# 1.4: Initial/Boundary conditions

**Definition 13**

An initial condition specifies the physical state at a particular time $t_0$.

- Example 1: $u_t = ku_{xx}$ with $u(x, t_0) = \phi(x)$.
- Example 2:
  $u_{tt} = ku_{xx}$ with $u(x, t_0) = \phi(x)$ and $u_t(x, t_0) = \phi(x)$.
- Example 3: $u_t = k(u_{xx} + u_{yy})$ with $u(x, y, t_0) = \phi(x, y)$.

# Boundary conditions

**Definition 14**

A boundary condition specifies the physical state on the boundary of the problem domain.

Example: $u_{tt} = ku_{xx}$ with the conditions
$u(x_0, t) = \phi(t)$ and $u(x_1, t) = \phi(t)$ for all $0 \leq t \leq T$.

# Boundary conditions

**Definition 15**

The three most important types of boundary conditions:

- Dirichlet: $u$ is specified. That is, $u = g$ for some given function $g$.
- Neumann: the normal derivative, which is $\frac{\partial u}{\partial \vec{n}}$ in 2D and 3D but $u_x$ in 1D, is specified. That is, $\frac{\partial u}{\partial \vec{n}} = g$ or $u_x = g$ for some given function $g$.
- Robin: $\frac{\partial u}{\partial \vec{n}} + au$ or $u_x + au$ specified where $a$ is a given function. That is, $\frac{\partial u}{\partial \vec{n}} + au = g$ or $u_x + au = g$ for some given function $g$.

# Boundary conditions

**Definition 16**

If $g = 0$, then we say the boundary condition is homogeneous. Otherwise, it is non-homogeneous.

- Example: Consider the boundary conditions for the displacement $u(x, t)$ of a vibrating string of length $l$.
- If the string is held fixed at both ends, as for a violin string, then

$$u(0, t) = u(l, t) = 0 \ \ \text{(Dirichlet)}.$$

- If the two ends of the string is moves in a specific way, then

$$u(0, t) = g_1(t) \ \text{ and } \ u(l, t) = g_2(t) \ \ \text{(Dirichlet)}.$$

- Assume the left end of the string is still held fixed. If the right end of the string is free to move transversally without any resistance, then

$$u(0,t) = 0 \ \text{(Dirichlet) and } \ u_x(l,t) = 0 \ \text{(Neumann)}$$

- Assume the left end of the string is still held fixed. If the right end of the string is free to move along a track but was attached to a coiled spring or rubber band, which tends to pull the end of the string to the equilibrium position, then

$$u(0,t) = 0 \ \text{(Dirichlet) and } \ u_x(l,t) + au(l,t) = g(t) \ \text{(Robin)}$$

Example : Consider the boundary conditions for the concentration $u$ of a diffusion problem in a tube.

- If the diffusing substance is enclosed in a container D so that none can escape or enter, then the concentration gradient in the normal direction must vanish:

$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial D \text{ (Neumann)}$$

where $\frac{\partial u}{\partial \vec{n}} = \mathbf{n} \cdot \nabla u$, $\mathbf{n}$ denotes unit noumal vector on bdy $D$.

- If the container is permeable and any substance that escapes to the boundary of the container is immediately washed away, then

$$u = 0 \text{ on } \partial D \text{ (Dirichlet).}$$

Example : Consider the boundary conditions for the temperature $u$ of a heat flow problem.

- If the object D, through which the heat is flowing, is perfectly insulated, then no heat flows across the boundary:

$$\frac{\partial u}{\partial \overrightarrow{n}} = 0 \text{ on } \partial D \ \text{(Neumann)}$$

  where $\frac{\partial u}{\partial \overrightarrow{n}} = \mathbf{n} \cdot \nabla u$, $\mathbf{n}$ denotes unit noumal vector on bdy $D$.
- If the object D is immersed in a large tank of specified temperature $g(t)$ and there were perfect thermal conduction, then

$$u = g(t) \text{ on } \partial D \ \text{(Dirichlet).}$$

# 1.5 Well-posedness

**Definition 17**

A well-posed problem consists of a PDE in a domain together with a set of initial and/or boundary conditions that satisfy the following fundamental properties:

- Existence of the solution.
- Uniqueness of the solution.
- Stability of the solution: A small change in the data only cause a small change of the solution.

**Definition 18**

A linear second order PDE in 2D is defined as follows.

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

**Definition 19 (Classification)**

A linear second order PDE can be classified as the following three cases.

- Elliptic equation: $D = a_{12}^2 - a_{11}a_{22} < 0$,
- Hyperbolic equation: $D = a_{12}^2 - a_{11}a_{22} > 0$,
- Parabolic equation: $D = a_{12}^2 - a_{11}a_{22} = 0$.

- Example: $u_{xx} - 5u_{xy} = 0$
  Since $a_{12} = -\frac{5}{2}$, $a_{11} = 1$, and $a_{22} = 0$
  $\Rightarrow D = a_{12}^2 - a_{11}a_{22} = \frac{25}{4} > 0$, so the equation is hyperbolic.
- Example: $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$
- Example: $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$