1.2 First order PDEs with constant coefficient

Geometric Method (Characteristic Lines):

Solve $au_x + bu_y = 0$ where a and b are constants not both zero.

- By using vector product, it can be written as $au_x + bu_y = (u_x, u_y) \cdot (a, b)$.
- Directional derivative of a function $u(x, y)$ in the direction of a vector $\overrightarrow{v} = (a, b)$: Then $\nabla u \cdot \overrightarrow{v} = (u_x, u_y) \cdot (a, b)$. Here $\nabla u = (u_x, u_y)$ is the gradient of u.
- We note that $\nabla u \cdot \overrightarrow{v} = 0$.
- This means that $u(x, y)$ must be constant in the direction of $\overrightarrow{v} = (a, b)$.
- How do we describe the direction of the vector $\overrightarrow{v} = (a, b)$?
- Characteristic lines: All the lines along the direction of the vector $\overrightarrow{v} = (a, b)$

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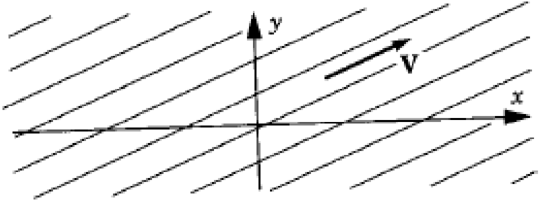
- sketch of characteristic lines



- $u(x, y)$ must be a constant along each characteristic line.
- However, $u(x; y)$ might be different constants along different characteristic lines.
- So $u(x, y)$ is actually a function of the characteristic lines: That is, each characteristic line decides the constant value of $u(x, y)$ along itself.
- Now let's work out the characteristic lines along the direction of the vector $\overrightarrow{v} = (a, b)$.

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- We know that the slope of the characteristic lines is $\frac{b}{a}$ if $a \neq 0$

  different characteristic lines.
- Then the equation for the characteristic lines is $\frac{dy}{dx} = \frac{b}{a}$
  $\Rightarrow y = \frac{b}{a}x + c_1$ if $a \neq 0$
- There is a uniform formula for both of the two cases:
  $bx - ay = c$ where $c = -ac1$ if $a \neq 0$
  characteristic line is uniquely determined by a constant c.
- So each c decides the constant value of $u(x, y)$ along the characteristic line determined by the c.

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- Therefore, the relationship between the constant values of $u(x, y)$ along the characteristic lines and the c is actually a function, say $f(c)$. That is $u(x, y) = f(c)$ along each characteristic line. Since $bx - ay = c$ along each characteristic line. Therefore, $u(x, y) = f(bx - ay)$.

Example

Example. Solve the PDE $4u_x - 3u_y = 0$ together with the auxiliary condition that $u(0, y) = y^3$.

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Coordinate Method: Using the coordinate method to solve $au_x + bu_y = 0$ where a and b are constants not both zero.

- First using change of variable: Let $\tilde{x} = ax + by$ and $\tilde{y} = bx - ay$.
- By the chain rule, we have
  $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial x} = a\frac{\partial u}{\partial \tilde{x}} + b\frac{\partial u}{\partial \tilde{y}}$
  $u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial y} + \frac{\partial u}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial y} = b\frac{\partial u}{\partial \tilde{x}} - a\frac{\partial u}{\partial \tilde{y}}$

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- Then
  $0 = au_x + bu_y = a\left(a\frac{\partial u}{\partial \tilde{x}} + b\frac{\partial u}{\partial \tilde{y}}\right) + \mathrm{b}\left(b\frac{\partial u}{\partial \tilde{x}} - a\frac{\partial u}{\partial \tilde{y}}\right)$
  $\quad = (a^2 + b^2)\frac{\partial u}{\partial \tilde{x}}.$
- Since a and b are constants not both zero, then $\frac{\partial u}{\partial \tilde{x}} = 0$
- So $u = f(\tilde{y}) = f(bx - ay)$

First order PDEs with variable coefficients

- Consider, $a(x,y)u_x + b(x,y)u_y = 0$
  find the equation of the characteristic curves by solving the ODE $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$
- We always obtain a constant c when we solve the ODE, which can uniquely determines the characteristic curves.
- Solve the equation of the characteristic curves for the constant c in terms of x and y. That is, $c = g(x,y)$ for some function $g(x,y)$.
- Then $u(x,y) = f(c) = f(g(x,y))$.
- Use the auxiliary condition to determine the function f. Then obtain the unique solution and its domain.

Example

Solve

1) $u_x + yu_y = 0$ with the auxiliary condition $u(x, 1) = x^2$.
2) $yu_x + xu_y = 0$ with the auxiliary condition $u(x, 0) = x^4$.
3) $u_x + 2xy^2 u_y = 0$.