x}{l}\right) \\
&= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l} - \int_{-l}^{l} sin\left(\frac{n\pi x}{l}\right) dx\right] \\
&= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l} + \frac{l}{n\pi} cos\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l}\right].
\end{aligned}$$

Full Fourier series

- Then

$$
\begin{aligned}
A_n &= \frac{2}{n\pi}\left[x\ sin\left(\frac{n\pi x}{l}\right)\right]|_{x=-l}^{x=l} + \frac{2l}{n^2\pi^2}\left[cos\left(\frac{n\pi x}{l}\right)\right]|_{x=-l}^{x=l} \\
&= \frac{2}{n\pi}[l\ sin(n\pi) - (-l)sin(-n\pi)] \\
&\quad + \frac{2l}{n^2\pi^2}[cos(n\pi) - cos(-n\pi)] \\
&= \frac{2}{n\pi}[l\ sin(n\pi) - sin(n\pi)] + \frac{2l}{n^2\pi^2}[cos(n\pi) - cos(n\pi)] \\
&= 0.
\end{aligned}
$$

Full Fourier series

- For $n = 1, 2, 3, \cdots,$

$$\begin{aligned}
B_n &= \frac{2}{l}\int_{-l}^{l}\phi(x)sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_{-l}^{l}x\ sin\left(\frac{n\pi x}{l}\right)dx \\
&= -\frac{l}{n\pi}\frac{2}{l}\int_{-l}^{l}x\ dcos\left(\frac{n\pi x}{l}\right) \\
&= -\frac{2}{n\pi}\int_{-l}^{l}x\ dcos\left(\frac{n\pi x}{l}\right) \\
&= -\frac{2}{n\pi}\left[x\ cos\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l} - \int_{-l}^{l}cos\left(\frac{n\pi x}{l}\right)dx\right] \\
&= -\frac{2}{n\pi}\left[x\ cos\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l} - \frac{l}{n\pi}sin\left(\frac{n\pi x}{l}\right)|_{x=-l}^{x=l}\right].
\end{aligned}$$

Full Fourier series

- Then

$$
\begin{aligned}
B_n &= -\frac{2}{n\pi}\left[x\ cos\left(\frac{n\pi x}{l}\right)\right]\Big|_{x=-l}^{x=l} + \frac{2l}{n^2\pi^2}\left[sin\left(\frac{n\pi x}{l}\right)\right]\Big|_{x=-l}^{x=l} \\
&= -\frac{2}{n\pi}[l\ cos(n\pi) - (-l)cos(-n\pi)] \\
&\quad + \frac{2l}{n^2\pi^2}[sin(n\pi) - sin(-n\pi)] \\
&= -\frac{2}{n\pi}[l\ cos(n\pi) + l\ cos(n\pi)] \\
&= -\frac{2}{n\pi}2l\ cos(n\pi) \\
&= -\frac{4l}{n\pi}(-1)^n \\
&= \frac{4l}{n\pi}(-1)^{n+1}.
\end{aligned}
$$

Full Fourier series

- Hence

$$
\begin{aligned}
x &= \phi(x) \\
&= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n cos\left(\frac{n\pi x}{l}\right) + B_n sin\left(\frac{n\pi x}{l}\right)\right] \\
&= 0 + \sum_{n=1}^{\infty}\left[0\ cos\left(\frac{n\pi x}{l}\right) + \frac{4l}{n\pi}(-1)^{n+1} sin\left(\frac{n\pi x}{l}\right)\right].
\end{aligned}
$$

Summary

Definition (Fourier sine series)

The Fourier sine series in the interval $(0, l)$ is defined by

$$\phi(x) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{2}{l}\int_0^l \phi(x) sin\left(\frac{n\pi x}{l}\right) dx, \;\; n = 1, 2, 3, \cdots.$$

Summary

Definition (Fourier cosine series)

The Fourier cosine series in the interval $(0, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{2}{l}\int_0^l \phi(x) cos\left(\frac{n\pi x}{l}\right) dx, \ \ n = 0, 1, 2, 3, \cdots.$$

Summary

Definition (Full Fourier series)

The full Fourier series in the interval $(-l, l)$ is defined by

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n cos\left(\frac{n\pi x}{l}\right) + B_n sin\left(\frac{n\pi x}{l}\right)\right],$$

where

$$\begin{aligned} A_n &= \frac{1}{l}\int_{-l}^{l}\phi(x) cos\left(\frac{n\pi x}{l}\right) dx, \ \ n = 0, 1, 2, 3, \cdots, \\ B_n &= \frac{1}{l}\int_{-l}^{l}\phi(x) sin\left(\frac{n\pi x}{l}\right) dx, \ \ n = 1, 2, 3, \cdots. \end{aligned}$$