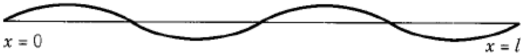
Vibrating string

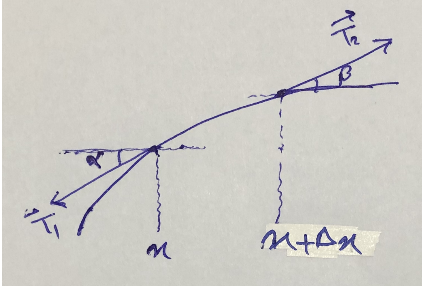
Drive one dimensional wave equation

- Consider a flexible, elastic homogeneous string or thread of length l , which undergoes relatively small transverse vibrations. For instance, it could be a guitar string or a plucked violin string. Assume that it remains in a plane, the motion is purely transverse and the air resistance is negligible. Let $u(x,t)$ be its displacement from equilibrium position at time t and position x. Because the string is perfectly flexible, the tension (force) is directed tangentially along the string. Let $T(x,t)$ be the magnitude of this tension and ρ be the density of the string. Derive a PDE model to describe the vibration of the string.



Vibrating string

- Homogeneous string: the density of the string ρ is a constant.
- Let's take a look at the section of the string from x to $x + \Delta x$ for arbitrary x.
- What's the principle or law governing the vibration?
- Newton's second law: $F = ma$.
- Note that the slope of the string at x is $u_x(x, t)$.



- Where $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ denote the tension respectively at the end points x and $x + \Delta x$ of our small string portion.
- Because our assumption that there is no movement in the horizontal direction, the horizontal components of the tension must be constant. Thus we obtain

$$T_1\cos(\alpha) = T_2\cos(\beta) = T = constant \tag{1}$$

- Recall that the motion is purely transverse. So, in the vertical direction there are two forces, the vertical components of $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are $-T_1\sin(\alpha)$ and $T_2\sin(\beta)$; respectively (The minus sign leads to the component at x is pointing downwards). Now, the Newton's second law tells us the resultant of the two vertical forces equal to the mass of the string portion $\rho\Delta x$ times the acceleration u_{tt}. That is

$$T_2\sin(\beta) - T_1\sin(\alpha) = \rho\Delta x\ u_{tt} \tag{2}$$

Vibrating string

- Using (1) in (2) we obtain

$$\frac{T_2 \sin(\beta)}{T_2 \cos(\beta)} - \frac{T_1 \sin(\alpha)}{T_1 \cos(\alpha)} = \tan(\beta) - \tan(\alpha) = \frac{\rho \Delta x \ u_{tt}}{T}.$$

- Note that the tan values are the corresponding slops of the string at x and $x + \Delta x$. That is, $\tan(\alpha) = u_x(x, t)$ and $\tan(\beta) = u_x(x + \Delta x, t)$.

- Therefore, $\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} = \frac{\rho u_{tt}}{T}$.

- If we let $\Delta x \to 0$ then we obtain the PDE as follows:

$$u_{tt} - c^2 u_{xx} = 0,$$

where $c = \sqrt{\frac{T}{\rho}}$. This is the wave equation and c is the wave speed.

Vibrating string

There are many variations of this equation:

- If significant air resistance r is present, then

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

  where $r > 0$

- If there is a transverse elastic force, then

$$u_{tt} - c^2 u_{xx} + k u = 0,$$

  where $k > 0$

- If there is an externally applied force, it appears as an extra term, thus:

$$u_{tt} - c^2 u_{xx} = f(x, t),$$

  which makes the equation in-homogeneous.

Gradient

Definition 9

The gradient of a function u is defined by

$$\nabla u = (u_x, u_y) \text{ in } 2D,$$

and

$$\nabla u = (u_x, u_y, u_z) \text{ in } 3D$$

Divergent

Definition 10

The divergence of a vector $\overrightarrow{v}$ is defined by

$$\nabla \cdot \overrightarrow{v} = \frac{\partial \overrightarrow{v_1}}{\partial x} + \frac{\partial \overrightarrow{v_2}}{\partial y} \quad \text{for } \overrightarrow{v} = (v_1, v_2) \text{ in } 2D$$

and

$$\nabla \cdot \overrightarrow{v} = \frac{\partial \overrightarrow{v_1}}{\partial x} + \frac{\partial \overrightarrow{v_2}}{\partial y} + \frac{\partial \overrightarrow{v_3}}{\partial z} \quad \text{for } \overrightarrow{v} = (v_1, v_2, v_3) \text{ in } 3D$$

Definition 11 (Laplace operator)

- 2D Laplace operator:
  $\Delta u = \nabla \cdot \nabla u = \nabla \cdot (u_x, u_y) = u_{xx} + u_{yy}$,
- 3D Laplace operator:
  $\Delta u = \nabla \cdot \nabla u = \nabla \cdot (u_x, u_y, u_z) = u_{xx} + u_{yy} + u_{zz}$.

Definition 12 (normal derivative)

Assume $\overrightarrow{n}$ is the unit outward normal vector on ∂D, then the normal derivative of a function u on ∂D is defined by

$$\frac{\partial u}{\partial \overrightarrow{n}} = \nabla u \cdot \overrightarrow{n}$$

- We can similarly get the 2D wave equation:

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}) = 0$$

where $c = \sqrt{\frac{T}{\rho}}$.

- The 3D wave equation:

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy} + u_{zz}) = 0,$$

where $c = \sqrt{\frac{T}{\rho}}$.