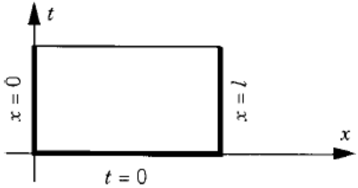
Target problem

- Consider the diffusion/heat equation:

$$u_t = k u_{xx}$$

in a rectangle $\{0 \leq x \leq l, 0 \leq t \leq T\}$ in the space-time plane.



Theorems

Theorem (Maximum/Minimum Principle)

If $u(x,t)$ satisfies the diffusion/heat equation in a rectangle $\{0 \leq x \leq l, 0 \leq t \leq T\}$ in the space-time plane, then the maximum and minimum values of $u(x,t)$ are assumed either initially (on the bottom $t = 0$) or on the lateral sides ($x = 0$ or $x = l$).

Applications

- Example 1: Prove the uniqueness for the solution of the following Dirichlet problem of the diffusion/heat equation:

$$\begin{cases} u_t - ku_{xx} = f(x,t), & 0 < x < l, \;\; t > 0 \\ u(x,0) = \phi(x), & \\ u(0,t) = g(t), \;\; u(l,t) = h(t) & \end{cases}$$

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The solution of the diffusion equation

- Consider the following initial value problem:

$$\begin{cases} u_t = ku_{xx}, & \text{for } -\infty < x < \infty, , \ 0 < t < \infty \\ u(x,0) = \phi(x) \end{cases}$$

- Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) \, dy \\ &= \int_{-\infty}^{\infty} S(x-y)\phi(y) \, dy \end{aligned}$$

where

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-x^2/4kt}.$$

The solution of the diffusion equation

- Example 1: Solve the diffusion equation with the initial condition $u(x,0) = e^{-x}$.
- Then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt-y}\, dy. \end{aligned}$$

- Hint: $\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$ (Homework).

The solution of the diffusion equation

- So by completing square, we obtain

$$
\begin{aligned}
& -(x-y)^2/4kt - y \\
= & -(x-y)^2/4kt - 4kty/4kt \\
= & -\frac{(x-y)^2 + 4kty}{4kt} \\
= & -\frac{x^2 - 2xy + y^2 + 4kty}{4kt} \\
= & -\frac{y^2 + 2(2kt - x)y + x^2}{4kt} \\
= & -\frac{y^2 + 2(2kt - x)y + (2kt - x)^2 - (2kt - x)^2 + x^2}{4kt} \\
= & -\frac{(y + 2kt - x)^2 - (4k^2t^2 - 4ktx + x^2) + x^2}{4kt}
\end{aligned}
$$

The solution of the diffusion equation

- So

$$
\begin{aligned}
& -(x-y)^2/4kt - y \\
= & -\frac{(y+2kt-x)^2 - 4k^2t^2 + 4ktx - x^2 + x^2}{4kt} \\
= & -\frac{(y+2kt-x)^2 - 4k^2t^2 + 4ktx}{4kt} \\
= & -\frac{(y+2kt-x)^2}{4kt} - \frac{-4k^2t^2 + 4ktx}{4kt} \\
= & -\frac{(y+2kt-x)^2}{4kt} + kt - x.
\end{aligned}
$$

The solution of the diffusion equation

- Hence

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt - y}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt} + kt - x}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} e^{kt-x}\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}}\, dy. \end{aligned}$$

- Let $p = \frac{y+2kt-x}{\sqrt{4kt}}$.
- Then $p^2 = \frac{(y+2kt-x)^2}{4kt}$.
- And $\frac{dp}{dy} = \frac{1}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}dp$.

The solution of the diffusion equation

- Hence

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}}\, dy \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \sqrt{4kt}\, dp \\
&= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2}\, dp \\
&= \frac{1}{\sqrt{\pi}} e^{kt-x} \sqrt{\pi} \\
&= e^{kt-x}.
\end{aligned}
$$