Chapter 6 April 5, 2022 P2

Numerical methods for PDEs.

- Finite difference
- Finite element
- Finite volume

Theorem: Suppose that $u(x)$ is a $(n+1)^{th}$ differentiable function on $[a,b]$ and $x_0 \in [a,b]$. Then for any $x \in [a,b]$, we have the following Taylor's expansion of u at x_0:

$$u(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} u^{(k)}(x_0) (x-x_0)^k$$

$$= u(x_0) + u'(x_0)(x-x_0) + \frac{1}{2!} u''(x_0)(x-x_0)^2 + \cdots$$

$$+ \frac{1}{n!} u^{(n)}(x_0)(x-x_0)^n,$$

and

$$R_n(x) = \frac{1}{(n+1)!} u^{(n+1)}(\xi)(x-x_0)^{n+1} \quad \text{for some } \xi \in [x_0, x]$$

or

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} u^{(n+1)}(s)(x-s)^n \, ds.$$

Pick $n=1$ in Taylor's expansion:

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + \frac{1}{2}u''(\xi)(x-x_0)^2$$

So, if we replace x by $x+\Delta x$ and x_0 by x;

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{1}{2}u''(\xi)(\Delta x)^2 \quad —①$$

First Consider the discretization of the first derivative $u'(x)$. Then

$$u'(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x} - \frac{1}{2}u''(\Delta x)$$

$$= \frac{u(x+\Delta x) - u(x)}{\Delta x} + O(\Delta x).$$

* Assume that we have a uniform partition of $[a,b]$ into N elements with <u>mesh size</u> $\Delta x = \frac{b-a}{N}$

* The mesh nodes are $x_j = a + j\Delta x$, $j = 0, 1, 2, \dots, N$, then

$$u'(x_j) = \frac{u(x_j+\Delta x) - u(x_j)}{\Delta x} + O(\Delta x)$$

$$= \frac{u(x_{j+1}) - u(x_j)}{\Delta x} + O(\Delta x)$$

$\Rightarrow$

$$u'(x_j) \simeq \frac{u_{j+1} - u_j}{\Delta x}, \quad j = 0, 1, 2, \cdots, N-1.$$

Here u_j is the approximation of $u(x_j)$. This is called forward difference.

- Recall the Taylor's expansion with $n=1$

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{1}{2} u''(\xi)(x - x_0)^2$$

Replace x by $x - \Delta x$ and x_0 by x_j

$\Rightarrow$

$$u(x - \Delta x) = u(x) - u'(x)\Delta x + \frac{1}{2} u''(\xi)(\Delta x^2) \quad -②$$

$$\Rightarrow u'(x) = \frac{u(x) - u(x - \Delta x)}{\Delta x} + \frac{1}{2} u''(\xi)(\Delta x)$$

$$= \frac{u(x) - u(x - \Delta x)}{\Delta x} + O(\Delta x)$$

Consider the same partition as above (page 3)

Then
$$u'(x_j) = \frac{u(x_j) - u(x_j - \Delta x)}{\Delta x} + O(\Delta x)$$

$$= \frac{u(x_j) - u(x_{j-1})}{\Delta x} + O(\Delta x)$$

$$\simeq \frac{u_j - u_{j-1}}{\Delta x}, \quad j = 1, 2, \cdots, N$$

This is called Backward difference.

* Recall the Taylor's expansion with $n=2$ to obtain a better approximation

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + \frac{1}{2}u''(x_0)(x-x_0)^2 + \frac{1}{6}u'''(s)(x-x_0)^3$$

~~* Recall~~ $\Rightarrow$

$$u(x+\Delta x) = u(x) + u'(x)(\Delta x) + \frac{1}{2}u''(x)(\Delta x)^2 + \frac{1}{6}u'''(s_1)(x-x_0)^3$$

$$u(x-\Delta x) = u(x) - u'(x)(\Delta x) + \frac{1}{2}u''(x)(\Delta x)^2 - \frac{1}{6}u'''(s_2)(x-x_0)^3$$

Subtracting the two expressions

$$u(x+\Delta x) - u(x-\Delta x) = 2u'(x)(\Delta x) + O(\Delta x^3)$$

$$\Rightarrow u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{2(\Delta x)} + O(\Delta x^2) \quad \text{(second order error)}$$

$$\Rightarrow u'(x_j) = \frac{u(x_j+\Delta x) - u(x_j-\Delta x)}{2(\Delta x)} + O(\Delta x)^2$$

$$\Rightarrow u'(x_j) = \frac{u(x_{j+1}) - u(x_{j-1})}{2(\Delta x)} + O(\Delta x)^2$$

$$\Rightarrow u'_j \simeq \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \quad j = 1, 2, \ldots, N-1$$

this is called Central-difference

Now, lets turn to the discretization of the second derivate $u''(x)$:

pick $n=3$ in The Taylor's expansion:

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + \frac{1}{2}u''(x_0)(x-x_0)^2 + \frac{1}{6}u'''(x_0)(x-x_0)^3 + \frac{1}{24}u''''(s)(x-x_0)^4$$

First Replace x by $x+\Delta x$ and x_0 by x, Second

~~and~~ x by $x-\Delta x$ & x_0 by x

$$u(x+\Delta x) = u(x) + u'(x)(\Delta x) + \frac{1}{2}u''(x)(\Delta x)^2 + \frac{1}{6}u'''(x)(\Delta x)^3 + \frac{1}{24}u^{(4)}(s_1)(\Delta x)^4$$

$$u(x-\Delta x) = u(x) - u'(x)(\Delta x) + \frac{1}{2}u''(x)(\Delta x)^2 - \frac{1}{6}u'''(x)(\Delta x)^3 + \frac{1}{24}u^{(4)}(s_2)(\Delta x)^4$$

Adding the two expressions

$$u(x+\Delta x) + u(x-\Delta x) = 2u(x) + u''(x)(\Delta x^2) + O(\Delta x^4)$$

$$\Rightarrow u''(x) = \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x^2)} + O(\Delta x^2)$$

$$\Rightarrow u''(x_j) = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{\Delta x^2} + O(\Delta x^2)$$

$$\Rightarrow u''_j \simeq \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \quad ; \quad j = 1, 2, \dots, N-1$$

This is called Central second difference formula.

Summarize

- Forward difference for $u'_j \simeq \frac{u_{j+1} - u_j}{\Delta x}$
- Backward difference for $u'_j \simeq \frac{u_j - u_{j-1}}{\Delta x}$
- Central difference for $u'_j \simeq \frac{u_{j+1} - u_{j-1}}{2\Delta x}$
- Central Second difference for $u''_j \simeq \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$

Example: Let $u(x) = x\ln(x)$. Estimate $u'(2)$ and $u''(2)$ by using above methods, with $\Delta x = 0.1$

- Forward difference $u'(2) \simeq \frac{u(2.1) - u(2)}{\Delta x} = \boxed{1.7177}$ appr.

  $|\text{exact} - \text{appr}| = \boxed{0.02}$ error.

- Backward difference $u'(2) \simeq \frac{u(2) - u(1.9)}{\Delta x} = \boxed{1.6677}$ appr

  $|\text{exact} - \text{appr}| = \boxed{0.025}$ error.

- Central difference $u'(2) \simeq \frac{u(2.1) - u(1.9)}{2\Delta x} = \boxed{1.6927}$ appr.

  $|\text{exact} - \text{appr}| = \boxed{0.0004}$ error.

- Central Second difference $u''(2) \simeq \frac{u(2.1) - 2u(2) + u(1.9)}{(\Delta x)^2}$

  $= \boxed{0.50021}$ appr.