Salahaddin University-Erbil
College of Science-Department of Mathematics

Numerical Analysis
3rd Year Second Semester
2023-2024

Lecturer: Dr. Jabar S. Hassan

Numerical Methods

Intermediate value Theorem: If a function f(x) is continuous in closed interval [a,b] and satisfies $f(a)f(b) < 0$ then there exists atleast one real root of the equation f(x) = 0 in open interval (a,b).

Algebraic equations are equations containing algebraic terms (different powers of x). For example $x^2-7x+6=0$

Transcendental equations are equations containing non-algebraic terms like trigonometric, exponential, logarithmic terms. For example $\sin x - e^x = 0$

A. **Fixed point iteration method for solving equation f(x) = 0**
   Procedure

   Step-I We rewrite the equation f(x) = 0 of the form x = h(x), x=g(x), x = D(x)

   We find the interval (a,b) containing the solution (called root).

   Step-II We choose that form say x = h(x) which satisfies $| h'(x) | < 1$ in interval (a,b) containing the solution (called root).

   Step-III We take $x_{n+1} = h(x_n)$ as the successive formula to find approximate solution (root) of the equation f(x) = 0

   Step-III Let $x=x_0$ be initial guess or initial approximation to the equation f(x) = 0

Then $x_1=h(x_1)$, $x_2=h(x_2)$, $x_3=h(x_3)$ and so on.We will continue this process till we get solution (root) of the equation $f(x) = 0$ up to desired accuracy.

Convergence condition for Fixed point iteration method

If $x=a$ is a root of the equation $f(x) = 0$ and the root is in interval (a, b). The function $h'(x)$ and $h(x)$ defined by $x = h(x)$ Is continuous in (a,b) .Then the approximations $x_1=h(x_1)$, $x_2=h(x_2)$, $x_3=h(x_3)$ .... converges to the root $x=a$ provided $|h'(x)| < 1$ in interval (a,b) containing the root for all values of x.

Problems

1. **Solve x^3 - sin x -1 =0 correct to two significant figures by fixed point iteration method correct up to 2 decimal places.**

Solution: $x^3 - \sin x - 1 = 0$........ ....(1)

Let $f(x) = x^3 - \sin x - 1$

$f(0) = -1$, $f(1) = -0.8415$, $f(2)=6.0907$

As $f(1)f(2)< 0$ by Intermediate value Theorem the root of real root of the equation $f(x) = 0$ lies between 1 and 2

Let us rewrite the equation $f(x) = 0$ of the form $x = h(x)$

$x= (1 + \text{Sin } x)^{1/3} = h_1(x)$ and $x = \text{Sin}^{-1}(x^3 - 1)= h_2(x)$

We see that $|h_1'(x)| < 1$ in interval (1,2) containing the root for all values of x.

We use $x_{n+1}= (1 + \text{Sin } x_n)^{1/3}$ as the successive formula to find approximate solution (root) of the equation (1).

Let $x_0 =1.5$ be initial guess to the equation (1).

Then $x_1= (1 + \text{Sin } x_0)^{1/3} = (1 + \text{Sin } 1.5)^{1/3} = 1.963154$

$x_2= (1 + \text{Sin } x_1)^{1/3} = (1 + \text{Sin } 1.963154)^{1/3} = 1.460827$

$x_3= (1 + \text{Sin } x_2)^{1/3} = (1 + \text{Sin } 1.460827)^{1/3} = 1.440751$

$x_4= (1 + \text{Sin } x_3)^{1/3} = (1 + \text{Sin } 1.440751)^{1/3} = 1.441289$

which is the root of equation (1) correct to two decimal places.

Newton Raphson Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation $f(x) = 0$.

Step-II Let $x=x_0$ be initial guess or initial approximation to the equation $f(x) = 0$

Step-III We use $x_{n+1} = x_n - [f(x_n) / f'(x_n)]$ as the successive formula to find approximate solution (root) of the equation $f(x) = 0$

Step-III Then x_1, x_2, x_3 .... and so on are calculated and we will continue this process till we get root of the equation $f(x) = 0$ up to desired accuracy.

2. Solve x - 2sin x - 3 = 0 correct to two significant figures by Newton Raphson method correct up to 5 significant digits.

Solution: $x - 2\sin x - 3 = 0$........ ....(2)

Let $f(x) = x - 2\sin x - 3$

$f(0) = -3$, $f(1) = -2 - 2 \text{ Sin } 1$, $f(2) = -1 - 2 \text{ Sin } 2$, $f(3) = -2 \text{ Sin } 3$, $f(4) = 1 - 2 \text{ Sin } 4$

$f(-2) = -5 + 2 \text{ Sin } 2$, $f(-1) = -4 + 2 \sin 1$

As $f(3)f(4) < 0$ by Intermediate value Theorem the root of real root of the equation $f(x) = 0$ lies between 3 and 4

Let Let $x_0 = 4$ be the initial guess to the equation (2).

Then $x_1 = x_0 - [f(x_0) / f'(x_0)] = 2 - f(2)/f'(2) = 3.09900$

$x_2 = x_1 - [f(x_1) / f'(x_1)] = -1.099 - f(-1.099)/f'(-1.099) = 3.10448$

$x_3 = x_2 - [f(x_2) / f'(x_2)] = 3.10450$

$x_4 = x_3 - [f(x_3) / f'(x_3)] = 3.10451$

which is the root of equation (2) correct to five significant digits.

Secant Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation $f(x) = 0$.

Step-II Let $x = x_0$ be initial guess or initial approximation to the equation $f(x) = 0$

Step-III We use $x_{n+1} = x_n - [(x_n - x_{n-1})f(x_n)] / [f(x_n) - f(x_{n-1})]$ as the successive formula to find approximate solution (root) of the equation $f(x) = 0$

Step-III Then x_1, x_2, x_3 .... and so on are calculated and we will continue this process till we get root of the equation $f(x) = 0$ up to desired accuracy.

3 . Solve Cos x = x e^x correct to two significant figures by Secant method correct up to 2 decimal places.

Solution: $\cos x = x e^x$ ....(3)

Let $f(x) = \cos x - x e^x$

f(0) = 1, f(1)= Cos 1 – e = - 2 .178

As f(0)f(1)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 0 and 1

Let Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses to the equation (3).

Then

$$x_2 = x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)} = 1 - \frac{(1-0) f(1)}{f(1) - f(0)} = 1 - \frac{2.178}{-3.178} = 0.31465$$

$f(x_2)$= f (0.31465)= Cos (0.31465) - 0.31465 $e^{0.31465}$ = 0.51987

$$x_3 = x_2 - \frac{(x_2 - x_1) f(x_2)}{f(x_2) - f(x_1)} = 0.31465 - \frac{(0.31465 - 1) f(0.31465)}{f(0.31465) - f(1)} = 0.44672$$

$$x_4 = x_3 - \frac{(x_3 - x_2) f(x_3)}{f(x_3) - f(x_2)} = 0.64748$$

$$x_5 = x_4 - \frac{(x_4 - x_3) f(x_4)}{f(x_4) - f(x_3)} = 0.44545$$

which is the root of equation (3) correct to two decimal places.

4.   Solve x^4 - x - 7 = 0 correct to two significant figures by Newton- Raphson method correct up to 6 significant digits.

Solution: x^4 - x - 7 = 0........   ....(4)

Let f(x) = x^4 - x - 7

f(0) = -7, f(1)= -7 , f(2)= 5

As f(1)f(2)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 1 and 2

Let Let x_0 = 1.5   be the initial guess to the equation (2).

Then x_1= x_0 - [f(x_0) / f′(x_0)]   = 1.5 -   f(1.5)/ f′(1.5) =   1.78541

x_2= x_1 - [f(x_1) / f′(x_1)] = 1.7854-   f1.7854)/ f′(1.7854) = 1.85876

x_3= x_2 - [f(x_2) / f′(x_2)] =   1.85643

x_4= x_3 - [f(x_3) / f′(x_3)]   = 1.85632

which is the root of equation (2) correct to 6S.

INTERPOLATION

Interpolation is the method of finding value of the dependent variable y at any point x using the following given data.

x	x_0	x_1	x_2	x_3	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	y_n

This means that for the function y = f(x) the known values at x = x_0, x_1, x_2,........., x_n are respectively

y = y_0, y_1, y_2,.........,y_n and we want to find value of y at any point x.

For this purpose we fit a polynomial to these datas called interpolating polynomial. After getting the polynomial p(x) which is an approximation to f(x), we can find the value of y at any point x.

Finite difference operators

Let us take equispaced points x_0, x_1, x_2,........., x_n

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Forward difference operator $\Delta y_n = y_{n+1} - y_n$

Backward difference operator $\nabla y_n = y_n - y_{n-1}$

Central difference operator $\delta y_i = y_{i+1/2} - y_{i-½}$

Shift Operator $E\, y_i = y_{i+1}$

Newton's Forward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\Delta y_n = y_{n+1} - y_n$ is called the first Forward difference

i.e. $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$ and so on.

$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$ is called the second Forward difference

i.e. $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ and so on.

Newton's Forward difference Interpolation formula is

$$Pn(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$$

$$+ \dots\dots + [p(p-1)(p-2)\dots\dots(p-n-1)/n!]\Delta^n y_0$$

Where $p = (x - x_0)/h$

Problems

5. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=5**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$,

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 5 lies between 0 and 10 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (x - 0)/10 = 0.1x$

$P_n(x) = y_0 + p\Delta y_0 + [\,p(p-1)/2!\,]\Delta^2 y_0 + [\,p(p-1)(p-2)/3!\,]\Delta^3 y_0$

$+ [\,p(p-1)(p-2)(p-3)/4!\,]\Delta^4 y_0$

$= 7 + 0.1x\,(11) + [0.1x(0.1x - 1)/2!\,]\,(3) + [0.1x(0.1x - 1)\,(0.1x - 2)/3!\,]\,(2)$

$+ [0.1x(0.1x - 1)\,(0.1x - 2)\,(0.1x - 3)/4!\,]\,(10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 + 0.2x)/3$

$+ 0.416\,(0.0001x^4 - 0.006x^3 + 0.11x^2 - 0.6x)$

$P_n(x) = 0.0000416\,x^4 - 0.0022\,x^3 + 0.05x^2 + 1.26\,x + 7$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x=5 we put x=5 in the interpolating polynomial to get

$y(5) = P_n(5) = 0.0000416\,(5)^4 - 0.0022\,(5)^3 + 0.05(5)^2 + 1.26\,(5) + 7 = 14.301$

6. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=24**

x	20	35	50	65	80
y	3	11	24	50	98

Solution

Here $x_0 = 20, x_1 = 35, x_2 = 50, x_3 = 65, x_4 = 80,$

$x_1 - x_0 = 15 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 24 lies between 20 and 35 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Here $x_0 = 20$, $y_0 = 3$, $h = x_1 - x_0 = 35 - 20 = 15$

$\Delta y_0 = 8$, $\Delta^2 y_0 = 5$,

$\Delta^3 y_0 = 8$, $\Delta^4 y_0 = 1$

$p = (x - x_0)/h = (x - 20)/15 = 0.0666\,x - 1.333333$

Forward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
20	**3**				
		8			
35	11		**05**		
		13		**08**	
50	24		13		0**1**
		26		9	
65	50		22		
		48			
80	98				

$P_n(x) = y_0 + p \Delta y_0 + [p(p-1)/2!] \Delta^2 y_0 + [p(p-1)(p-2)/3!] \Delta^3 y_0$

$+ [p(p-1)(p-2)(p-3)/4!] \Delta^4 y_0$

$= 3 + 8(0.0666x - 1.333333) + 5[(0.0666x - 1.333333)(0.0666x - 2.333333)/2!]$

$+ 8[(0.0666x - 1.333333)(0.0666x - 2.333333)(0.0666x - 3.333333)/3!]$

$+ [(0.0666x - 1.333333)(0.0666x - 2.333333)(0.0666x - 3.333333)(0.0666x - 4.333333)/4!]$

$= 3 + 0.53333333x - 10.666666 + 0.01111x^2 - 0.16666666x + 7.777777$

$+ [(0.5333333x - 10.66666)(0.0666x - 2.333333)(0.011111x - 0.5555555)]$

$+ [(0.0666x - 1.333333)(0.0666x - 2.333333)(0.011111x - 0.5555555)(0.01666x - 1.083333)]$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x = 24 we put x = 24 in the interpolating polynomial to get

$y(24) = P_n(24) = 3 + (0.53333333)24 - 10.666666 + 0.01111(24^2) - (0.16666666)24 + 7.777777$

$+ [(0.5333333(24) - 10.66666)(0.0666(24) - 2.333333)(0.011111(24) - 0.5555555)]$

$+ [(1.59999 - 1.333333)(1.59999 - 2.333333)(0.266666 - 0.5555555)(0.399999 - 1.083333)]$

Newton's Backward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\nabla y_n = y_n - y_{n-1}$ is called the first backward difference

i.e. $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ and so on.

$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$ is called the second backward difference

i.e. $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$ and so on.

Newton's backward difference Interpolation formula is

$$Pn (x) = y_n + p \nabla y_n + [p(p + 1)/2!] \nabla^2 y_n + [p(p + 1) (p + 2)/3!] \nabla^3 y_n$$

$$+ .... + [p(p + 1) (p+ 2).......(p +n - 1)/n!] \nabla^n y_n$$

Where $p = (x - x_n)/h$

7. Using following data to find the value of y at x = 35

x	0	10	20	30	40
y	7	18	32	48	85

Solution :

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$,

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 35 lies between 3 0 and 40 and at the end of the table and given data is equispaced ,we have to use Newton's Backward difference Interpolation.

Here $x = 35$, $x_n = 40$, $y_n = 87$, $h= x_1 - x_0 = 10-0 = 10$

$\nabla y_n = 36$, $\nabla^2 y_n = 17$,

$\nabla^3 y_n = 12$, $\nabla^4 y_n=10$

$p = (x - x_n)/h = (35 - 40)/10 = -0.5$

Backward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
0	7				
		11			
10	18		03		
		14		02	
20	32		05		**10**
		19		**12**	
30	51		**17**		
		36			
40	**87**				

$P_n(x) = y_n + p \nabla y_n + [p(p+1)/2!] \nabla^2 y_n + [p(p+1)(p+2)/3!] \nabla^3 y_n$

$+ [p(p+1)(p+2)(p+3)/4!] \nabla^4 y_n$

$= 87 + (-0.5)(36) + (-0.5)(-0.5+1)(17)/2! + (-0.5)(-0.5+1)(-0.5+2)(12)/3!$

$+ (-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(10)/4!$

$= 87 - 18 - 0.25(8.5) - 0.25(18)/6 - 0.25(15)(2.5)/24$

$= 65.734375$

This is the approximate value of y at x=35

$y(35) = P_n(35) = 65.734375$

Inverse Interpolation

The process of finding the independent variable x for given values of f(x) is called Inverse Interpolation .

8. Solve ln x = 1.3 by inverse Interpolation using x= G(y) with G(1)=2.718 ,G(1.5)= 4.481 , G(2)= 7.387 ,G(2.5)= 12.179 and find value of x

Forward difference table

y	x	Δ y	Δ^2 y	Δ^3 y
1	**2.718**			
		1.763		
1.5	4.481		**1.143**	
		2.906		**0.743**
2	7.387		1.886	
		4.792		
2.5	12.179			

Here $y_0 = 1$, $h=y_1 - y_0 = 1.5 - 1 = 0.5$

$x_0 = 2.718$, $\Delta x_0 = 1.763$, $\Delta^2 x_0 = 1.143$,

$\Delta^3 x_0 = 0.743$

$p = (y - y_0)/h = (1.3 - 1)/0.5 = 0.6$

Newton's Forward difference Interpolation formula is

$P_n(y) = x_0 + p\,\Delta x_0 + [\,p(p-1)/2!\,]\,\Delta^2 x_0 + [\,p(p-1)(p-2)/3!\,]\,\Delta^3 x_0$

$= 2.718 + 0.6(1.763) + 0.6(0.6-1)1.143/2 + 0.6(0.6-1)(0.6-2)0.743/6$

$= 3.680248$

Lagrange Interpolation (data may not be equispaced)

Lagrange Interpolation can be applied to arbitrary spaced data.

Linear interpolation is interpolation by the line through points (x_1,y_1) and (x_0,y_0)

Linear interpolation is $P_1(x)= l_0 y_0 + l_1 y_1$

Where $l_0 = (x- x_1) /(x_0- x_1)$ and $l_1 = (x- x_0) /(x_1- x_0)$

Quadratic Lagrange Interpolation is the Interpolation through three given points (x_2,y_2) , (x_1,y_1) and (x_0,y_0) given by the formula

$P_2(x)= l_0 y_0 + l_1 y_1 + l_2 y_2$

Where $l_0 = \dfrac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)}$, $l_1 = \dfrac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)}$ $and\ l_2 = \dfrac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)}$

9. Using quadratic Lagrange Interpolation find the Lagrange interpolating polynomial $P_2(x)$ and hence find value of y at x=2 Given y(0) = 15, y(1) = 48, y(5) = 85

Solution :

Here $x_0 = 0, x_1 = 1, x_2 = 5$ and $y_0 = 15, y_1 = 48, y_2 = 85$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 4$

The given data is not equispaced.

$$l_0 = \frac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)} = \frac{(x-5)(x-1)}{(0-5)(0-1)} = \frac{(x^2-6x+5)}{5}$$
$$l_1 = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} = \frac{(x-5)(x-0)}{(1-5)(1-0)} = \frac{(x^2-5x)}{(-4)}$$
$$and\ l_2 = \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)} = \frac{(x-1)(x-0)}{(5-1)(5-0)} = \frac{(x^2-x)}{20}$$

$$y = l_0\, y_0 + l_1\, y_1 + l_2\, y_2 = \frac{(x^2-6x+5)}{5}15 + \frac{(x^2-5x)}{(-4)}48 + \frac{(x^2-x)}{20}85$$
$$= -4.75x^2 + 37.75x + 15$$

Which is the Lagrange interpolating polynomial $P_2(x)$

Hence at x=2 the value is $P_2(2) = -4.75(2^2)+37.75(2)+15 = 71.5$

General Lagrange Interpolation is the Interpolation through n given points (x_0,y_0), (x_1,y_1) , (x_2,y_2)...................... , (x_n,y_n) given by the formula

$P_n(x)= l_0 y_0 + l_1 y_1 + l_2 y_2 + \ldots\ldots\ldots\ldots + l_n y_n$

Where $l_0 = \dfrac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_2)(x-x_1)}{(x_0-x_n)\ldots\ldots\ldots\ldots(x_0-x_2)(x_0-x_1)}$

$$l_1 = \frac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_2)(x-x_0)}{(x_1-x_n)\ldots\ldots\ldots\ldots(x_1-x_2)(x_1-x_0)}$$

$$l_2 = \frac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_1)(x-x_0)}{(x_2-x_n)\ldots\ldots\ldots\ldots(x_2-x_1)(x_2-x_0)}$$

………

……….

$$and\ l_n = \frac{(x-x_{n-1})\ldots\ldots\ldots\ldots(x-x_1)(x-x_0)}{(x_n-x_{n-1})\ldots\ldots\ldots\ldots(x_2-x_1)(x_2-x_0)}$$

10. Using Lagrange Interpolation find the value of y at x=8

Given y(0) = 18, y(1) = 42, y(7) = 57 and y(9) = 90

Solution :

Here $x_0 = 0, x_1 = 1, x_2 = 7, x_3 = 9$ and $y_0 = 26, y_1 = 40, y_2 = 75, y_3 = 90$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 6$

The given data is not equispaced.

$$l_0 = \frac{(x-x_3)(x-x_2)(x-x_1)}{(x_0-x_3)(x_0-x_2)(x_0-x_1)} = \frac{(8-9)(8-7)(8-1)}{(0-9)(0-7)(0-1)} = \frac{-7}{-63} = \frac{1}{9}$$

$$l_1 = \frac{(x-x_3)(x-x_2)(x-x_0)}{(x_1-x_3)(x_1-x_2)(x_1-x_0)} = \frac{(8-9)(8-7)(8-0)}{(1-9)(1-7)(1-0)} = \frac{(-8)}{(48)} = \frac{1}{6}$$

$$l_2 = \frac{(x-x_3)(x-x_1)(x-x_0)}{(x_2-x_3)(x_2-x_1)(x_2-x_0)} = \frac{(8-9)(8-1)(8-0)}{(7-9)(7-1)(7-0)} = \frac{-56}{-84} = \frac{2}{3}$$

$$and\ \ l_3 = \frac{(x-x_2)(x-x_1)(x-x_0)}{(x_3-x_2)(x_3-x_1)(x_3-x_0)} = \frac{(8-7)(8-1)(8-0)}{(9-7)(9-1)(9-0)} = \frac{56}{144} = \frac{7}{18}$$

$$y = l_0\ y_0 + l_1\ y_1 + l_2\ y_2 + l_3\ y_3 = \frac{1}{9}(18) + \frac{1}{6}(42) + \frac{2}{3}(57) + \frac{7}{18}(90)$$

$$= 2 + 7 + 38 + 35 = 82$$

Which is the value of y at x=8

Newton divided difference Interpolation (data may not be equispaced)

Newton divided difference Interpolation can be applied to arbitrary spaced data.

The first divided difference is $f[x_0 , x_1] = (y_1 - y_0)/ (x_1 - x_0)$

$$f[x_1 , x_2] = (y_2 - y_1)/ (x_2 - x_1)$$

The second divided difference is

$$f[x_0 , x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

The third divided difference is

$$f[x_0 , x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0 , x_1, x_2]}{x_3 - x_0}$$

The nth divided difference is

$$f[x_0 , x_1, x_2, x_3, \ldots\ldots, x_n] = \frac{f[x_1, x_2, x_3, \ldots\ldots, x_n] - f[x_0 , x_1, x_2, \ldots\ldots, x_{n-1}]}{x_n - x_0}$$

Newton divided difference Interpolation formula is

$$Y = y_0 + (x-x_0)\ f[x_0 , x_1] + (x-x_0)(x-x_1)\ f[x_0 , x_1, x_2] + \ldots\ldots\ldots\ldots$$

$$+ (x-x_0)(x-x_1) \ldots\ldots\ldots (x-x_{n-1})\ f[x_0 , x_1, x_2, \ldots\ldots\ldots\ldots, x_n]$$

Problems

11. Using following data find the Newton's divided difference interpolating polynomial and also find the value of y at x= 15

x	0	6	20	45
y	30	48	88	238

Newton's divided difference table

x	y	First divided difference	Second divided difference	Third divided difference
0	**30**			
		(48-30)/6=**3**		
6	48		(8-3)/11=**0.45**	
		(88-48)/5=8		(0.1 -0.45)/26 = **-0.0136**
11	88		(10-8)/20=0.1	
		(238-88)/15=10		
26	238			

$$Y = y_0 + (x-x_0)\ f[x_0, x_1] + (x-x_0)(x-x_1)\ f[x_0, x_1, x_2]$$

$$+ (x-x_0)(x-x_1)(x-x_2)\ f[x_0, x_1, x_2, x_3]$$

$$= 30 + 3x + x(x-6)(0.45) + x(x-6)(x-11)(-0.0136)$$

The value of y at x= 15

= 30 +3(15) +15(9)(0.45)+ 15(9)(4)(-0.0136) = 128.406

NUMERICAL DIFFERENTIATION

When a function y = f(x) is unknown but its values are given at some points like (x_0, y_0), (x_1, y_1), ... (x_n, y_n) or in form of a table, then we can differentiate using numerical differentiation.

Sometimes it is difficult to differentiate a composite or complicated function which can be done easily in less time and less number of steps by numerical differentiation.

We use following methods for numerical differentiation.

(i) Method based on finite difference operators

(ii) Method based on Interpolation

(i) Method based on finite difference operators

Newton's forward difference Interpolation formula is

$Pn(x) = y_0 + p\Delta y_0 + [\, p(p-1)/2!\,]\, \Delta^2 y_0 + [\, p(p-1)(p-2)/3!\,]\, \Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$

where $p = (x - x_0)/h$

Newton's backward difference Interpolation formula is

$Pn(x) = y_n + p\nabla y_n + [\, p(p+1)/2!\,]\, \nabla^2 y_n + [\, p(p+1)(p+2)/3!\,]\, \nabla^3 y_n$

$+ \ldots\ldots\ldots\ldots\ldots\ldots + [\, p(p+1)(p+2)\ldots\ldots(p+n-1)/n!\,]\, \nabla^n y_n$

where $p = (x - x_n)/h$

Using forward difference the formula for numerical differentiation is

$y'(x_0) = (1/h)\, [\, \Delta y_0 - \Delta^2 y_0/2 + \Delta^3 y_0/3 + \ldots\ldots\ldots\ldots\ldots\ldots\,]$

$y''(x_0) = (1/h^2)\, [\, \Delta^2 y_0 - \Delta^3 y_0 + (11/12)\, \Delta^4 y_0 \ldots\ldots\ldots\ldots\,]$

Using backward difference the formula for numerical differentiation is

$y'(x_n) = (1/h)\, [\, \nabla y_n + \nabla^2 y_n/2 + \nabla^3 y_n/3 + \ldots\ldots\ldots\ldots\ldots\ldots\,]$

$y''(x_n) = (1/h^2)\, [\, \nabla^2 y_n + \nabla^3 y_n + (11/12)\, \nabla^4 y_n \ldots\ldots\ldots\ldots\ldots\,]$

If we consider the first term only the formula becomes

$y'(x_0) = (1/h)\, [\, \Delta y_0\,] = (y_1 - y_0)/h$

$y''(x_0) = (1/h^2)\, [\, \Delta^2 y_0\,] = (\Delta y_1 - \Delta y_0)/h^2$

$= [\, (y_2 - y_1) - (y_1 - y_0)\,]/h^2 = [\, y_2 - 2y_1 + y_0\,]/h^2$

12. **Using following data find the first and second derivative of y at x=0**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$

Forward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (4 - 0)/10 = 0.4$

$y'(x_0) = (1/h)\ [\ \Delta y_0 - \Delta^2 y_0/2 + \Delta^3 y_0/3 - \Delta^4 y_0/4 + \ldots\ldots\ldots\ldots\ldots\ldots\ ]$

$= 0.1\ [\ 11 - 3/2 + 2/3 - 10/4\ ] = 0.7666$

$y''(x_0) = (1/h^2)\ [\ \Delta^2 y_0 - \Delta^3 y_0 + (11/12)\ \Delta^4 y_0\ \ldots\ldots\ldots\ldots\ ]$

$= (1/100)\ [\ 3 - 2 + (11/12)\ 10\ ] = 0.10166$

(ii) Method based on Interpolation

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4y_1 - y_2)/(2h)$

$y'(x_1) = (y_2 - y_0)/(2h)$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h)$

The second derivative is constant i.e. same at all points because of quadratic interpolation and the interpolating polynomial is of degree two. Hence we must have

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_1) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_2) = (y_0 - 2y_1 + y_2)/(2h)$

Problems

13. **Using following data find the value of first and second derivatives of y at x=30**

x	10	30	50
y	42	64	88

Solution

Here $x_0 = 10$, $x_1 = 30$, $x_2 = 50$, $h = x_1 - x_0 = 30 - 10 = 20$

$y_0 = 42$, $y_1 = 64$, $y_2 = 88$

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{64-42}{30-10} = 1.1$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4y_1 - y_2)/(2h) = [-3(42) + 4(64) - 88]/40 = 1.05$

$y'(x_1) = (y_2 - y_0)/(2h) = (88 - 42)/40 = 1.15$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h) = (42 - 256 + 264)/40 = 1.25$

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h) = (42 - 128 + 88)/40 = 0.05$

14. **Using following data find the value of first and second derivatives of y at x=12**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0, x_1 = 10, x_2 = 20, x_3 = 30, x_4 = 40,$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (x - 0)/10 = 0.1x$

$P_n(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$

$+ [p(p-1)(p-2)(p-3)/4!]\Delta^4 y_0$

$= 7 + 0.1x(11) + [0.1x(0.1x - 1)/2!](3) + [0.1x(0.1x - 1)(0.1x - 2)/3!](2)$

$+ [0.1x(0.1x - 1)(0.1x - 2)(0.1x - 3)/4!](10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 + 0.2x)/3$

$+ 0.416(0.0001x^4 - 0.006x^3 + 0.11x^2 - 0.6x)$

$y = P_n(x) = 0.0000416x^4 - 0.0022x^3 + 0.05x^2 + 1.26x + 7$ ....(1)

Differentiating (1) w.r. to x we get

$y' = 0.0001664x^3 - 0.0066x^2 + 0.1x + 1.26$ .... (2)

$y'(12) = 1.7971392$ at $x = 12$

Differentiating (2) w.r. to x we get

$y'' = 0.0004992\, x^2 - 0.0132\, x + 0.1$

$y''(12) = 0.0134848$ at $x = 12$

NUMERICAL INTEGRATION

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us divide the interval [a, b] into n number of equal subintervals so that length of each subinterval

is $h = (b - a)/n$

The end points of subintervals are $a = x_0, x_1, x_2, x_3, \ldots\ldots, x_n = b$

Trapezoidal Rule of integration

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$. Then from x=a to x=b the area under curve of y = f(x) is approximately equal to sum of the areas of n trapezoids of each n subintervals.

So the integral $I = \int_a^b f(x)\, dx = (h/2)[y_0 + y_1] + (h/2)[y_1 + y_2] + (h/2)[y_2 + y_3]$

$+ \ldots\ldots\ldots + (h/2)[y_{n-1} + y_n]$

$= (h/2)[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1} + y_n]$

$= (h/2)[y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$

Which is called trapezoidal rule.

The error in trapezoidal rule is $-\frac{b-a}{12} h^2 f''(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 1/3rd rule)

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots\ldots\ldots, (x_n, y_n)$.

We are taking two strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons rule of Numerical integration must be even.

The length of each subinterval is h = (b – a)/(2m)

The formula is

$$I = \int_a^b f(x)\, dx = (h/3)\,[\, y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\,]$$

The error in Simpson 1/3rd rule is $-\frac{b-a}{180} h^4 f^{\prime v}(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 3/8th rule)

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

We are taking three strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons $3/8^{th}$ rule of Numerical integration must be multiple of 3.

The length of each subinterval is h = (b – a)/(3m)

The formula is

$$I = \int_a^b f(x)\, dx = (3h/8)\,[\, y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{3m-3})\,]$$

The error in Simpson 1/3rd rule is $-\frac{b-a}{80} h^4 f^{\prime v}(\theta)$ where $a < \theta < b$

15. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6**

$$\int_0^6 e^{(-x^2)}\, dx$$

Solution:

Here integrand $y = f(x) = \exp(-x^2)$

a=0, b=6, h= (b-a)/n= (6-0)/6=1

x	0	1	2	3	4	5	6
Y= $\exp(-x^2)$	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	e^{-36}
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) **Using Trapezoidal rule**

$$I = (h/2)[\, y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots\ldots + y_{n-1})\,]$$

$$= (1/2)[\, y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\,]$$

$$= 0.5\,[\,1 + e^{-36} + 2(e^{-1} + e^{-4} + e^{-9} + e^{-16} + e^{-25})\,]$$

(ii) **Using Simpsons rule**

$$I = (h/3)\,[\, y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\,]$$

$$= (h/3)\,[\, y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\,]$$

$$= (1/3)\,[\,1 + e^{-36} + 4(e^{-1} + e^{-9} + e^{-25}) + 2(e^{-4} + e^{-16})\,]$$

(iii) **Using Simpsons 3/8th rule**

$$I = (3h/8)\,[\, y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{3m-3})\,]$$

$$= (3h/8)\,[\, y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\,]$$

$$= (3/8)\,[\,1 + e^{-36} + 3(e^{-1} + e^{-4} + e^{-16} + e^{-25}) + 2(e^{-9})\,]$$

16. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =8 and compare the result**

$$\int_0^{0.8} \frac{dx}{4 + x^2}$$

Solution:

Here integrand $y = f(x) = (4 + x^2)^{-1}$

a=0, b= 0.8 ,   h= (b-a)/n= (0.8-0)/8= 0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Y= $(4 + x^2)^{-1}$	1/4	1/4.01	1/4.04	1/4.09	1/4.16	1/4.25	1/4.36	1/4.49	1/4.64
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6	Y_7	y_8

(i) **Using Trapezoidal rule**

$$I = (h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \dots\dots + y_{n-1})\ ]$$

$$= (0.1/2)[\ y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\ ]$$

= 0.05 [0.25+ 1/4.64+   2(1/4.01+ 1/4.04+ 1/4.09+1/4.16+ 1/4.25+ 1/4.36+1/4.49)]

(ii) **Using Simpsons rule**

$$I = (h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \dots\dots + y_{2m-1}) + 2(\ y_2 + y_4 + \dots\dots + y_{2m-2})\ ]$$

$$= (h/3)\ [\ y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(\ y_2 + y_4 + y_6)\ ]$$

= (0.1/3) [0.25+ 1/4.64+ 4(1/4.01+ 1/4.09+ 1/4.25+ 1/4.49)

+2(1/4.04+1/4.16+ 1/4.36)]

By direct integration we get

$$\int_0^{0.8} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{0.8} = 0.5\left[\tan^{-1}0.4 - \tan^{-1}0\right] = 0.5\tan^{-1}0.4$$

=10.900704743176

Comparing the result we get error in Trapezoidal and Simpsons rule.

17. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6**

I = $\int_{0}^{0.6} \frac{dx}{\sqrt{1+x}}$

Solution:

Here integrand y = f(x) = $\frac{1}{\sqrt{1+x}}$

a=0, b= 0.6 , h= (b-a)/n = (0.6-0)/6 = 0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6
Y= $\frac{1}{\sqrt{1+x}}$	1	$\frac{1}{\sqrt{1.1}}$ =0.953462	$\frac{1}{\sqrt{1.2}}$ =0.912871	$\frac{1}{\sqrt{1.3}}$ =0.877058	$\frac{1}{\sqrt{1.4}}$ =0.845154	$\frac{1}{\sqrt{1.5}}$ =0.816496	$\frac{1}{\sqrt{1.6}}$ =0.790569
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) **Using Trapezoidal rule**

I = $(h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \dots\dots + y_{n-1})\ ]$

= $(0.1/2)[\ y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\ ]$

= 0.05 [1+ 0.790569+ 2(0.953462 + 0.912871 + 0.877058+0.845154+0.816496)]

(ii) **Using Simpsons rule**

I = $(h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \dots\dots + y_{2m-1}) + 2(y_2 + y_4 + \dots\dots + y_{2m-2})\ ]$

= $(h/3)\ [\ y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\ ]$

= (0.1/3) [1+ 0.790569+ 4(0.953462 + 0.877058+0.816496)+2(0.912871 + 0.845154)]

(iii) **Using Simpsons 3/8th rule**

$I = (3h/8) \; [\, y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{3m-3}) \,]$

$= (3h/8) \; [\, y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) \,]$

$= (0.3/8) \; [\, 1+ 0.790569+$

$3(0.953462 + 0.912871 + 0.845154 + 0.816496)+2(0.877058)]$

UNIT-II

Linear System: Solution by iteration

Gauss-Seidal iteration method

This is an iterative method used to find approximate solution of a system of linear equations.

Some times in iterative method convergence is faster where matrices have large diagonal elements. In this case Gauss elimination method require more number of steps and more row operations. Also sometimes a system has many zero coefficients which require more space to store zeros for example 30 zeros after or before decimal point. In such cases Gauss-Seidal iteration method is very useful to overcome these difficulties and find approximate solution of a system of linear equations.

Procedure:

We shall find a solution x of the system of equations Ax=b with given initial guess x^0.

A is an n x n matrix with non-zero diagonal elements

Step-I Rewrite the given equations in such a way that in first equation coefficient of x_1 is maximum, in second equation coefficient of x_2 is maximum, in third equation coefficient of x_3 is maximum and so on.

Step-II From first equation write x_1 in terms of other variables x_2 , x_3 , x_4 etc.

From the second equation write x_2 in terms of other variables x_1 , x_3 , x_4 etc.

From third equation write x_3 in terms of other variables x_1 , x_2 , x_4 etc.

And so on write all equations in this form.

Step-III

If initial guess is given we take that value otherwise we assume X = (1, 1, 1) as initial guess.

Put $x_2 = 1, x_3 = 1$ in first equation to get x_1 ....(1)

Put $x_3 = 1$ and put value of x_1 obtained in (1) in the second equation to get value of x_2. ....(2)

Put values of x_1, x_2 obtained in (1) and (2) in the third equation to get value of x_3.

Step-IV

We repeat this procedure up to desired accuracy and up to desired number of steps.

18. **Solve following linear equations using Gauss-Seidal iteration method starting from 1, 1, 1**

$x_1 + x_2 + 2x_3 = 8$

$2x_1 + 3x_2 + x_3 = 12$

$5x_1 + x_2 + x_3 = 15$

Solution Rewrite the given equations so that each equation for the variable that has coefficient largest we get

$5x_1 + x_2 + x_3 = 15$ ....(1)

$2x_1 + 3x_2 + x_3 = 12$ ....(2)

$x_1 + x_2 + 2x_3 = 10$ ....(3)

From equation (1) we get x_1 in terms of other variables x_2 and x_3 as

$5x_1 = 15 - x_2 - x_3$

$x_1 = (15 - x_2 - x_3)/5 = 3 - 0.2x_2 - 0.2x_3$ ....(4)

From equation (2) we get x_2 in terms of other variables x_1 and x_3 as

$2x_1 + 3x_2 + x_3 = 12$

$x_2 = 4 - (2x_1 + x_3)/3$ ....(5)

From equation (3) we get x_3 in terms of other variables x_1 and x_2 as

$x_1 + x_2 + 2x_3 = 10$

$x_3 = 5 - 0.5x_1 - 0.5x_2$ ....(6)

Step-1

Putting $x_2 = 1$, $x_3 = 1$ in equation (4) we get

$x_1 = 3 - 0.2x_2 - 0.2x_3 = 3 - 0.2 - 0.2 = 2.6$

Putting $x_1 = 2.6$, $x_3 = 1$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 - (5.2+1)/3 = 1.93333$

Putting $x_2 = 1.93333$, $x_1 = 2.6$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.6) - 0.5\,(1.93333) = 2.73333$

Step-2

Putting $x_2 = 1.93333$, $x_3 = 2.73333$ in equation (4) we get

$x_1 = 3 – 0.2\,x_2 – 0.2\,x_3 = 3 – 0.2(1.93333) – 0.2\,(2.73333) = 2.066666$

Putting $x_1 = 2.06666$, $x_3 = 2.73333$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 – (4.13333 + 2.73333)/3 = 1.71111$

Putting $x_2 = 1.71111$, $x_1 = 2.066666$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.066666) - 0.5\,(1.71111) = 3.11111$

Step-3

Putting $x_2 = 1.71111$, $x_3 = 3.11111$ in equation (4) we get

$x_1 = 3 – 0.2\,x_2 – 0.2\,x_3 = 3 – 0.2(1.71111) – 0.2\,(3.11111) = 2.035555$

Putting $x_1 = 2.035555$, $x_3 = 3.11111$in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 – (4.07111 + 3.11111)/3 = 1.605925$

Putting $x_2 = 1.605925$, $x_1 = 2.035555$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.035555) - 0.5\,(1.605925) = 3.17926$

Step-4

Putting $x_2 = 1.605925$, $x_3 = 3.17926$ in equation (4) we get

$x_1 = 3 – 0.2\,x_2 – 0.2\,x_3 = 3 – 0.2(1.605925) – 0.2\,(3.17926) = 2.042962$

Putting $x_1 = 2.042962$, $x_3 = 3.17926$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 – (4.08592 + 3.17926)/3 = 1.57827$

Putting $x_2 = 1.57827$, $x_1 = 2.042962$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.042962) - 0.5\,(1.57827) = 3.18938$

Eigen values and Eigen vectors by Power method

This is an iterative method used to find approximate value of Eigen values and Eigen vectors of an n x n non-singular matrix A.

Procedure:

We start with any non-zero vector x_0 of n components and compute followings.

$x_1 = Ax_0$

$x_2 = A x_1$

$x_3 = A x_2$

....

....

....

$x_n = A x_{n-1}$

For any n x n non-singular matrix A we can apply this method and we get a dominant

eigen value λ such that absolute value of this eigen value λ is greater than that of other eigen values.

Theorem: Let A be an n x n real symmetric matrix. Let $x \neq 0$ be any real vector with n components. Let $y=Ax$, $m_0 = x^T x$, $m_1=x^Ty$, $m_2=y^Ty$

Then the ratio $r = m_1 / m_0$ called Rayleigh quotient is an approximate eigen value λ of A.

Assuming $r = \lambda - \epsilon$ we have $|\epsilon| \leq \sqrt{\frac{m_2}{m_0} - r^2}$

where ϵ is the error of ratio $r = m_1 / m_0$

19 . **Find the eigen values and eigen vectors of the matrix** $\begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$ **by Power method**

taking $x_0= [\ 1\ 1\ ]^T$

Solution Let $A = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$. Given $x_0= [\ 1\ \ 1\ ]^T$

$x_1 = Ax_0$

$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} = 9\begin{bmatrix} 1 \\ 5/9 \end{bmatrix}$

Dominated eigen value is 9 and and eigen vector is $\begin{bmatrix} 1 \\ 5/9 \end{bmatrix}$

$x_2 = A\, x_1$

$$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5/9 \end{bmatrix} = \begin{bmatrix} 7.666 \\ 4.111 \end{bmatrix} = 7.666 \begin{bmatrix} 1 \\ 0.536 \end{bmatrix}$$

Dominated eigen value is 7.666 and and eigen vector is $\begin{bmatrix} 1 \\ 0.536 \end{bmatrix}$

$x_3 = A\, x_2$

$$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.536 \end{bmatrix} = \begin{bmatrix} 7.608 \\ 4.072 \end{bmatrix} = 7.608 \begin{bmatrix} 1 \\ 0.535 \end{bmatrix}$$

Dominated eigen value is 7.608 and and eigen vector is $\begin{bmatrix} 1 \\ 0.535 \end{bmatrix}$

20 . **Find the eigen values and eigen vectors of the matrix** $\begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ **by Power method taking $x_0 = [\,1\ \ 1\ \ 1\,]^T$**

Solution Let $A = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$. Given $x_0 = [\,1\ \ 1\ \ 1]^T$

$x_1 = Ax_0$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$$

Dominated eigen value is 10 and and eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$

$x_2 = A\, x_1$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.5 \\ 4 \\ 8 \end{bmatrix} = 8.5 \begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix}$$

Dominated eigen value is 8.5 and and eigen vector is $\begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix}$

$x_3 = A x_2$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix} = \begin{bmatrix} 8.3526 \\ 3.941 \\ 7.5875 \end{bmatrix} = 8.3526 \begin{bmatrix} 1 \\ 0.4718 \\ 0.9084 \end{bmatrix}$$

Dominated eigen value is 8.3526 and and eigen vector is $\begin{bmatrix} 1 \\ 0.4718 \\ 0.9084 \end{bmatrix}$

Unit III: Solution of IVP by Euler's method, Heun's method and Runge-Kutta fourth order method. Basic concept of optimization, Linear programming, simplex method, degeneracy, and Big-M method.

Numerical Solution of Differential Equation:

Introduction:

We consider the first order differential equation

$$y' = f(x, y)$$

With the initial condition

$$y(x_o) = y_0$$

The sufficient conditions for the existence of unique solution on the interval $[x_0, b]$ are the well-known Lipschitz conditions. However in 'Numerical Analysis', one finds values of y at successive steps, $x = x_1, x_2, \ldots, x_n$ with spacing h. There are many numerical methods available to find solution of IVP, such as : Picards method, Euler's method, Taylor' series method, Runge-Kutta method etc.

In the present section we will solve the ode

$$y' = f(x, y), \quad y(x_0) = y_0 \text{ in the interval } I = (x_0, x_n) \qquad (1)$$

using a numerical scheme applied to discrete node $x_n = x_0 + nh$, where h is the step-size by Euler's method, Heun's method and Runge-Kutta method.

- In Euler's method we use the slope evaluated at the current level (x_n, y_n) and use that value as an approximation of the slope throughout the interval (x_n, x_{n+1}).
- Hune' method samples the slope at beginning and at the end and uses the average as the final approximation of the slope. It is also known as Runge-kutta method of order-2.
- Runge-kutta method of order-4 improve on Euler' s method looking at the slope at multiple points.

The necessary formula for solution of (1) by **Euler' s method** is:

$$y_{j+1} = y_j + hf(x_j, y_j), \ j = 0, 1, 2, \ldots n - 1.$$

The necessary formula for solution of (1) by **Hune' s method** is:

$$y_{j+1} = y_j + \frac{1}{2}(k_1 + k_2), \ j = 0, 1, 2, \ldots n - 1.$$

Where $k_1 = hf(x_j, y_j), k_2 = hf(x_j + h, y_j + k_1)$

The necessary formula for solution of (1) by **Runge – Kutta method of order-4** is:

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \ j = 0, 1, 2, \ldots, n - 1.$$

Where $k_1 = hf(x_j, y_j)$

$$k_2 = hf(x_j + \frac{1}{2}h, y_j + \frac{1}{2}k_1)$$

$$k_3 = hf(x_j + \frac{1}{2}h, y_j + \frac{1}{2}k_2)$$

$$k_4 = hf(x_j + h, y_j + k_3)$$

Example : Use the Euler method to solve numerically the initial value problem

$$u' = -2tu^2, u(0) = 1$$

With h = 0.2 on the interval [0, 1]. Compute u (1.0)

We have

$u_{j+1} = u_j - 2ht_j u_j^2, \qquad j = 0,1,2,3,4.$ *[Here x and y are replaced by t and u respectively]*

With h=0.2. The initial condition gives $u_0=1$

For j = 0: $t_0 = 0$, $u_0 = 1$

u (0.2) = $u_1 = u_0 - 2ht_0u_0^2 = 1.0$.

For j = 1: $t_1 = 0.2$, $u_1 = 1$

u (0.4) = $u_2 = u_1 - 2ht_1u_1^2 = 0.92$.

For j = 2: $t_2 = 0.4$, $u_2 = 0.92$

u (0.6) = $u_3 = u_2 - 2ht_2u_2^2 = 0.78458$.

For j = 3: $t_3 = 0.6$, $u_3 = 0.78458$

u(0.8) = $u_4 = 0.63684$.

Similarly, we get

u(1.0) = $u_5 = 0.50706$.

Note: In the similar way IVP can be solved by Heun's method and Runge-Kutta fourth order method.