case are shown. As shown at (iii), the principle of superposition and the two loading cases may be applied simultaneously to the truss, producing the combined support reactions indicated.

Supplementary problems

S1.1 Determine the reactions at the supports of the frame shown in Figure S1.1.

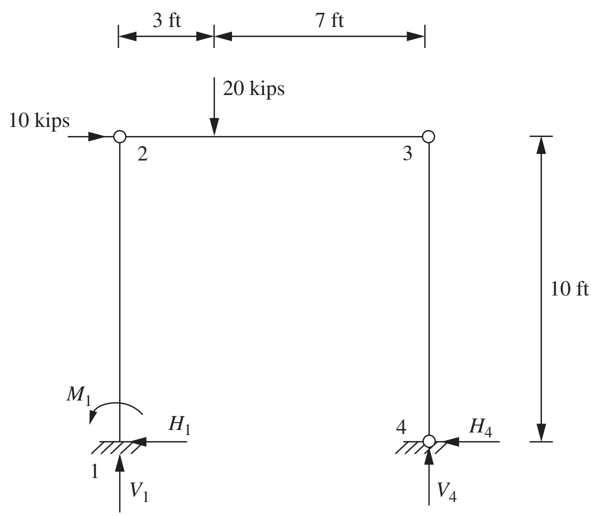


Figure S1.1

S1.2 Determine the reactions at supports 1 and 2 of the bridge girder shown in Figure S1.2. In addition, determine the bending moment in the girder at support 2.

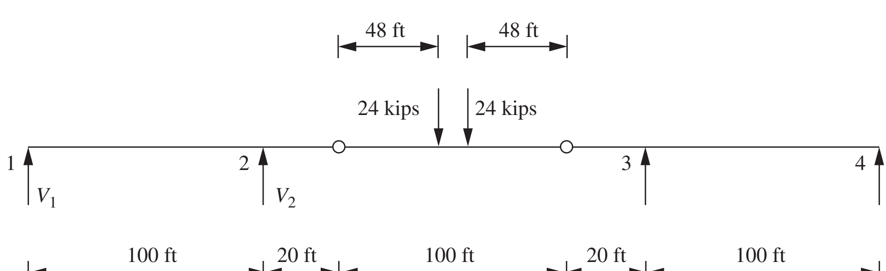


Figure S1.2

S1.3 Determine the reactions at the supports of the frame shown in Figure S1.3. In addition, determine the bending moment in member 32.

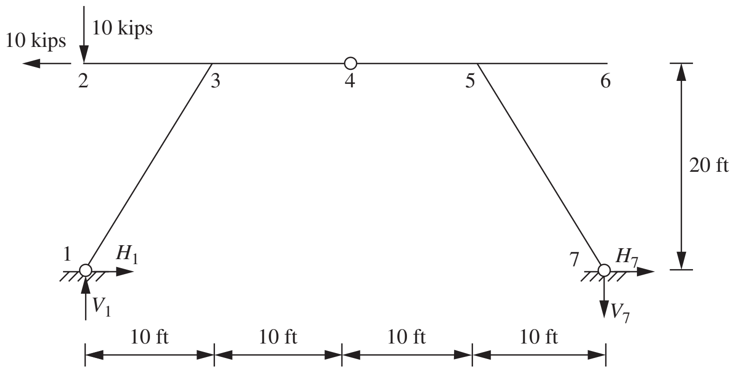


Figure S1.3

S1.4 Determine the reactions at the supports of the derrick crane shown in Figure S1.4. In addition, determine the forces produced in the members of the crane.

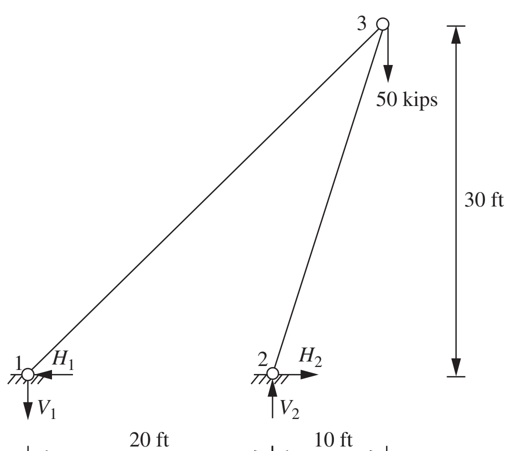


Figure S1.4

S1.5 Determine the reactions at the supports of the pin-jointed frame shown in Figure S1.5. In addition, determine the force produced in member 13.

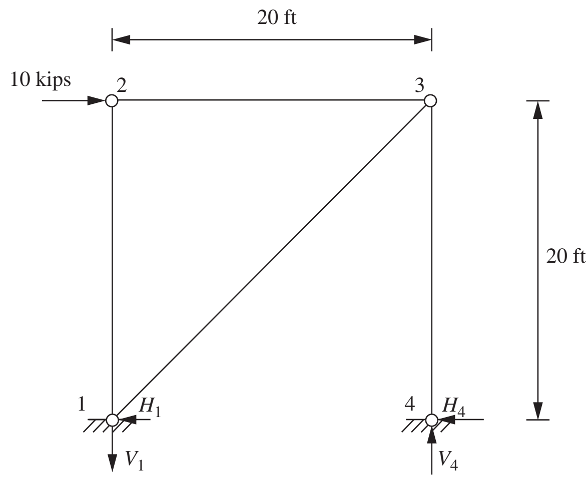


Figure S1.5

S1.6 Determine the reactions at the supports of the bent shown in Figure S1.6. In addition, determine the bending moment produced in the bent at node 3.

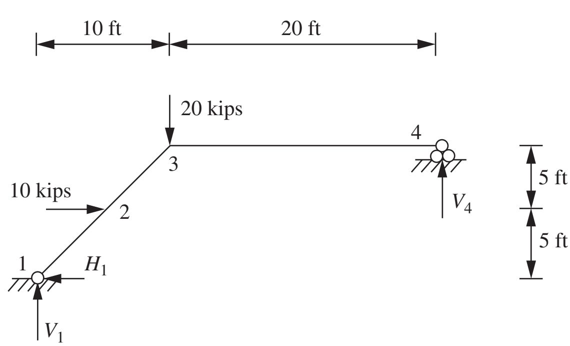


Figure S1.6

S1.7 Determine the reactions at the supports of the pin-jointed truss shown in Figure S1.7.

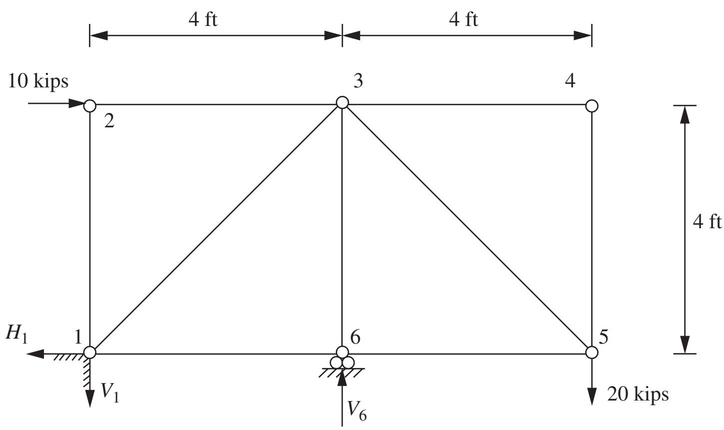


Figure S1.7

S1.8 Determine the reactions at the supports of the bent shown in Figure S1.8. The applied loading consists of the uniformly distributed load indicated.

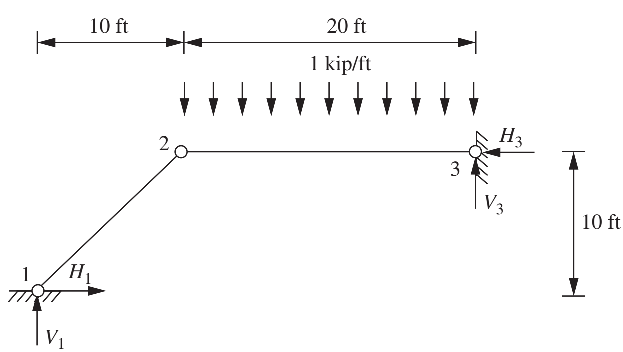


Figure S1.8

S1.9 Determine the reactions at the support of the cantilever shown in Figure S1.9. The applied loading consists of the distributed triangular force shown.

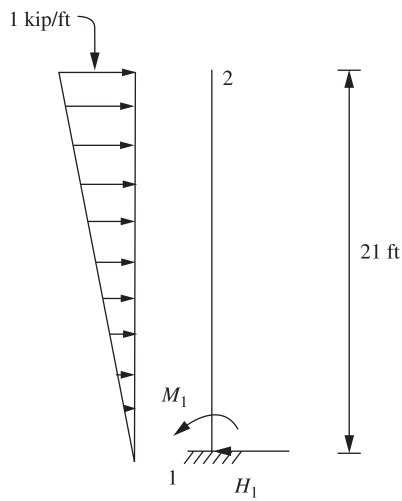


Figure S1.9

S1.10 Determine the reactions at the support of the jib crane shown in Figure S1.10. In addition, determine the force produced in members 24 and 34.

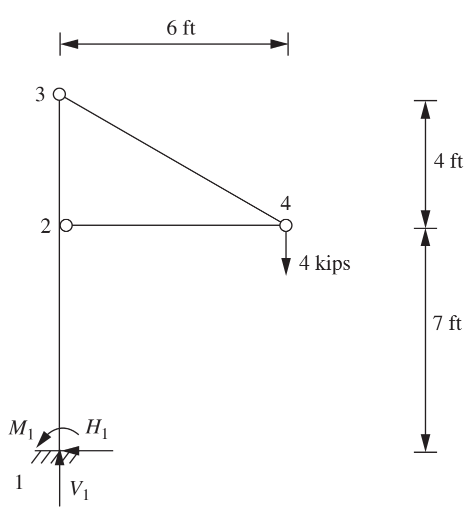


Figure S1.10

2 Statically determinate pin-jointed frames

Notation

a	panel width
F	force
H	horizontal force
h	truss height
l	length of member
M	bending moment
P	axial force in a member due to applied loads
P'	axial force in a member of the modified truss due to applied loads
R	support reaction
u	axial force in a member due a unit virtual load applied to the modified truss
V	vertical force
W_{LL}	concentrated live load
θ	angle of inclination

2.1 Introduction

A simple truss consists of a triangulated planar framework of straight members. Typical examples of simple trusses are shown in Figure 2.1 and are customarily used in bridge and roof construction. The basic unit of a truss is a triangle formed from three members. A simple truss is formed by adding members, two at a time, to form additional triangular units. The top and bottom members of a truss are referred to as chords, and the sloping and vertical members are referred to as web members.

Simple trusses may also be combined, as shown in Figure 2.2, to form a compound truss. To provide stability, the two simple trusses are connected at the apex node and also by means of an additional member at the base.

Simple trusses are analyzed using the equations of static equilibrium with the following assumptions:

- all members are connected at their nodes with frictionless hinges
- the centroidal axes of all members at a node intersect at one point so as to avoid eccentricities

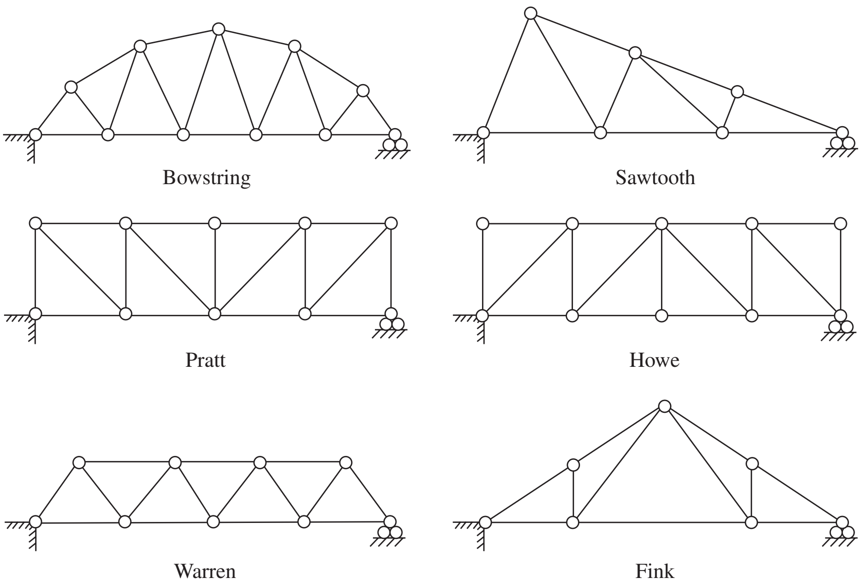


Figure 2.1

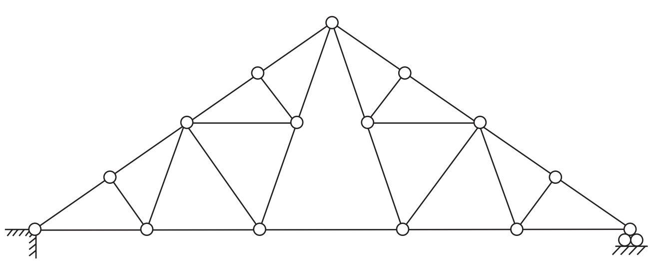


Figure 2.2

- all loads, including member weight, are applied at the nodes
- members are subjected to axial forces only
- secondary stresses caused by axial deformations are ignored

2.2 Statical determinacy

A statically determinate truss is one in which all member forces and external reactions may be determined by applying the equations of equilibrium. In a simple truss, external reactions are provided by either hinge supports or roller supports, as shown in Figure 2.3 (i) and (ii). The roller support provides only one degree of restraint, in the vertical direction, and both horizontal and

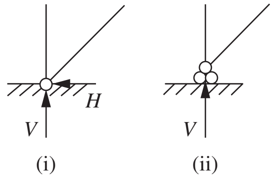


Figure 2.3

rotational displacements can occur. The hinge support provides two degrees of restraint, in the vertical and horizontal directions, and only rotational displacement can occur. The magnitudes of the external restraints may be obtained from the three equations of equilibrium. Thus, a truss is externally indeterminate when it possesses more than three external restraints and is unstable when it possesses less than three.

In a simple truss with j nodes, including the supports, $2j$ equations of equilibrium may be obtained, since at each node:

$$\Sigma H = 0$$

and $$\Sigma V = 0$$

Each member of the truss is subjected to an unknown axial force; if the truss has n members and r external restraints, the number of unknowns is $(n + r)$. Thus, a simple truss is determinate when the number of unknowns equals the number of equilibrium equations or:

$$n + r = 2j$$

A truss is statically indeterminate, as shown in Figure 2.4, when:

$$n + r > 2j$$

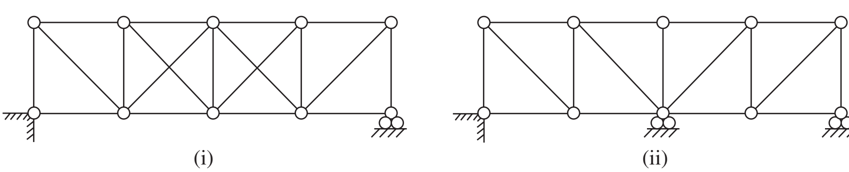


Figure 2.4

The truss at (i) is internally redundant, and the truss at (ii) is externally redundant.

A truss is unstable, as shown in Figure 2.5, when:

$$n + r < 2j$$

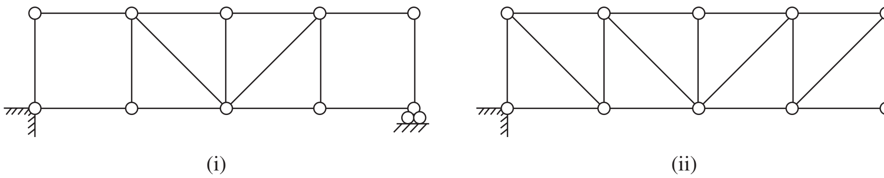


Figure 2.5

The truss at (i) is internally deficient, and the truss at (ii) is externally deficient.

However, a situation can occur in which a truss is deficient even when the expression $n + r = 2j$ is satisfied. As shown in Figure 2.6, the left-hand side of the truss has a redundant member, while the right-hand side is unstable.

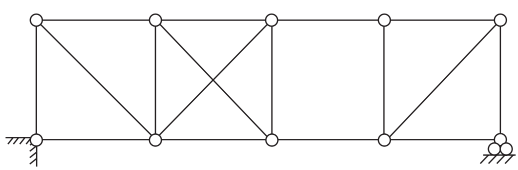


Figure 2.6

2.3 Sign convention

As indicated in Figure 2.7, a tensile force in a member is considered positive, and a compressive force is considered negative. Forces are depicted as acting from the member on the node; the direction of the member force represents the force the member exerts on the node.

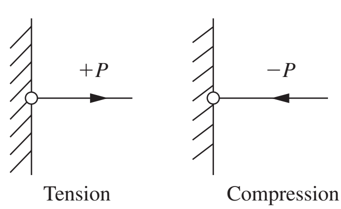


Figure 2.7

2.4 Methods of analysis

Several methods of analysis are available, each with a specific usefulness and applicability.

(a) Method of resolution at the nodes

At each node in a simple truss, the forces acting are the applied loads or support reactions and the forces in the members connected to the node. These forces constitute a concurrent, coplanar system of forces in equilibrium, and, by applying the equilibrium equations $\Sigma H = 0$ and $\Sigma V = 0$, the unknown forces in a maximum of two members may be determined. The method consists of first determining the support reactions acting on the truss. Then, each node, at which not more than two unknown member forces are present, is systematically selected in turn and the equilibrium equations applied to solve for the unknown forces.

It is not essential to resolve horizontally and vertically at all nodes; any convenient rectangular coordinate system may be adopted. Hence, it follows that for an unloaded node, when two of the three members at the node are collinear, the force in the third member is zero.

For the truss shown in Figure 2.8, the support reactions V_1 and V_9, caused by the applied load W_{LL}, are first determined. Selecting node 1 as the starting point and applying the equilibrium equations, by inspection it is apparent that:

$$P_{13} = 0$$

and $$P_{12} = V_1 \ldots \text{compression}$$

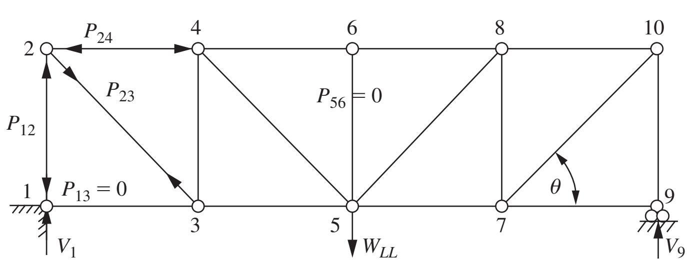


Figure 2.8

Node 2 now has only two members with unknown forces, which are given by:

$$P_{23} = P_{12}/\sin\theta \ldots \text{tension}$$

and $$P_{24} = P_{23}\cos\theta \ldots \text{compression}$$

Nodes 3 and 4 are now selected in sequence, and the remaining member forces are determined. Since members 46 and 68 are collinear, it is clear that:

$$P_{56} = 0$$

This technique may be applied to any truss configuration and is suitable when the forces in all the members of the truss are required.

Example 2.1

Determine the forces produced by the applied loads in the members of the saw-tooth truss shown in Figure 2.9.

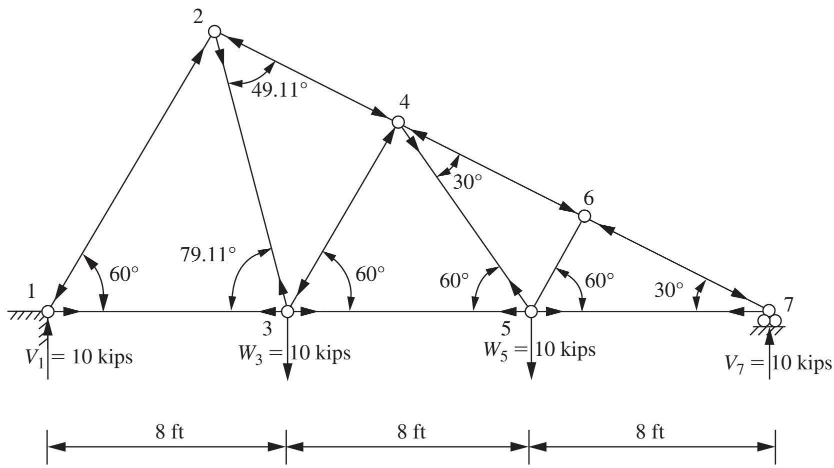


Figure 2.9

Solution

The support reactions are calculated, and the directions of the member forces are assumed as indicated.

Resolving forces at node 7:

$$\begin{aligned} P_{67} &= V_7/\sin 30^\circ \\ &= 20 \text{ kips} \ldots \text{compression} \\ P_{57} &= V_7/\tan 30^\circ \\ &= 17.32 \text{ kips} \ldots \text{tension} \end{aligned}$$

Resolving forces at node 6:

$$\begin{aligned} P_{46} &= P_{67} \\ &= 20 \text{ kips} \ldots \text{compression} \\ P_{56} &= 0 \end{aligned}$$

Resolving forces at node 5:

$$\begin{aligned} P_{45} &= W_5/\sin 60^\circ \\ &= 11.55 \text{ kips} \ldots \text{tension} \\ P_{35} &= P_{57} - W_5/\tan 60^\circ \\ &= 11.55 \text{ kips} \ldots \text{tension} \end{aligned}$$

Resolving forces at node 4:

$$\begin{aligned} P_{34} &= P_{45}/\sin 30° \\ &= 5.77 \text{ kips} \ldots \text{compression} \\ P_{24} &= P_{46} - P_{45} \ \sin 60° \\ &= 10 \text{ kips} \ldots \text{compression} \end{aligned}$$

Resolving forces at node 2:

$$\begin{aligned} P_{23} &= P_{24}/\cos 49.11° \\ &= 15.28 \text{ kips} \ldots \text{tension} \end{aligned}$$

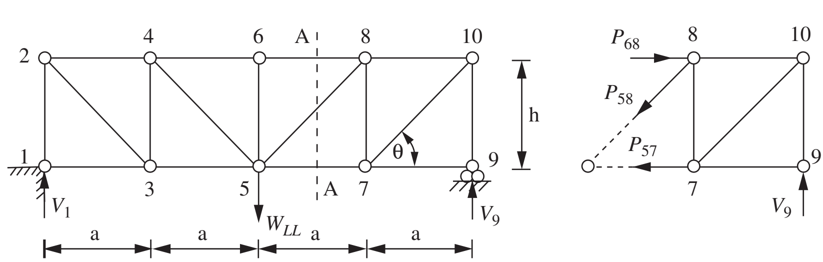
Resolving forces at node 1:

$$\begin{aligned} P_{12} &= V_1/\sin 60° \\ &= 11.55 \text{ kips} \ldots \text{compression} \\ P_{13} &= V_1/\tan 60° \\ &= 5.77 \text{ kips} \ldots \text{tension} \end{aligned}$$

(b) Method of sections

This method uses the concept of a free body diagram to determine the member forces in the members of a specific panel of a truss. The method consists of first determining the support reactions acting on the truss. Then, a free body diagram is selected so as to cut through the panel; the forces acting on the free body consist of the applied loads, support reactions, and forces in the cut members. These forces constitute a coplanar system of forces in equilibrium; by applying the equilibrium equations $\Sigma M = 0$, $\Sigma H = 0$ and $\Sigma V = 0$, the unknown forces in a maximum of three members may be determined.

For the truss shown in Figure 2.10, the support reactions V_1 and V_9, caused by the applied load W_{LL}, are first determined. To determine the forces in



(i) Applied loads

(ii) Free body diagram

Figure 2.10

members 68, 57, and 58 of the truss, the structure is cut at section A-A, and the right-hand portion separated as shown at (ii).

Resolving forces vertically gives the force in member 58 as:

$$P_{58} = V_9/\sin\theta \ldots \text{tension}$$

Taking moments about node 8 gives the force in member 57 as:

$$P_{57} = aV_9/h \ldots \text{tension}$$

Taking moments about node 5 gives the force in member 68 as:

$$P_{68} = 2aV_9/h \ldots \text{compression}$$

This technique may be applied to any truss configuration and is suitable when the forces in selected members of the truss are required. For the case of a truss with non-parallel chords, by taking moments about the point of intersection of two of the cut members the force in the third cut member may be obtained.

Example 2.2

Determine the forces produced by the applied loads in members 46, 56 and 57 of the sawtooth truss shown in Figure 2.11.

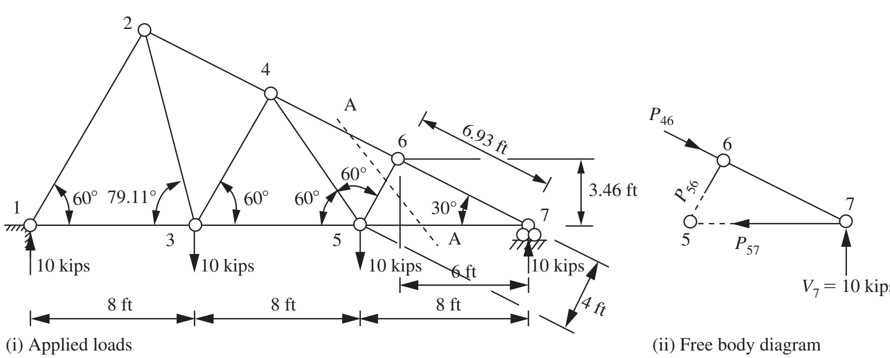


Figure 2.11

Solution

The support reactions are calculated and the truss cut at section A-A as shown. The directions of the member forces on the right-hand free body diagram are assumed as indicated in (ii).

Taking moments about node 7 gives the force in member 56 as:

$$\begin{aligned} P_{56} &= 0/6.93 \\ &= 0 \end{aligned}$$

Taking moments about node 5 gives the force in member 46 as:

$$\begin{aligned} P_{46} &= 8V_7/4 \\ &= 8 \times 10/4 \\ &= 20 \text{ kips} \ldots \text{compression} \end{aligned}$$

Taking moments about node 6 gives the force in member 57 as:

$$\begin{aligned} P_{57} &= 6V_7/3.46 \\ &= 6 \times 10/3.46 \\ &= 17.32 \text{ kips} \ldots \text{tension} \end{aligned}$$

(c) Method of force coefficients

This is an adaptation of the method of sections, which simplifies the solution of trusses with parallel chords and with web members inclined at a constant angle. The forces in chord members are determined from the magnitude of the moment of external forces about the nodes of a truss. The forces in web members are obtained from knowledge of the shear force acting on a truss. The method constitutes a routine procedure for applying the method of sections.

The shear force diagram for the truss shown in Figure 2.12 is obtained by plotting at any section the cumulative vertical force produced by the applied loads on one side of the section. The force in a vertical web member is given directly, by the method of sections, as the magnitude of the shear force at the location of the member. Thus, the force in member 12 is equal to the shear force at node 1 and:

$$\begin{aligned} P_{12} &= V_1 \\ &= 3W \ldots \text{compression} \end{aligned}$$

The force in member 34 is equal to the shear force at node 3 and:

$$\begin{aligned} P_{34} &= V_1 - 2W \\ &= W \ldots \text{compression} \end{aligned}$$

The force in member 56, by inspection of node 6, is:

$$P_{56} = 0$$

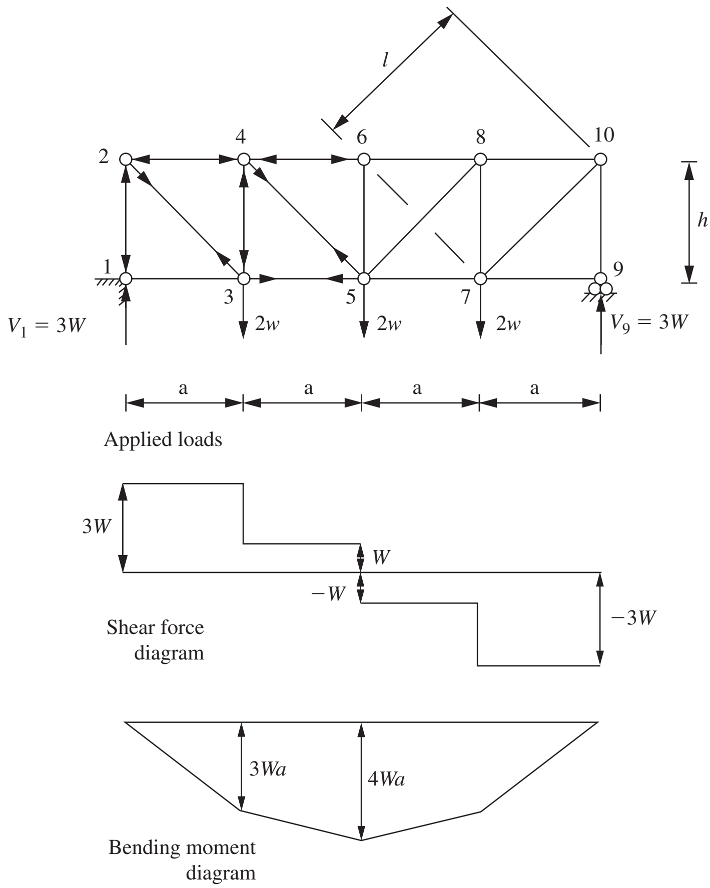


Figure 2.12

The force in a diagonal web member is given directly, by the method of sections, as the magnitude of the shear force in the corresponding panel multiplied by the coefficient l/h. Thus, the force in member 23 is equal to the shear force in the first panel multiplied by l/h and:

$$
\begin{aligned}
P_{23} &= V_1 \times l/h \\
&= 3W \times l/h \ldots \text{tension}
\end{aligned}
$$

The force in member 45 is equal to the shear force in the second panel multiplied by l/h and:

$$
\begin{aligned}
P_{45} &= (V_1 - 2W) \times l/h \\
&= W \times l/h \ldots \text{tension}
\end{aligned}
$$

The bending moment diagram for the truss shown in Figure 2.12 is obtained by plotting at each node the cumulative moment produced by the applied loads on one side of the node. The force in the bottom chord of a particular panel is given directly, by the method of sections, as the magnitude of the moment at the point of intersection of the top chord and the diagonal member in the panel multiplied by the coefficient $1/h$. Thus, the force in member 13 is equal to the bending moment at node 2 multiplied by $1/h$ and:

$$\begin{aligned} P_{13} &= M_2 \times 1/h \\ &= 0 \end{aligned}$$

The force in member 35 is equal to the bending moment at node 4 multiplied by $1/h$ and:

$$\begin{aligned} P_{35} &= M_4 \times 1/h \\ &= V_1 \times a \times 1/h \\ &= 3Wa/h \ldots \text{tension} \end{aligned}$$

The force in the top chord of a particular panel is given directly, by the method of sections, as the magnitude of the moment at the point of intersection of the bottom chord and the diagonal member in the panel multiplied by the coefficient $1/h$. Thus, the force in member 24 is equal to the bending moment at node 3 multiplied by $1/h$ and:

$$\begin{aligned} P_{24} &= M_3 \times 1/h \\ &= V_1 \times a \times 1/h \\ &= 3Wa/h \ldots \text{compression} \end{aligned}$$

The force in member 46 is equal to the bending moment at node 5 multiplied by $1/h$ and:

$$\begin{aligned} P_{46} &= M_5 \times 1/h \\ &= (V_1 \times 2a - 2W \times a) \times 1/h \\ &= 4Wa/h \ldots \text{compression} \end{aligned}$$

This technique may be applied to any truss with parallel chords and with web members at a constant angle of inclination. It may be used when the forces in all members or in selected members of the truss are required.

Example 2.3

Determine the forces produced by the applied loads in the members of the truss shown in Figure 2.13.

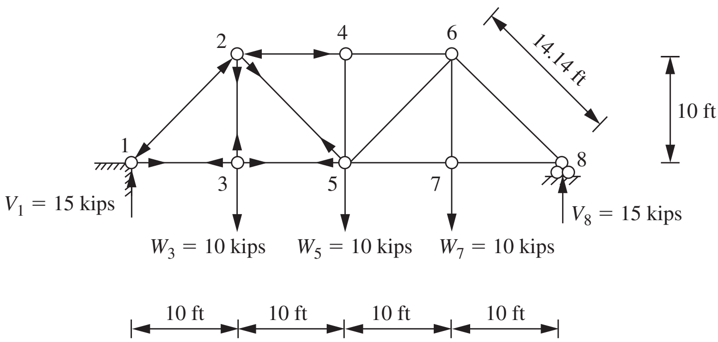


Figure 2.13

Solution

The support reactions are calculated as shown. By inspection, the force in the vertical web members is obtained directly. Thus, the force in member 23 is given as:

$$
\begin{aligned}
P_{23} &= W_3 \\
&= 10 \text{ kips} \ldots \text{tension} \\
P_{45} &= 0
\end{aligned}
$$

The force in the diagonal web member 12 is given by the magnitude of the shear force in the first panel multiplied by the coefficient *l*/*h*. Thus, the force in member 12 is:

$$
\begin{aligned}
P_{12} &= V_1 \times l/h \\
&= 15 \times 14.14/10 \\
&= 21.21 \text{ kips} \ldots \text{compression}
\end{aligned}
$$

The force in the diagonal web member 25 is given by the magnitude of the shear force in the second panel multiplied by the coefficient *l*/*h*. Thus, the force in member 25 is:

$$
\begin{aligned}
P_{25} &= (V_1 - W_3) \times l/h \\
&= 5 \times 14.14/10 \\
&= 7.07 \text{ kips} \ldots \text{tension}
\end{aligned}
$$

The force in the bottom chord member 13 is given by the magnitude of the moment at node 2 multiplied by the coefficient $1/h$. Thus, the force in member 13 is:

$$\begin{aligned} P_{13} &= M_2 \times 1/h \\ &= V_1 \times a \times 1/h \\ &= 150 \times 1/10 \\ &= 15 \text{ kips} \ldots \text{tension} \end{aligned}$$

Similarly, the force in the bottom chord member 35 is given by the magnitude of the moment at node 2 multiplied by the coefficient $1/h$. Thus, the force in member 35 is:

$$\begin{aligned} P_{35} &= M_2 \times 1/h \\ &= V_1 \times a \times 1/h \\ &= 150 \times 1/10 \\ &= 15 \text{ kips} \ldots \text{tension} \end{aligned}$$

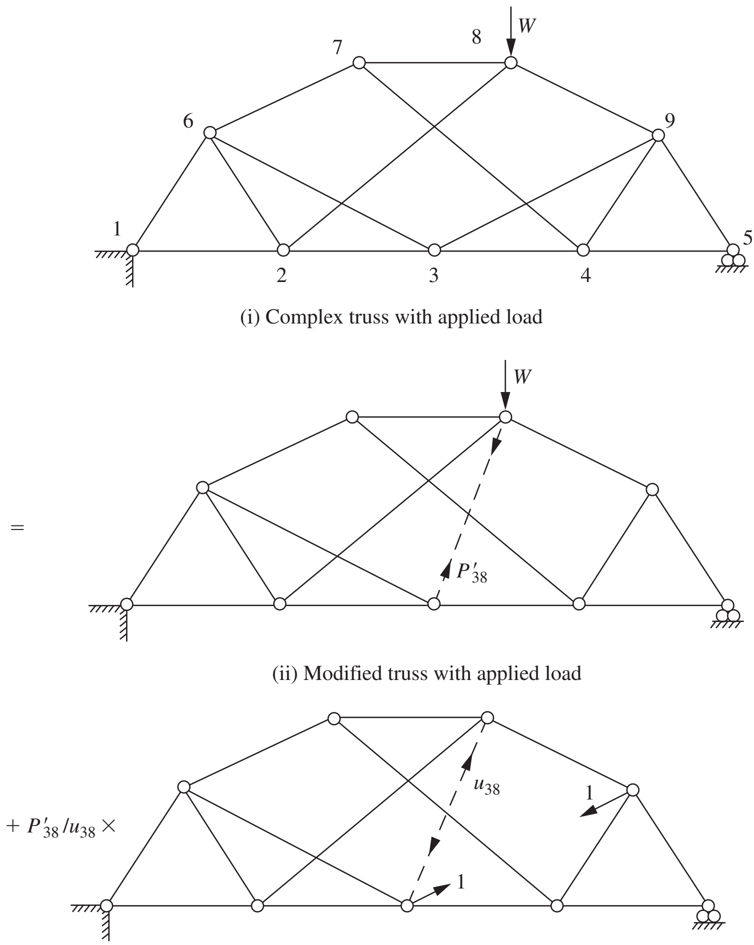
The force in the top chord member 24 is given as the magnitude of the moment at node 5 multiplied by the coefficient $1/h$. Thus, the force in member 24 is:

$$\begin{aligned} P_{24} &= M_5 \times 1/h \\ &= (V_1 \times 2a - W_3 \times a) \times 1/h \\ &= (300 - 100) \times 1/10 \\ &= 20 \text{ kips} \ldots \text{compression} \end{aligned}$$

(d) Method of substitution of members

A complex truss, as shown in Figure 2.14 (i), has three or more connecting members at a node, all with unknown member forces. This precludes the use of the method of sections or the method of resolution at the nodes as a means of determining the forces in the truss. The technique consists of removing one of the existing members at a node so that only two members with unknown forces remain and substituting another member so as to maintain the truss in stable equilibrium.

The forces in members 45 and 59 are obtained by resolution of forces at node 5. However, at nodes 4 and 9, three unknown member forces remain, and these cannot be determined by resolution or by the method of sections. As shown at (ii), member 39 is removed, leaving only two unknown forces at node 9, which may be determined. To maintain stable equilibrium, a substitute member 38 is added to create a modified truss, and the original applied loads are applied to the modified truss. The forces P' in all the



(i) Complex truss with applied load

(ii) Modified truss with applied load

(iii) Modified truss with unit virtual load

Figure 2.14

remaining members of the modified truss may now be determined. The force in member 38 is P'_{38}.

The applied loads are now removed, and unit virtual loads are applied to the modified truss along the line of action of the original member 39, as shown at (iii). The forces u in the modified truss are determined; the force in member 38 is $-u_{38}$. Multiplying the forces in system (iii) by P'_{38}/u_{38} and adding them to the forces in system (ii) gives the force in member 38 as:

$$
\begin{aligned}
P_{38} &= P'_{38} + (-u_{38})P'_{38}/u_{38} \\
&= 0
\end{aligned}
$$

In effect, the substitute member 38 has been eliminated from the truss.

Hence, by applying the principle of superposition, the final forces in the original truss are obtained from the expression:

$$P = P' + uP'_{38} / u_{38}$$

where tensile forces are positive and compressive forces are negative. The final force in member 39 is:

$$\begin{aligned} P_{39} &= 1 \times P'_{38}/\mathrm{u}_{38} \\ &= P'_{38}/u_{38} \end{aligned}$$

Example 2.4

Determine the forces produced by the applied loads in members 49, 39, and 89 of the complex truss shown in Figure 2.15.

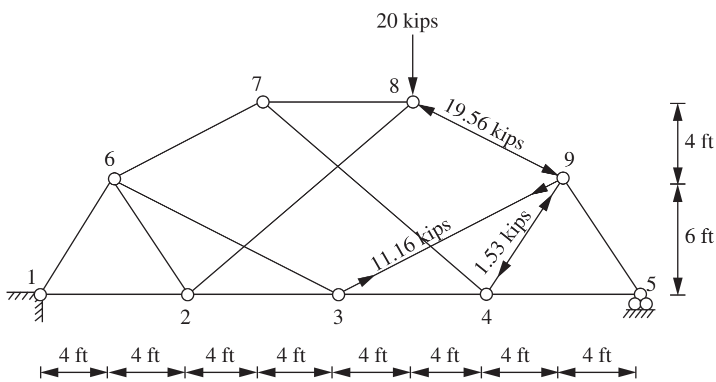


Figure 2.15

Solution

The modified truss shown in Figure 2.16 is created by removing member 39, adding the substitute member 38, and applying the 20 kips load. The member forces in the modified truss may now be determined; the values obtained are:

$$P'_{49} = 7.51 \text{ kips} \ldots \text{tension}$$

$$P'_{89} = -13.98 \text{ kips} \ldots \text{compression}$$

$$P'_{38} = 21.54 \text{ kips} \ldots \text{tension}$$

The 20 kips load is removed from the modified truss, and unit virtual loads are applied at nodes 3 and 9 in the direction of the line of action of the force in

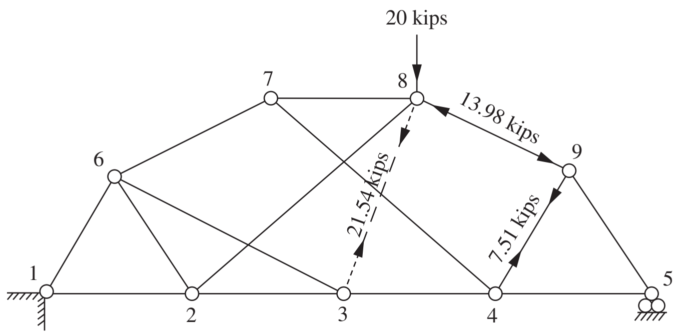


Figure 2.16

member 39. The member forces for this loading condition may now be determined; the values obtained are:

$$u_{49} = -0.81 \text{ kips} \ldots \text{compression}$$

$$u_{89} = -0.50 \text{ kips} \ldots \text{compression}$$

$$u_{38} = -1.93 \text{ kips} \ldots \text{compression}$$

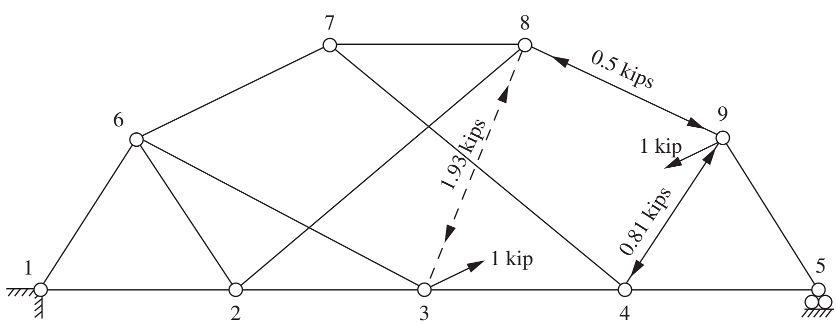


Figure 2.17

The multiplying ratio is given by:

$$\begin{aligned} P'_{38}/u_{38} &= 21.54/1.93 \\ &= 11.16 \end{aligned}$$

The final member forces in the original truss are:

$$\begin{aligned} P_{49} &= 7.51 + 11.16(-0.81) \\ &= -1.53 \text{ kips} \ldots \text{compression} \end{aligned}$$

$$P_{89} = -13.98 + 11.16(-0.50)$$
$$= -19.56 \text{ kips} \ldots \text{compression}$$
$$P_{39} = P'_{38}/u_{38}$$
$$= 11.16 \text{ kips} \ldots \text{tension}$$

Supplementary problems

S2.1 Determine in Figure S2.1 the reactions at the supports of the roof truss shown and the forces in members 34, 38, and 78 caused by the applied loads.

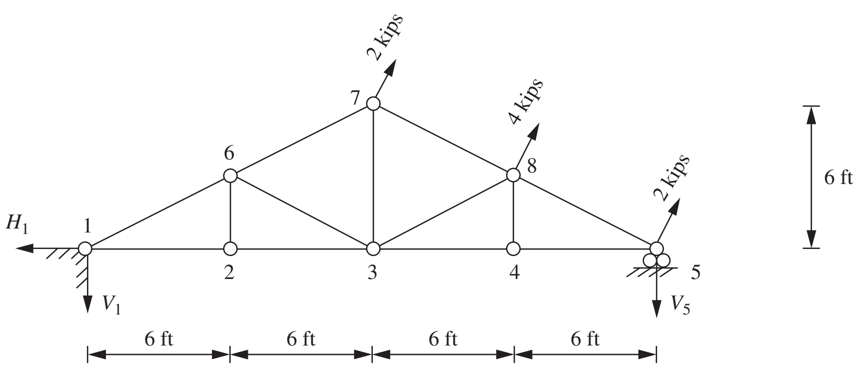


Figure S2.1

S2.2 For the pin-jointed truss shown in Figure S2.2 determine the forces in members 45, 411, and 1011 caused by the applied loads.

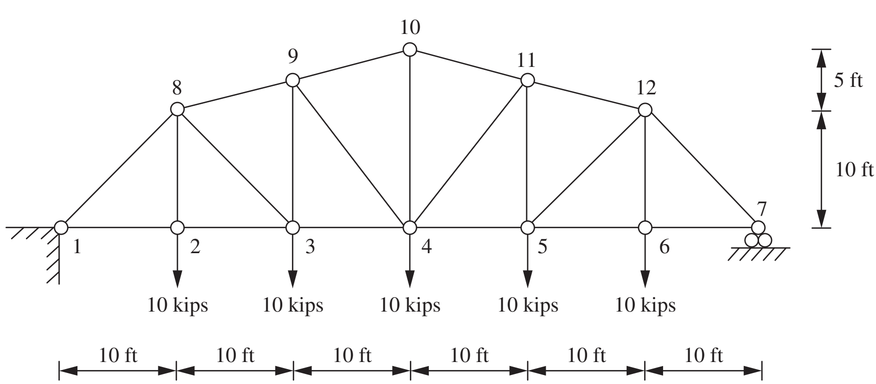


Figure S2.2

S2.3 For the roof truss shown in Figure S2.3 determine the forces in members 23, 27 and 67 caused by the applied loads.

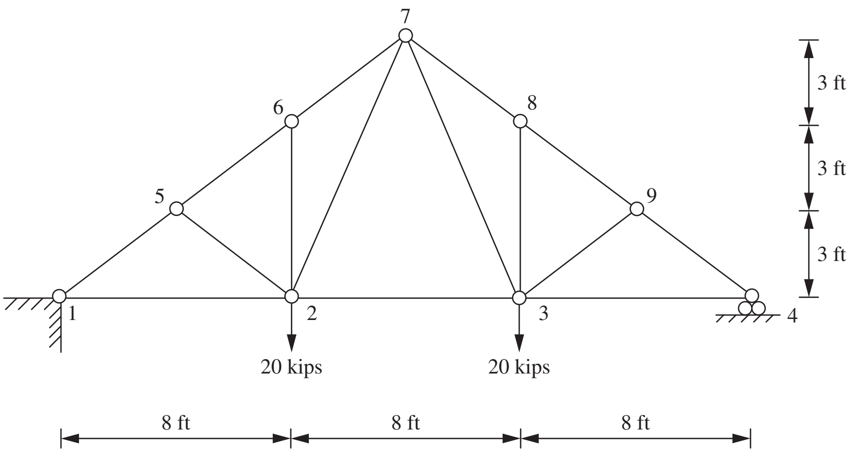


Figure S2.3

S2.4 For the pin-jointed truss shown in Figure S2.4 determine the forces in all members caused by the applied loads.

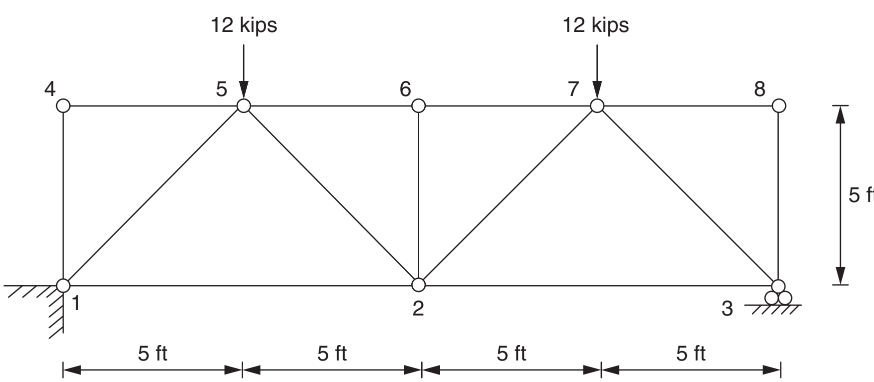


Figure S2.4

S2.5 For the roof truss shown in Figure S2.5 determine the forces in all members due to the applied loads.

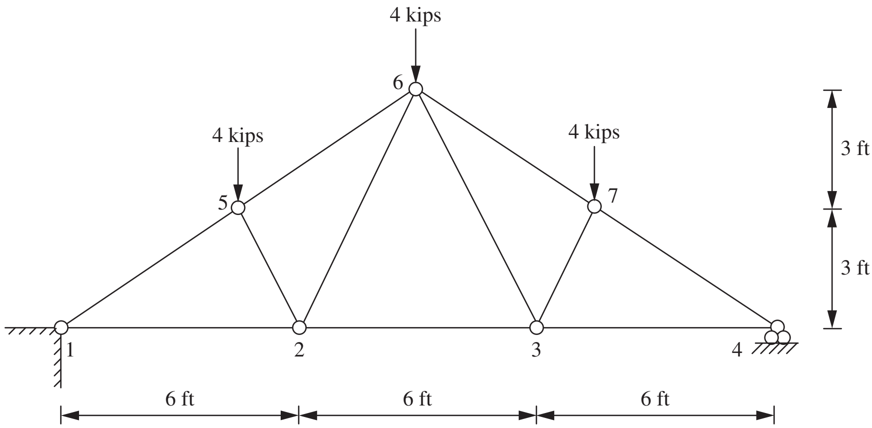


Figure S2.5

S2.6 For the truss shown in Figure S2.6 determine the forces in members 12, 114, 110, 23, 310, 315, 1014, and 1415 caused by the applied loads.

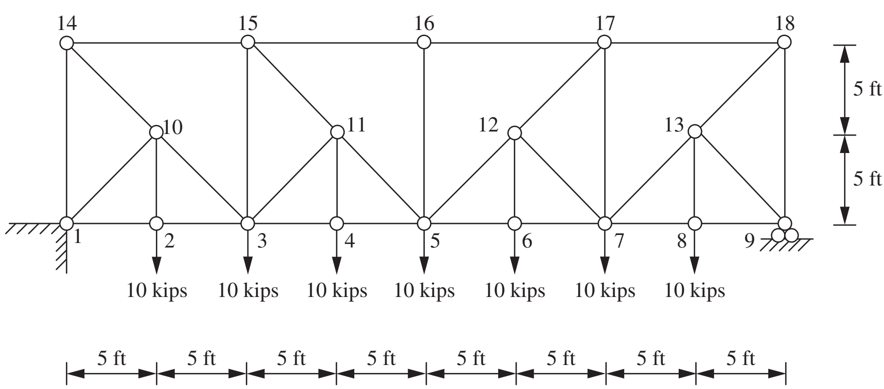


Figure S2.6

S2.7 For the roof truss shown in Figure S2.7 determine the forces in members 23, 27, 37, 78, and 67 caused by the applied loads.

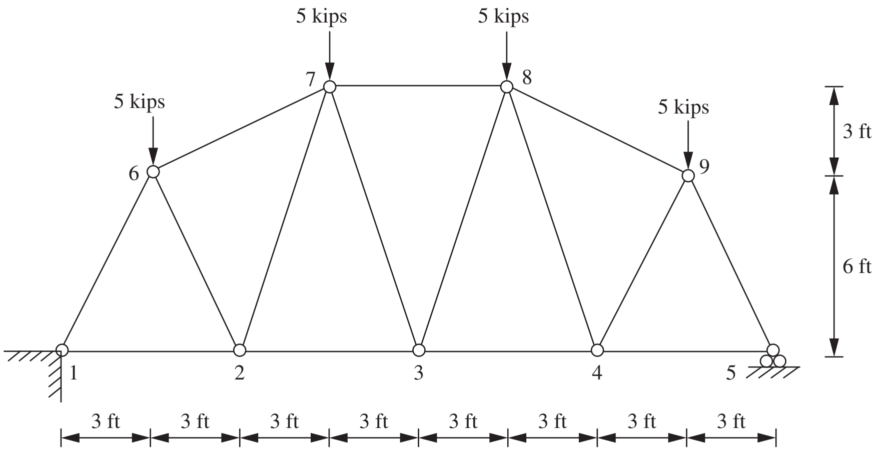


Figure S2.7

S2.8 For the roof truss shown in Figure S2.8 determine the forces in all members due to the applied loads.

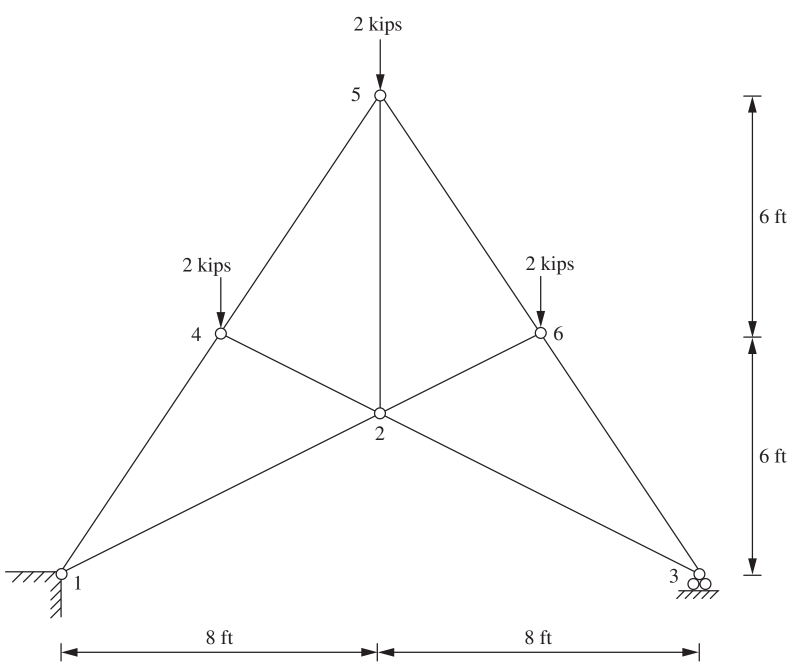


Figure S2.8

S2.9 For the roof truss shown in Figure S2.9 determine the forces in members 23, 26, 27, 67, and 37 caused by the applied loads.

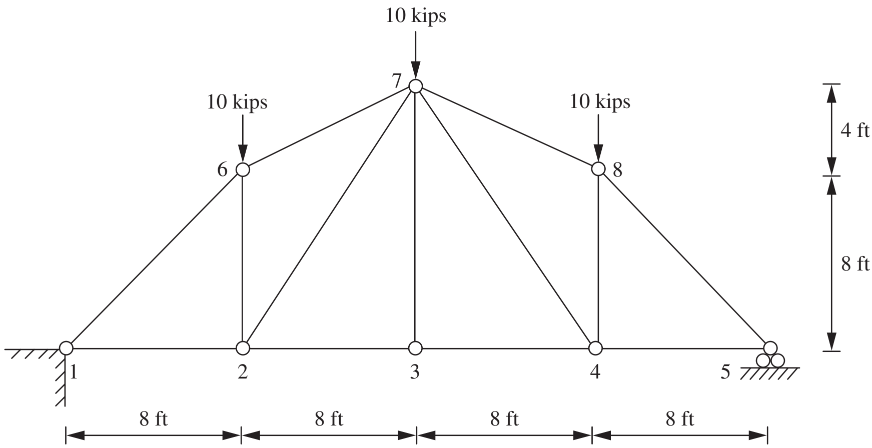


Figure S2.9

S2.10 For the roof truss shown in Figure S2.10 determine the force in members 49, 59, 89, and 78 due to the applied loads.

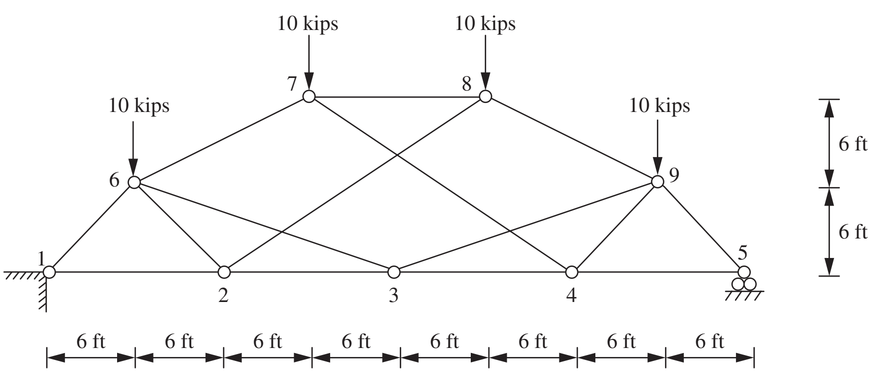


Figure S2.10

3 Elements in flexure

Notation

c	rise of an arch
F	force
H	horizontal force, support reaction
h	height of a rigid frame
l	length of span
M	bending moment
P	axial force in a member
Q	shear force in a member
R	support reaction
V	vertical force, support reaction
W	concentrated load
w	distributed load
δ	displacement
θ	rotation

3.1 Load intensity, shear force, and bending moment diagrams

A uniformly distributed load of magnitude $-w$ is applied to a simply supported beam as shown in Figure 3.1 (i). The sign convention adopted is that forces acting upward are defined as positive. The support reaction at end 1 of the beam is obtained by considering moment equilibrium about end 2. Hence:

$$\begin{aligned} M_2 &= 0 \\ &= lV_1 + (-w)l^2/2 \end{aligned}$$

and:

$$V_1 = wl/2$$

Similarly:

$$V_2 = wl/2$$

The load intensity diagram is shown at (ii) and consists of a horizontal line of magnitude $-w$.

The shear force acting on any section A-A at a distance x from end 1 is defined as the cumulative sum of the vertical forces acting on one side of the

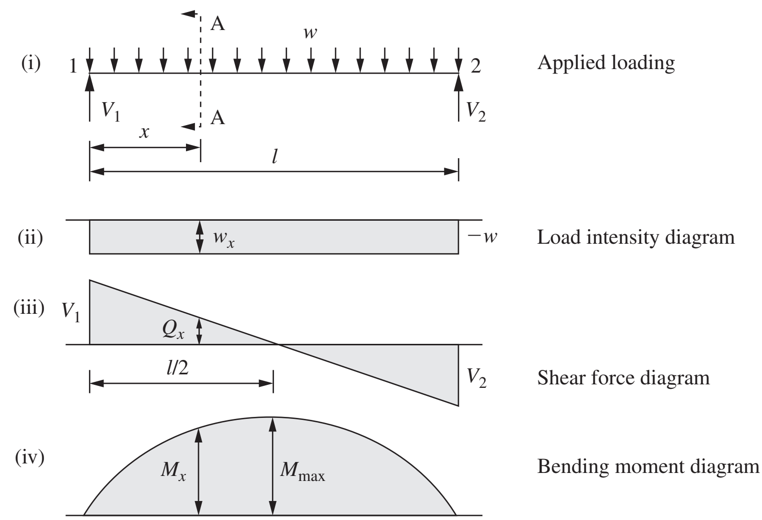


Figure 3.1

section. The vertical forces consist of the applied loads and support reactions. Considering the segment of the beam on the left of section A-A; the shear force is given by:

$$\begin{aligned} Q_x &= V_1 + (-w)x \\ &= V_1 - wx \end{aligned}$$

where $(-wx)$ represents the area of the load intensity diagram on the left of section A-A from $x = 0$ to $x = x$. In general, the change in shear force between two sections of a beam equals the area of the load intensity diagram between the same two sections.

The variation of shear force along the length of the beam may be illustrated by plotting a shear force diagram as shown at (iii). The sign convention adopted is that resultant shear force upward on the left of a section is positive. The maximum shear force occurs at the location of zero load intensity, which is at the ends of the beam.

The bending moment acting on any section A-A at a distance x from end 1 is defined as the cumulative sum of the moments acting on one side of the section. Considering the segment of the beam on the left of section A-A, the bending moment is given by:

$$\begin{aligned} M_x &= V_1x + (-w)x^2/2 \\ &= V_1x + (Q_x - V_1)x/2 \\ &= (V_1 + Q_x)x/2 \end{aligned}$$

where $(V_1 + Q_x)x/2$ represents the area of the shear force diagram on the left of section A-A from $x = 0$ to $x = x$. In general, the change in bending moment between two sections of a beam equals the area of the shear force diagram between the same two sections.

The variation of bending moment along the length of the beam may be illustrated by plotting a bending moment diagram as shown at (iv). The sign convention adopted is that a bending moment producing tension in the bottom fibers of the beam is positive. The maximum bending moment occurs at the location of zero shear force in the beam.

Example 3.1

For the simply supported beam shown in Figure 3.2 (i) draw the load intensity, shear force, and bending moment diagrams.

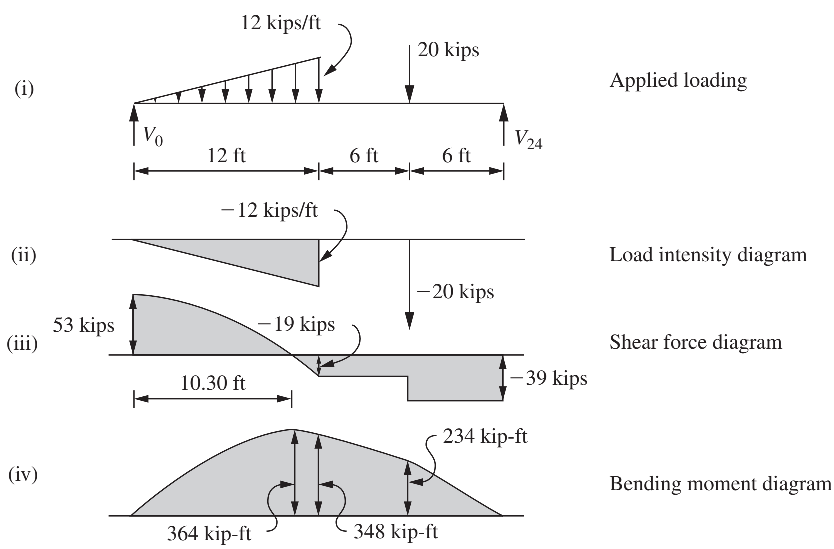


Figure 3.2

Solution

The intensity of loading diagram is shown at (ii) and consists of triangularly shaped sections varying from zero to −12 kips/ft over the left half of the beam and a concentrated force of −20 kips acting at $x = 18$ ft.

The support reaction at the left end of the beam is derived by taking moments about the right end of the beam to give:

$$\begin{aligned} M_{24} &= 0 \\ &= 24V_0 + (12/3 + 12)(-12 \times 12/2) + (-20 \times 6) \end{aligned}$$

and:

$$
\begin{aligned}
V_0 &= 53 \text{ kips} \\
V_{24} &= 72 + 20 - V_0 \\
&= 39 \text{ kips}
\end{aligned}
$$

The ordinate of the shear force diagram at $x = 0$ equals the support reaction at the left end of the beam. The change in shear force between $x = 0$ and $x = 12$ ft equals the area of the load intensity diagram over the initial 12 ft of the beam. Hence, the ordinate of the shear force diagram at $x = 12$ ft is given by:

$$
\begin{aligned}
Q_{12} &= V_0 - 12 \times 12/2 \\
&= 53 - 72 \\
&= -19 \text{ kips}
\end{aligned}
$$

The shear force diagram crosses the base line at a distance x from the end of the beam given by:

$$
\begin{aligned}
Q_x &= 0 \\
&= V_0 - w_x x/2 \\
&= 53 - x^2/2
\end{aligned}
$$

and:

$$
\begin{aligned}
x^2 &= 106 \\
x &= 10.30 \text{ ft}
\end{aligned}
$$

Between $x = 12$ ft and $x = 18$ ft the ordinate of the shear force diagram remains constant at a value of -19 kips since the load intensity is zero over this segment of the beam. At $x = 18$ ft, the location of the concentrated load of 20 kips, the ordinate of the shear force diagram reduces to the value:

$$
\begin{aligned}
Q_{18} &= -19 - 20 \\
&= -39 \text{ kips}
\end{aligned}
$$

Between $x = 18$ ft and $x = 24$ ft the ordinate of the shear force diagram remains constant at a value of -39 kips. At $x = 24$ ft the ordinate of the shear force diagram changes to the value:

$$
\begin{aligned}
Q_{24} &= -39 + 39 \\
&= 0 \text{ kips}
\end{aligned}
$$

The bending moment ordinates at each end of the beam are zero. Between $x = 0$ and $x = 12$ ft the ordinate of the bending moment diagram is given by:

$$
\begin{aligned}
M_x &= V_0 x - (w_x x/2)(x/3) \\
&= 53x - x^3/6
\end{aligned}
$$

The maximum bending moment over this length occurs when:

$$\begin{aligned} \mathrm{d}M_x/\mathrm{d}x &= 0 \\ &= 53 - x^2/2 \end{aligned}$$

and:

$$x = 10.30\,\text{ft}$$

This is the same location at which the shear force diagram crosses the base line, and the maximum bending moment is:

$$\begin{aligned} M_{\max} &= 53 \times 10.30 - (10.30)^3/6 \\ &= 546 - 182 \\ &= 364\ \text{kip-ft} \end{aligned}$$

At $x = 12\,\text{ft}$ the bending moment is:

$$\begin{aligned} M_{12} &= 53 \times 12 - 12^3/6 \\ &= 636 - 288 \\ &= 348\ \text{kip-ft} \end{aligned}$$

At $x = 18\,\text{ft}$ the bending moment is:

$$\begin{aligned} M_{18} &= M_{12} + (-19) \times 6 \\ &= 348 - 114 \\ &= 234\ \text{kip-ft} \end{aligned}$$

and the bending moment decreases linearly between $x = 12\,\text{ft}$ and $x = 18\,\text{ft}$. At $x = 24\,\text{ft}$ the bending moment is:

$$\begin{aligned} M_{24} &= M_{18} + (-39) \times 6 \\ &= 234 - 234 \\ &= 0\ \text{kip-ft} \end{aligned}$$

and the bending moment decreases linearly between $x = 18\,\text{ft}$ and $x = 24\,\text{ft}$.

3.2 Relationships among loading, shear force, and bending moment

A small element of the beam shown in Figure 3.1 is taken at a distance x from end 1. The forces acting on the element are shown in Figure 3.3. Resolving forces vertically gives the expression:

$$Q = (Q + \delta Q) + w\,\delta x$$

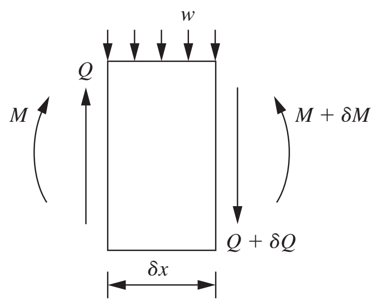


Figure 3.3

and:

$$\delta Q/\delta x = -w$$

The limiting condition is:

$$dQ/dx = -w$$

which indicates that the slope of the shear force diagram, at any section, equals the intensity of loading at that section. This also indicates that the shear force reaches a maximum or a minimum value where the load intensity diagram crosses the base line. Alternatively, since:

$$dQ = -w\ dx$$
$$\int dQ = \int -w\ dx$$
$$Q_2 - Q_1 = \int_{x_1}^{x_2} -w\ dx$$

where Q_1 = shear force in the beam at $x = x_1$, Q_2 = shear force in the beam at $x = x_2$, and the change in shear force between the two sections equals the area of the load intensity diagram between the two sections.

Taking moments about the lower right corner of the element gives the expression:

$$M = (M + \delta M) - Q\ \delta x + w\ (\delta x)^2/2$$

Neglecting the small value $(\delta x)^2$ gives:

$$\delta M/\delta x = Q$$

The limiting condition is:

$$dM/dx = Q$$

which indicates that the slope of the bending moment diagram at any section equals the force at that section. In addition, it indicates that the bending moment reaches a maximum or a minimum value where the shear force diagram crosses the base line. Alternatively, since:

$$\begin{aligned} dM &= Q\ dx \\ \int dM &= \int Q\ dx \\ M_2 - M_1 &= \int_{x_1}^{x_2} Q\ dx \end{aligned}$$

where M_1 = bending moment in the beam at $x = x_1$, M_2 = bending moment in the beam at $x = x_2$, and the change in bending moment between the two sections equals the area of the shear force diagram between the two sections.

Example 3.2

For the simply supported beam shown in Figure 3.2 use the load-shear-moment relationships to draw the shear force and bending moment diagrams.

Solution

The support reactions were determined in Example 3.1:

$$\begin{aligned} V_0 &= 53 \text{ kips} \\ V_{24} &= 39 \text{ kips} \end{aligned}$$

The change in shear force between $x = 0$ and $x = 12$ ft is:

$$\begin{aligned} Q_{12} - V_0 &= \int_0^{12} -w\ dx \\ &= \int_0^{12} -12x/12\ dx \\ &= \int_0^{12} -x\ dx \\ &= [-x^2/2]_0^{12} \\ &= -72 \text{ kips} \end{aligned}$$

and:

$$\begin{aligned} Q_{12} &= 53 - 72 \\ &= -19 \text{ kips} \end{aligned}$$

At intermediate sections between $x = 0$ and $x = 12$ ft, the ordinates of the shear force diagram are given by:

$$\begin{aligned} Q_x &= V_0 - wx/2 \\ &= 53 - x^2/2 \end{aligned}$$

which is the equation of a parabola.

The shear force diagram crosses the base line at a distance x from the end of the beam, given by:

$$\begin{aligned} Q_x &= 0 \\ &= V_0 - wx/2 \\ &= 53 - x^2/2 \end{aligned}$$

and:

$$\begin{aligned} x^2 &= 106 \\ x &= 10.30 \text{ ft} \end{aligned}$$

Between $x = 12$ ft and $x = 18$ ft the load intensity is zero, and the shear force diagram is a horizontal line. At $x = 18$ ft, a concentrated load of 20 kips is applied to the beam; this constitutes an infinite load intensity and produces a vertical step of -20 kips in the ordinate of the shear force diagram. The ordinate of the shear force diagram is given by:

$$\begin{aligned} Q_{18} &= V_{12} - 20 \\ &= -19 - 20 \\ &= -39 \text{ kips} \end{aligned}$$

The ordinate of the shear force diagram remains constant at a value of -39 kips to the end of the beam.

The change in bending moment between $x = 0$ and $x = 12$ ft is:

$$\begin{aligned} M_{12} - M_0 &= \int_0^{12} Q \,\mathrm{d}x \\ &= \int_0^{12} (V_0 - wx/2) \,\mathrm{d}x \\ &= \int_0^{12} (53 - x^2/2) \,\mathrm{d}x \\ &= [53x - x^3/6]_0^{12} \\ &= 636 - 288 \\ &= 348 \text{ kip-ft} \end{aligned}$$

At intermediate sections between $x = 0$ and $x = 12$ ft, the ordinates of the bending moment diagram are given by:

$$\begin{aligned} M_x &= V_0 x - (wx/2)(x/3) \\ &= 53x - x^3/6 \end{aligned}$$

which is the equation of a cubic parabola.

The maximum bending moment occurs when:

$$\begin{aligned} \mathrm{d}M_x/\mathrm{d}x &= 0 \\ &= 53 - x^2/2 \end{aligned}$$

and:

$$x = 10.30 \text{ ft}$$

The change in bending moment between $x = 12$ ft and $x = 18$ ft is:

$$\begin{aligned} M_{18} - M_{12} &= \int_{12}^{18} Q \, dx \\ &= \int_{12}^{18} -19 \, dx \\ &= [-19x]_{12}^{18} \\ &= -19 \times 6 \\ &= -114 \text{ kip-ft} \end{aligned}$$

Hence:

$$\begin{aligned} M_{18} &= 348 - 114 \\ &= 234 \text{ kip-ft} \end{aligned}$$

The bending moment decreases linearly between $x = 12$ ft and $x = 18$ ft.

At $x = 18$ ft a concentrated load of 20 kips is applied, and the change in bending moment between $x = 18$ ft and $x = 24$ ft is:

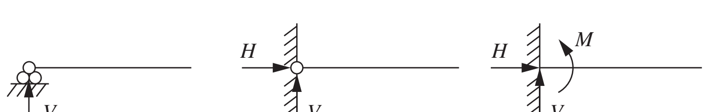
$$\begin{aligned} M_{24} - M_{18} &= \int_{18}^{24} Q \, dx \\ &= \int_{18}^{24} -39 \, dx \\ &= [-39x]_{18}^{24} \\ &= -39 \times 6 \\ &= -234 \text{ kip-ft} \end{aligned}$$

Hence:

$$\begin{aligned} M_{24} &= 234 - 234 \\ &= 0 \text{ kip-ft} \end{aligned}$$

3.3 Statical determinacy

A statically determinate beam or rigid frame is one in which all member forces and external reactions may be determined by applying the equations of equilibrium. In a beam or rigid frame external reactions are provided by either hinge or roller supports or by a fixed end, as shown in Figure 3.4. The roller support



Roller support   Hinge support   Fixed end

Figure 3.4

provides only one degree of restraint, in the vertical direction, and both horizontal and rotational displacements can occur. The hinge support provides two degrees of restraint, in the vertical and horizontal directions, and only rotational displacement can occur. The fixed end provides three degrees of restraint, vertical, horizontal, and rotational. Identifying whether a structure is determinate depends on the configuration of the structure.

(a) Beam or rigid frame with no internal hinges

In a rigid frame with j nodes, including the supports, $3j$ equations of equilibrium may be obtained since, at each node:

$$\sum H = 0$$
$$\sum V = 0$$

and:

$$\sum M = 0$$

Each member of the rigid frame is subjected to an unknown axial and shear force and bending moment. If the rigid frame has n members and r external restraints, the number of unknowns is $(3n + r)$. Thus, a beam or frame is determinate when the number of unknowns equals the number of equilibrium equations or:

$$3n + r = 3j$$

A beam is statically indeterminate, as shown in Figure 3.5, when:

$$3n + r > 3j$$

Figure 3.5

In this case:

$$\begin{aligned} 3j &= 3 \times 2 \\ &= 6 \end{aligned}$$

and:

$$\begin{aligned} 3n + r &= 3 \times 1 + 4 \\ &= 7 \\ &> 3j \end{aligned}$$

A rigid frame is unstable, as shown in Figure 3.6, when:

$$3n + r < 3j$$

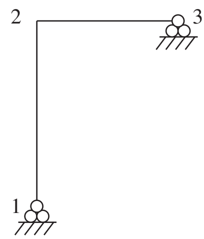


Figure 3.6

In this case:

$$\begin{aligned} 3j &= 3 \times 3 \\ &= 9 \end{aligned}$$

and:

$$\begin{aligned} 3n + r &= 3 \times 2 + 2 \\ &= 8 \\ &< 3j \end{aligned}$$

(b) Beam or rigid frame with internal hinges or rollers

The introduction of an internal hinge in a beam or rigid frame provides an additional equation of equilibrium at the hinge of $M = 0$. In effect, a moment release has been introduced in the member.

The introduction of a horizontal, internal roller provides two additional equations of equilibrium at the roller of $M = 0$ and $H = 0$. In effect, a moment release and a release of horizontal restraint have been introduced in the member. Thus, a beam or frame with internal hinges or rollers is determinate when:

$$3n + r = 3j + h + 2s$$

where n is the number of members, j is the number of nodes in the rigid frame before the introduction of hinges, r is the number of external restraints, h is the number of internal hinges, and s is the number of rollers introduced.

The compound beam shown in Figure 3.7 is determinate since:

$$\begin{aligned} 3n + r &= 3 \times 3 + 5 \\ &= 14 \end{aligned}$$

Figure 3.7

and:

$$\begin{aligned} 3j + h + 2s &= 3 \times 4 + 2 + 0 \\ &= 14 \\ &= 3n + r \end{aligned}$$

(c) Rigid frame with internal hinges at a node

The introduction of a hinge into i of the n members meeting at a node in a rigid frame produces i releases. The introduction of a hinge into all n members produces $(n-1)$ releases.

Thus, a rigid frame with hinges at the nodes is determinate when:

$$3n + r = 3j + c$$

where n is the number of members, j is the number of nodes in the rigid frame, r is the number of external restraints, and c is the number of releases introduced.

As shown in Figure 3.8, for four members meeting at a rigid node there are three unknown moments. The introduction of a hinge into one of the members produces one release, the introduction of a hinge into two members produces two releases, and the introduction of a hinge into all four members produces three releases.

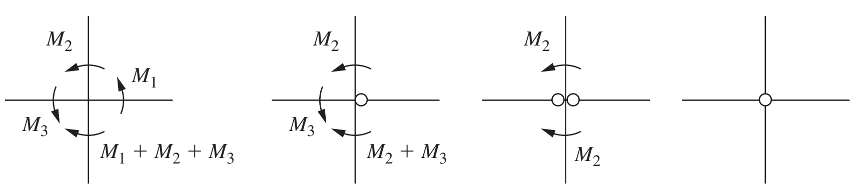


Figure 3.8

The rigid frame shown in Figure 3.9 is determinate since:

$$\begin{aligned} 3n + r &= 3 \times 3 + 4 \\ &= 13 \end{aligned}$$

and:

$$\begin{aligned} 3j + c &= 3 \times 4 + 1 \\ &= 13 \\ &= 3n + r \end{aligned}$$

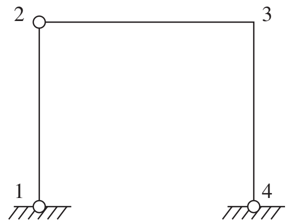


Figure 3.9

3.4 Beams

Beams are normally subject to transverse loads only, and roller and hinge supports are typically represented by vertical arrows. Typical examples of beams are shown in Figure 3.10. Beams may be analyzed using the equations of static equilibrium and the method of sections, as illustrated in Section 3.1. Alternatively, the principle of virtual work may be utilized to provide a simple and convenient solution.

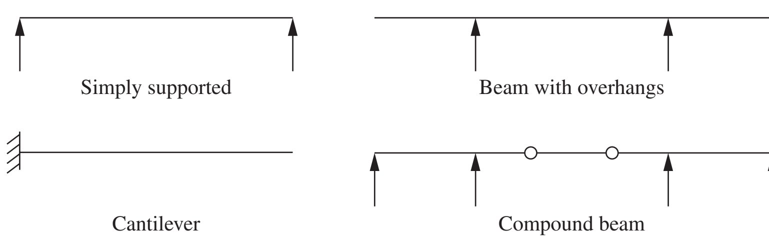


Figure 3.10

The principle of virtual work may be defined as follows: If a structure in equilibrium under a system of applied forces is subjected to a system of displacements compatible with the external restraints and the geometry of the structure, the total work done by the applied forces during these external displacements equals the work done by the internal forces, corresponding to the applied forces, during the internal deformations corresponding to the external displacements.

The expression "virtual work" signifies that the work done is the product of a real loading system and imaginary displacements or an imaginary loading system and real displacements.

For the simply supported beam shown in Figure 3.11 (i), the support reaction V_2, caused by the applied load W, may be determined by the principle of virtual work. As shown at (ii), a unit virtual displacement of $\delta = 1$ is imposed

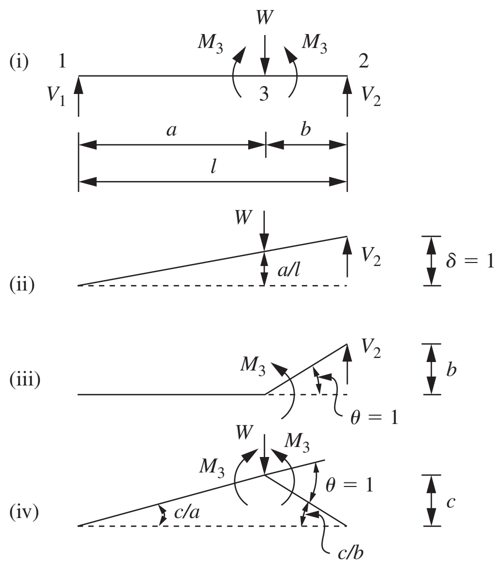


Figure 3.11

on end 2 in the direction of V_2 and the internal work done equated to the external work done. Then:

$$V_2 \times 1 = W \times a/l$$

and:

$$V_2 = Wa/l$$

Similarly, as shown at (iii), the bending moment produced at point 3 by the applied load may be determined by cutting the beam at 3 and imposing a unit virtual angular discontinuity of $\theta = 1$. Equating internal work done to external work done gives:

$$M_3 \times 1 = V_2 \times b$$

and:

$$M_3 = Wab/l$$

Alternatively, after cutting the beam at 3 and imposing a unit virtual angular discontinuity, the ends may be clamped together to produce the deformed shape shown in (iv). Equating internal work done to external work done gives:

$$M_3(c/a + c/b) = W \times c$$

and:

$$M_3 = Wab/l$$

Example 3.3

Use the virtual work method to determine the support reactions and significant bending moments for the compound beam shown in Figure 3.12 due to the applied loads indicated. Hence, draw the shear force and the bending moment diagrams for the beam.

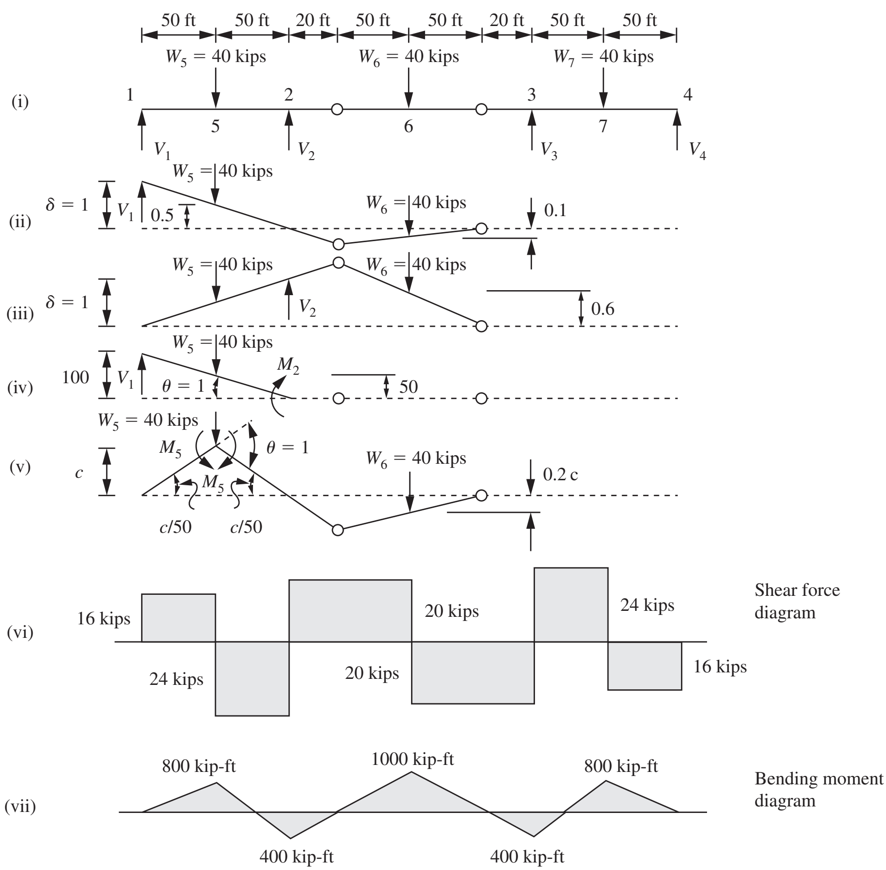


Figure 3.12

Solution

As shown at (ii), a unit virtual displacement of $\delta = 1$ is imposed on end 1 in the direction of V_1 and the internal work done equated to the external work done. Then:

$$V_1 \times 1 = W_5 \times 0.5 - W_6 \times 0.1$$

and:

$$V_1 = 40 \times 0.4$$
$$= 16 \text{ kips}$$

As shown at (iii), a unit virtual displacement of $\delta = 1$ is imposed at support 2 in the direction of V_2 and the internal work done equated to the external work done. Then:

$$V_2 \times 1 = W_5 \times 0.5 + W_6 \times 0.6$$

and:

$$V_2 = 40 \times 1.1$$
$$= 44 \text{ kips}$$

As shown at (iv), the beam is cut at 2 and a unit virtual angular discontinuity of $\theta = 1$ is imposed. The internal work done is equated to the external work done to give:

$$M_2 \times 1 = W_5 \times 50 - V_1 \times 100$$

and:

$$M_2 = 40 \times 50 - 16 \times 100$$
$$= 400 \text{ kip-ft ... tension in the top fibers}$$

As shown at (v), the beam is cut at 5, and a unit virtual angular discontinuity of $\theta = 1$ is imposed. The internal work done is equated to the external work done to give:

$$M_5(c/50 + c/50) = W_5 \times c - W_6 \times 0.2c$$

and:

$$M_5 = 25 \times 40 \times 0.8$$
$$= 800 \text{ kip-ft ... tension in the bottom fibers}$$

Similarly:

$$M_6 = 25 \times 40$$
$$= 1000 \text{ kip-ft ... tension in the bottom fibers}$$

The shear force and bending moment diagrams are shown at (vi) and (vii). The bending moment is drawn on the compression side of the beam.

3.5 Rigid frames

The support reactions of rigid frame structures may be determined using the equations of static equilibrium, and the internal forces in the members from a free body diagram of the individual members. The internal forces on a member are most conveniently indicated as acting from the node on the member: i.e., as support reactions at the node.

The vertical reaction at support 4 of the frame shown in Figure 3.13 is obtained by considering moment equilibrium about support 1. Hence:

$$\begin{aligned} M_1 &= 0 \\ &= lV_4 - Wh \end{aligned}$$

and:

$$V_4 = Wh/l \text{ ... upward}$$

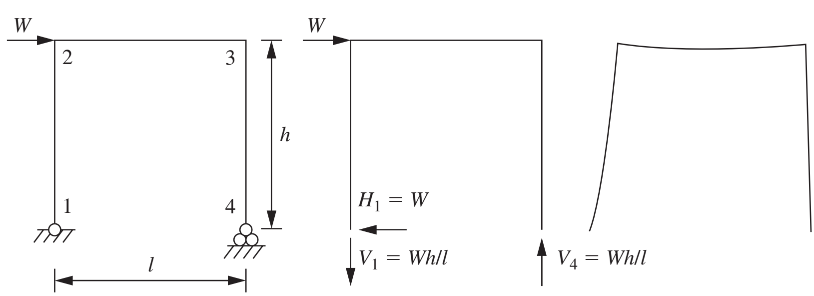
Resolving forces vertically gives:

$$\begin{aligned} V_1 &= -V_4 \\ &= -Wh/l \text{ ... downward} \end{aligned}$$

Resolving forces horizontally gives:

$$H_1 = -W \text{ ... to the left}$$

and the support reactions are shown at (ii). The deformed shape of the frame is shown at (iii).



(i) Applied loads (ii) Support reactions (iii) Deformed shape

Figure 3.13

The internal forces in member 12 are determined from a free body diagram of the member, as shown in Figure 3.14. Resolving forces vertically gives the internal force acting at node 2 as:

$$\begin{aligned} V_2 &= -V_1 \\ &= Wh/l \text{ ... upward} \end{aligned}$$

Considering moment equilibrium about node 2 gives the internal moment at node 2 as:

$$\begin{aligned} M_{21} &= -hH_1 \\ &= Wh \text{ ... counter-clockwise} \end{aligned}$$

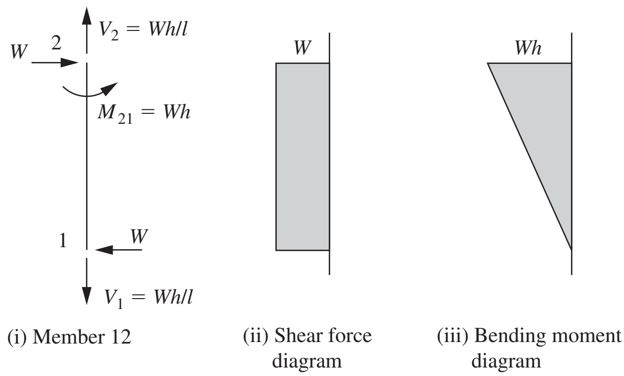


Figure 3.14

Hence, member 12 is subject to a tensile force of $P_{12} = Wh/l$, the shear force diagram is shown at (ii) and the bending moment diagram at (iii) with the moment drawn on the compression side of the member.

The internal forces in member 34 are determined from a free body diagram of the member, as shown in Figure 3.15. Resolving forces vertically gives the internal force acting at node 3 as:

$$\begin{aligned} V_3 &= -V_4 \\ &= -Wh/l \text{ ... downward} \end{aligned}$$

Figure 3.15

Considering moment equilibrium about node 3 gives the internal moment at node 3 as:

$$M_{34} = 0$$

Resolving forces horizontally gives the internal force acting at node 3 as:

$$H_3 = 0$$

Hence, member 34 is subject to a compressive force of $P_{34} = Wh/l$, and both the shear force and the bending moment are zero.

The internal forces in member 23 are determined from a free body diagram of the member, as shown in Figure 3.16. Resolving forces vertically gives the internal force acting at node 2 as:

$$\begin{aligned} V_2 &= -V_3 \\ &= -Wh/l \text{ ... downward} \end{aligned}$$

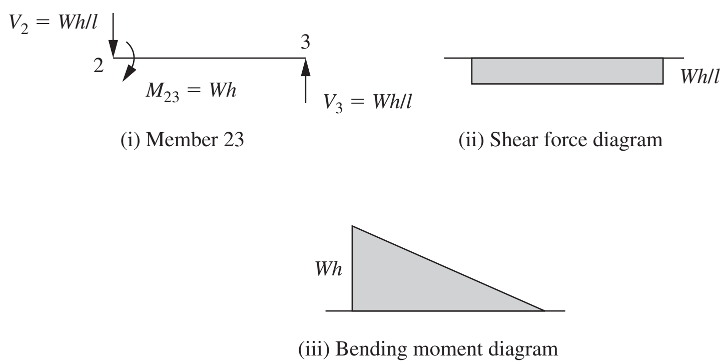


Figure 3.16

Considering moment equilibrium about node 2 gives the internal moment at node 2 as:

$$\begin{aligned} M_{23} &= -lV_3 \\ &= -Wh \text{ ... clockwise} \end{aligned}$$

Resolving forces horizontally gives the internal force acting at node 2 as:

$$H_2 = 0$$

Hence, member 23 has no axial force. The shear force diagram is shown at (ii) and the bending moment diagram at (iii), with the moment drawn on the compression side of the member.

Example 3.4

Determine the support reactions and member forces in the rigid frame shown in Figure 3.17 due to the applied loads indicated. Hence, draw the shear force and the bending moment diagrams for the members.

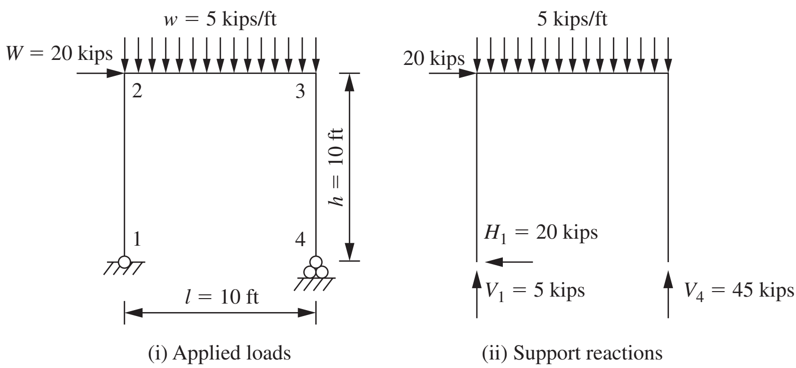


Figure 3.17

Solution

The vertical reaction at support 4 of the frame is obtained by considering moment equilibrium about support 1. Hence:

$$
\begin{aligned}
M_1 &= 0 \\
&= lV_4 - Wh - wl^2/2 \\
&= 10V_4 - 20 \times 10 - 5 \times 100/2
\end{aligned}
$$

and:

$$V_4 = 45 \text{ kips ... upward}$$

Resolving forces vertically gives:

$$
\begin{aligned}
V_1 &= -V_4 + wl \\
&= -45 + 5 \times 10 \\
&= 5\,\text{kips ... upward}
\end{aligned}
$$

Resolving forces horizontally gives:

$$
\begin{aligned}
H_1 &= -W \\
&= 20\,\text{kips ... to the left}
\end{aligned}
$$

and the support reactions are shown at (ii).

The internal forces in member 12 are determined from a free body diagram of the member, as shown in Figure 3.18. Resolving forces vertically gives the internal force acting at node 2 as:

$$
\begin{aligned}
V_2 &= -V_1 \\
&= -5\,\text{kips ... downward}
\end{aligned}
$$

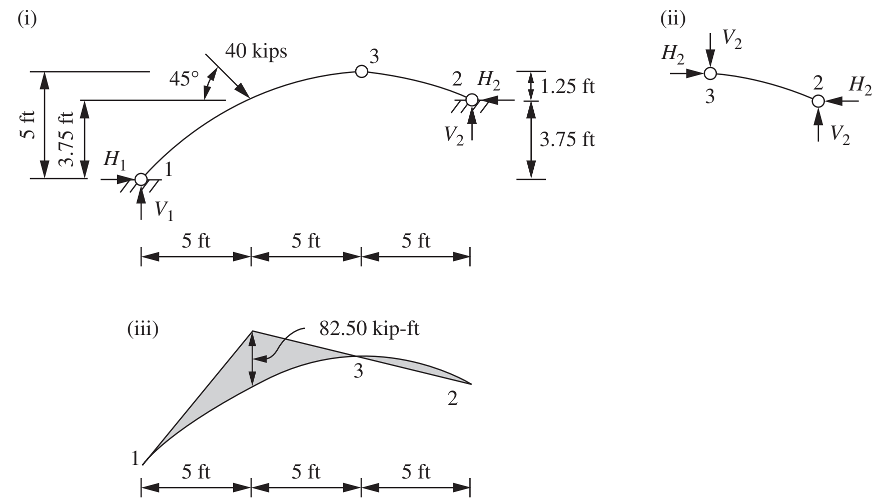


Figure 3.23

Resolving forces vertically gives:

$$\begin{aligned} 0 &= V_1 + V_2 - 40 \sin 45^\circ \\ &= V_1 + V_2 - 28.28 \end{aligned}$$

Resolving forces horizontally gives:

$$\begin{aligned} 0 &= H_1 - H_2 + 40 \cos 45^\circ \\ &= H_1 - H_2 + 28.28 \end{aligned}$$

Considering moment equilibrium about the crown hinge at 3 for the free body diagram of section 23 of the arch shown at (ii) gives:

$$\begin{aligned} M_3 &= 0 \\ &= 5V_2 - 1.25H_2 \end{aligned}$$

Solving these equations simultaneously gives:

$$\begin{aligned} H_1 &= 4.72 \\ H_2 &= 33.00 \\ V_1 &= 20.03 \\ V_2 &= 8.25 \end{aligned}$$

The maximum moment occurs at the location of the applied load and is:

$$\begin{aligned} M_{\max} &= 10V_2 \\ &= 10 \times 8.25 \\ &= 82.50 \text{ kip-ft} \end{aligned}$$

The bending moment is shown at (iii), drawn on the compression side of the arch.

Supplementary problems

S3.1 Determine whether the beams shown in Figure S3.1 are determinate, indeterminate, or unstable.

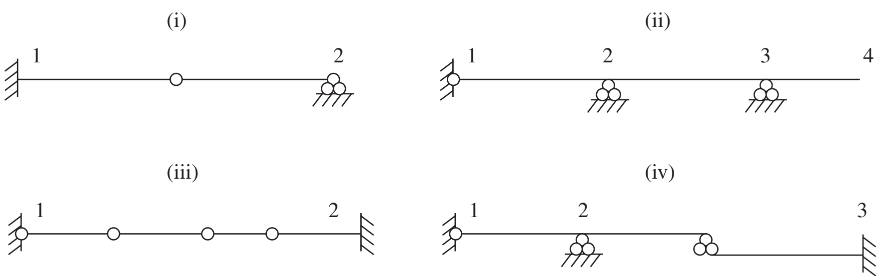


Figure S3.1

S3.2 Determine whether the rigid frames shown in Figure S3.2 are determinate, indeterminate, or unstable.

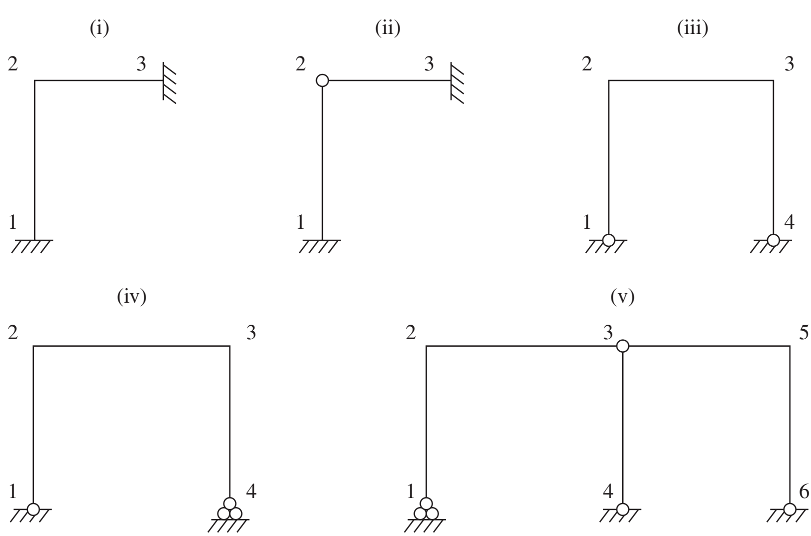


Figure S3.2

S3.3 Determine whether the arch structures shown in Figure S3.3 are determinate, indeterminate or unstable.

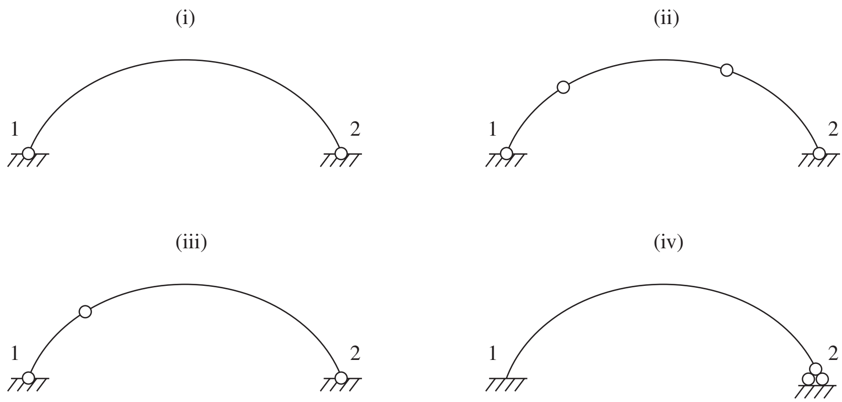


Figure S3.3

S3.4 For the beam shown in Figure S3.4, determine the support reactions and draw the shear force and bending moment diagrams.

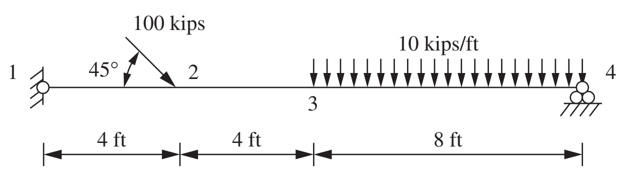


Figure S3.4

S3.5 For the beam shown in Figure S3.5, determine the support reactions and draw the shear force and bending moment diagrams.

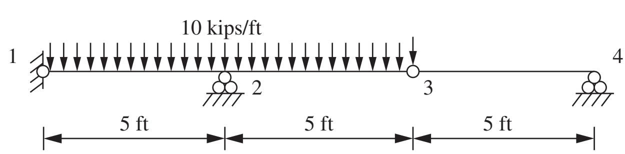


Figure S3.5

S3.6 For the beam shown in Figure S3.6, determine the support reactions and draw the shear force and bending moment diagrams.

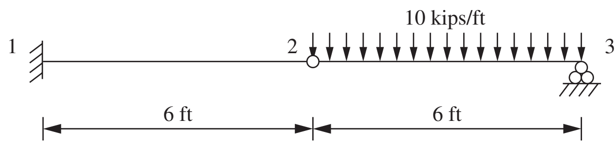


Figure S3.6

S3.7 For the frame shown in Figure S3.7, determine the support reactions and draw the shear force and bending moment diagrams.

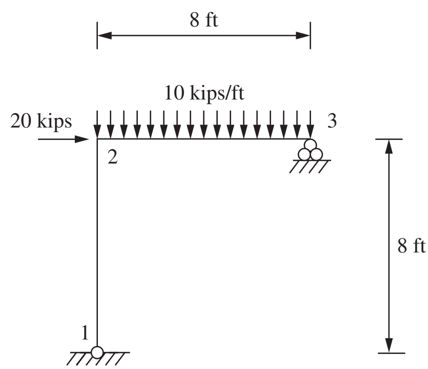


Figure S3.7

S3.8 For the frame shown in Figure S3.8, determine the support reactions and draw the shear force and bending moment diagrams.

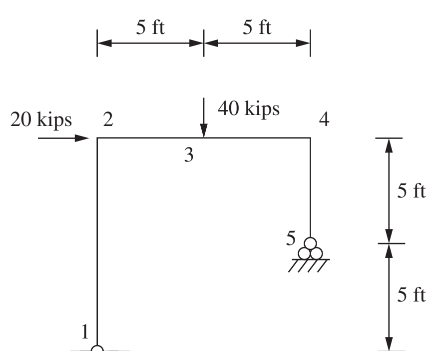


Figure S3.8

S3.9 For the frame shown in Figure S3.9, determine the support reactions and draw the shear force and bending moment diagrams.

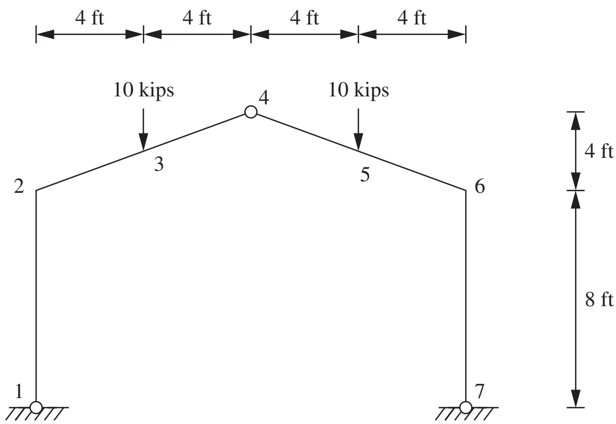


Figure S3.9

S3.10 For the three-hinged arch shown in Figure S3.10, determine the support reactions and the bending moment in the arch rib at the location $x = 8$ ft.

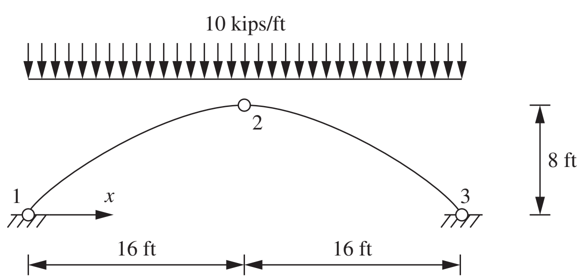


Figure S3.10

4 Elastic deformations

Notation

A	cross–sectional area of a member
c	rise of an arch
C	constant of integration
E	modulus of elasticity
F	force
G	modulus of torsional rigidity
h	height of a rigid frame
H	horizontal force
I	moment of inertia
l	length of member
m	bending moment in a member due to a unit virtual load
M	bending moment in a member due to the applied loads
M'	bending moment in a conjugate member due to the elastic load
P	axial force in a member due to the applied load
q	shear force in a member due to a unit virtual load
Q	shear force in a member due to the applied load
Q'	shear force in a conjugate member due to the elastic load
R	redundant force in a member, radius of curvature
u	axial force in a member due to a unit virtual load
V	vertical force
w	intensity of applied distributed load on a member
w'	intensity of elastic load on a conjugate member, expressed as M/EI
W	concentrated load, applied load on a member, expressed as $\int w\,\mathrm{d}x$
W'	elastic load on a conjugate member, expressed as $\int M\,\mathrm{d}x/EI$
x	horizontal deflection
y	vertical deflection
δ	deflection due to the applied load
δx	element of length of a member
$\delta\theta$	relative rotation between two sections in a member due to the applied loads
θ	rotation due to the applied loads
μ	form factor in shear

4.1 Deflection of beams

(a) Macaulay's method

Macaulay's method provides a simple and convenient method for determining the deflection of beams. It may be used to obtain an expression for the entire elastic curve over the whole length of a beam.

A small element δs of a beam is shown in Figure 4.1 and is assumed to be bent in the shape of an arc of a circle of radius R. The slope of the elastic curve

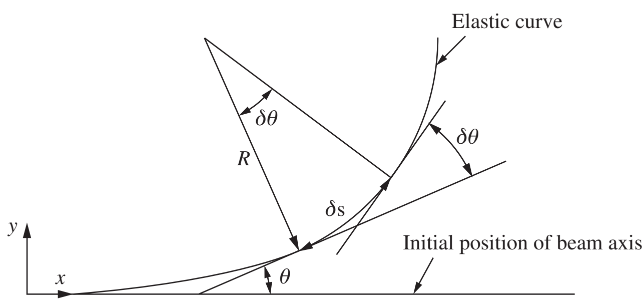


Figure 4.1

at one end of the element is θ. The change in slope of the elastic curve over the length of the element is $\delta\theta$, and the curvature, or rate of change of slope, over the element is:

$$\begin{aligned} \delta\theta/\delta s &= 1/R \\ &\approx \delta\theta/\delta x \ \ldots \text{ positive as shown} \end{aligned}$$

The slope of the beam is positive as shown and for small displacements is given by:

$$\begin{aligned} \theta &\approx \tan\theta \\ &\approx \delta y/\delta x \end{aligned}$$

In the limit:

$$\begin{aligned} \mathrm{d}\theta/\mathrm{d}x &= 1/R \\ \theta &= \mathrm{d}y/\mathrm{d}x \end{aligned}$$

and:

$$\begin{aligned} \mathrm{d}\theta/\mathrm{d}x &= \mathrm{d}/\mathrm{d}x(\mathrm{d}y/\mathrm{d}x) \\ &= \mathrm{d}^2y/\mathrm{d}x^2 \\ &= 1/R \\ &= M/EI \end{aligned}$$

and:

$$EI\ d^2y/dx^2 = M$$

Hence, by setting up an expression for M in terms of the applied loads on a beam and x and integrating this expression twice, an equation is obtained for the deflection of the beam.

Thus, the curvature of the elastic curve is given by the expression:

$$\begin{aligned} d^2y/dx^2 &= 1/R \\ &= M(x)/EI \end{aligned}$$

The slope of the elastic curve is given by the expression:

$$\begin{aligned} dy/dx &= \theta \\ &= \int M(x)/EI + C_1 \end{aligned}$$

The deflection of the elastic curve is given by the expression:

$$\begin{aligned} y &= \delta \\ &= \iint M(x)/EI + C_1x + C_2 \end{aligned}$$

where:

$$\begin{aligned} C_1 &= \text{constant of integration} \\ C_2 &= \text{constant of integration} \\ M(x) &= \text{bending moment at any point in the beam in terms of } x \\ R &= \text{radius of curvature} \end{aligned}$$

The expression for the bending moment at any point in the cantilever shown in Figure 4.2 is:

$$\begin{aligned} EI\,d^2y/dx^2 &= M \\ &= -W(l - x) \ \ldots \text{ tension in the top fiber of the cantilever} \\ &= -Wl + Wx \end{aligned}$$

Integrating this expression with respect to x gives:

$$EI\,dy/dx = -Wlx + Wx^2/2 + C_1$$

where:

$$\begin{aligned} C_1 &= \text{constant of integration} \\ &= 0 \end{aligned}$$

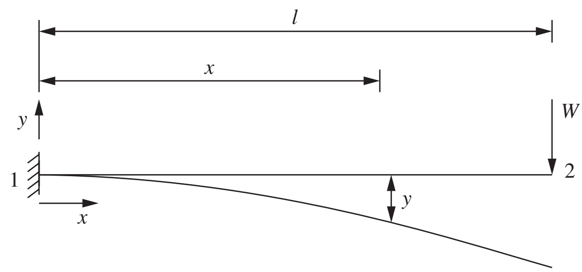


Figure 4.2

since:

$$dy/dx = 0 \text{ at } x = 0$$

Hence, the slope of the elastic curve is given by the expression:

$$EI\,dy/dx = -Wlx + Wx^2/2$$

Integrating this expression with respect to x gives

$$EIy = -Wlx^2/2 + Wx^3/6 + C_2$$

where:

$$\begin{aligned} C_2 &= \text{constant of integration} \\ &= 0 \end{aligned}$$

since:

$$y = 0 \text{ at } x = 0$$

Hence, the deflection of the elastic curve is given by the expression:

$$EIy = -Wlx^2/2 + Wx^3/6$$

At $x = l$ the deflection of the cantilever is:

$$\begin{aligned} y &= -Wl^3/2EI + Wl^3/6EI \\ &= -Wl^3/3EI \ \ldots \text{ downward} \end{aligned}$$

The expression for the bending moment at any point in the cantilever shown in Figure 4.3 is:

$$\begin{aligned} EI\ d^2y/dx^2 &= M \\ &= -W(a - x) - W[x - a] \end{aligned}$$

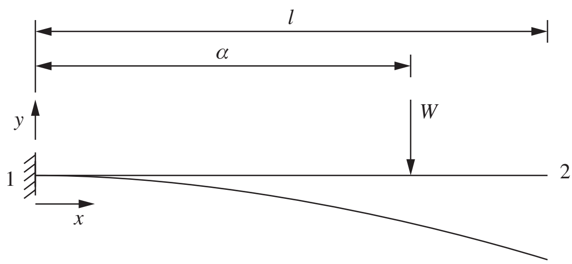


Figure 4.3

The term $[x - a]$ is valid only when positive: i.e., when $x > a$.
Hence:

$$EI\ d^2y/dx^2 = -Wa + Wx - W[x - a]$$

Integrating this expression with respect to x for the terms outside the brackets and with respect to $(x - a)$ for the term in brackets gives:

$$EI\ dy/dx = -Wax + Wx^2/2 - W[x - a]^2/2 + C_1$$

where:

$$C_1 = \text{constant of integration}$$
$$= 0$$

since:

$$dy/dx = 0 \text{ at } x = 0$$

Hence, the slope of the elastic curve is given by the expression:

$$EI\, dy/dx = -Wax + Wx^2/2 - W[x - a]^2/2$$

Integrating this expression gives:

$$EIy = -Wax^2/2 + Wx^3/6 - W[x - a]^3/6 + C_2$$

where:

$$C_2 = \text{constant of integration}$$
$$= 0$$

since:

$$y = 0 \text{ at } x = 0$$

Hence, the deflection of the elastic curve is given by the expression:

$$EIy = -Wax^2/2 + Wx^3/6 - W[x-a]^3/6$$

At $x = a$ the deflection of the cantilever is:

$$\begin{aligned} y &= -Wa^3/2EI + Wa^3/6EI \\ &= -Wa^3/3EI \ldots \text{ downward} \end{aligned}$$

At $x = l$ the deflection of the cantilever is:

$$\begin{aligned} y &= -Wal^2/2EI + Wl^3/6EI - W(l-a)^3/6EI \\ &= -\,Wal^2/2EI + Wl^3/6EI - W(l^3 - 3al^2 + 3a^2l - a^3)/6EI \\ &= -\,Wa^2(3l-a)/6EI \ldots \text{ downward} \end{aligned}$$

Example 4.1

For the simply supported beam shown in Figure 4.4, determine the maximum deflection due to the applied load. The flexural rigidity of the beam is $EI = 7 \times 10^6$ kip in^2.

Solution

The expression for the bending moment at any point in the beam shown in Figure 4.4 is:

$$\begin{aligned} EI\ \mathrm{d}^2y/\mathrm{d}x^2 &= V_1x - W[x-a] \\ &= 20x - 60[x-12] \end{aligned}$$

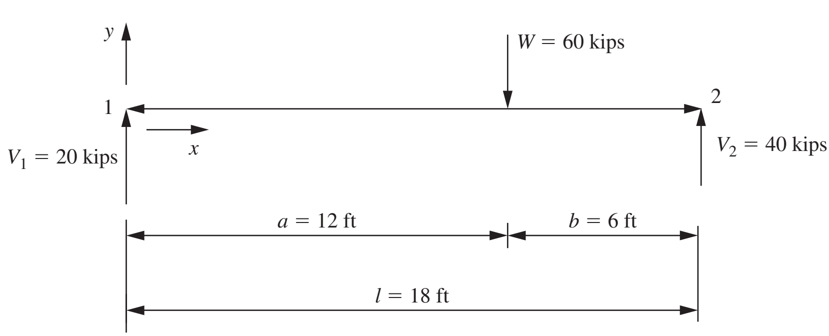


Figure 4.4

Integrating this expression with respect to x for the terms outside the brackets and with respect to $(x - a)$ for the term in brackets gives an expression for the slope of the elastic curve of:

$$EI\ \mathrm{d}y/\mathrm{d}x = 10x^2 - 30[x - 12]^2 + C_1$$

where:

$$C_1 = \text{constant of integration}$$

Integrating this expression gives an expression for the deflection of the elastic curve of:

$$EIy = 10x^3/3 - 10[x - 12]^3 + C_1x + C_2$$

where:

$$C_2 = \text{constant of integration}$$

At

$$x = 0, y = 0$$

and:

$$C_2 = 0$$

At

$$x = 18 \text{ ft}, y = 0$$

and:

$$0 = 10 \times 18^3/3 - 10(18 - 12)^3 + C_1 \times 18$$

then

$$C_1 = -960 \text{ kip ft}^2$$

Hence, the deflection of the elastic curve is given by the expression:

$$EIy = 10x^3/3 - 10[x - 12]^3 - 960x$$

The maximum deflection occurs where the slope of the elastic curve is zero and:

$$\begin{aligned} EI\ \mathrm{d}y/\mathrm{d}x &= 0 \\ &= 10x^2 - 960 \end{aligned}$$

and:

$$x = 9.80 \text{ ft}$$

The maximum deflection is given by:

$$\begin{aligned} EIy_{\max} &= 10 \times 9.8^3/3 - 960 \times 9.8 \\ &= -6273 \\ y_{\max} &= -6273 \times 144/(7 \times 10^6) \\ &= -0.129 \text{ ft} \\ &= -1.55 \text{ in} \ldots \text{ downward} \end{aligned}$$

(b) Virtual work method

The virtual work, or unit-load, method may be used to obtain the displacement of a single point in a beam. The principle may be defined as follows: if a structure in equilibrium under a system of applied forces is subjected to a system of displacements compatible with the external restraints and the geometry of the structure, the total work done by the applied forces during these external displacements equals the work done by the internal forces, corresponding to the applied forces, during the internal deformations, corresponding to the external displacements. The expression "virtual work" signifies that the work done is the product of a real loading system and imaginary displacements or an imaginary loading system and real displacements.

To the cantilever shown in Figure 4.5 (i), the external loads W are gradually applied. This results in the deflection of any point 3 a distance δ, while each load moves a distance y in its line of action. The loading produces a bending moment M and a relative rotation $\delta\theta$ to the ends of the element shown at (ii). From the principle of conservation of energy and ignoring the effects of axial and shear forces, the external work done during the application of the loads must equal the internal energy stored in the beam.

Then:

$$\sum Wy/2 = \sum M\delta\theta/2 \ldots \tag{1}$$

To the unloaded structure a unit virtual load is applied at 3 in the direction of δ as shown at (iii). This results in a bending moment m in the element.

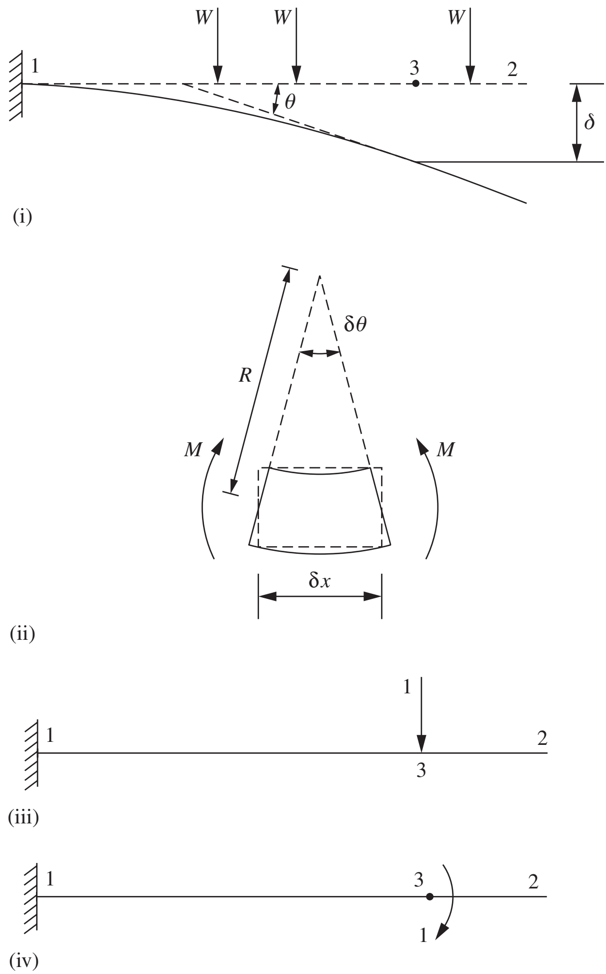


Figure 4.5

Now, while the virtual load is still in position, the real loads W are gradually applied to the structure. Again equating external work and internal energy:

$$\sum Wy/2 + 1 \times \delta = \sum M\,\delta\theta/2 + \sum m\,\delta\theta \ldots \qquad (2)$$

Subtracting expression (1) from expression (2):

$$\begin{aligned} 1 \times \delta &= \sum m\,\delta\theta \\ &= \sum m\,\delta x/R \\ &= \sum m\,\delta x/EI \end{aligned}$$

In the limit:

$$1 \times \delta = \int Mm\,\mathrm{d}x/EI$$

If it becomes necessary to include the deflection due to shear, the expression becomes:

$$1 \times \delta = \int Mm\,\mathrm{d}x/EI + \int Vv\,\mathrm{d}x/\mu AG$$

where M and V are the bending moment and shear force at any section due to the applied loads, and I, G, and A are the second moment of area, the rigidity modulus, and the area of the section; μ is the form factor; and m and v are the bending moment and shear force at any section due to the unit virtual load.

In a similar manner, the rotation θ of any point 3 of the structure may be obtained by applying a unit virtual bending moment at 3 in the direction of θ, as shown at (iv).

Then:

$$1 \times \theta = \int Mm\,\mathrm{d}x/EI + \int Vv\,\mathrm{d}x/\mu AG$$

where m and v are the bending moment and shear force at any section due to the unit virtual moment.

For a beam, moments produced by the virtual load or moment are considered positive, and moments produced by the applied loads, which are of opposite sense, are considered negative. A positive value for the displacement indicates that the displacement is in the same direction as the virtual force or moment.

The deflection and slope at the free end of the cantilever shown in Figure 4.6 may be obtained by the virtual work method. Taking the origin of coordinates at point 3, the expression for the bending moment due to the applied load is obtained from Figure 4.6 (ii) as:

$$M = Wx$$

A unit vertical load is applied at the end of the cantilever and the function m derived from (iii) as:

$$m = l - a + x$$

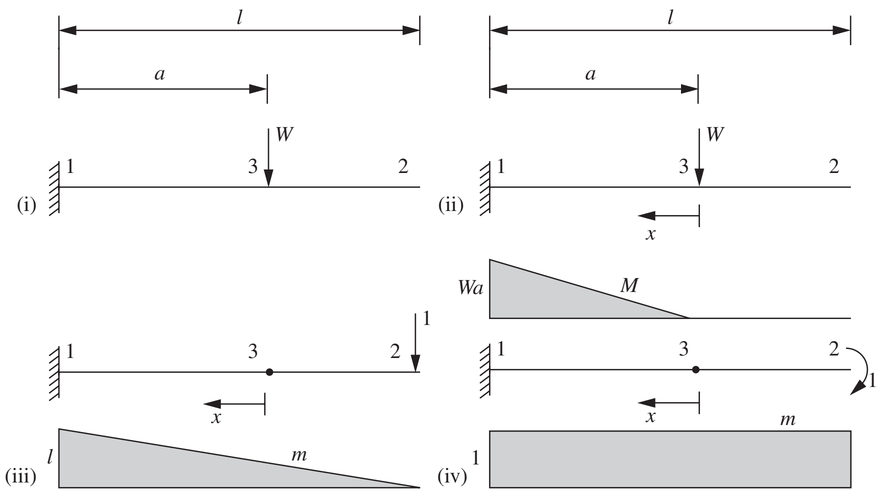


Figure 4.6

The vertical deflection at 2 is given by:

$$
\begin{aligned}
1 \times \delta &= \int_0^a Mm \, \mathrm{d}x/EI \\
&= W\int_0^a x(l - a + x) \, \mathrm{d}x/EI \\
&= W[x^2l/2 - x^2a/2 + x^3/3]_0^a/EI \\
&= W(a^2l/2 - a^3/2 + a^3/3)/EI \\
&= Wa^2(3l - a)/6EI \ldots \text{ downward}
\end{aligned}
$$

To determine the slope at the end of the cantilever, a unit clockwise rotation is applied at the end of the cantilever and the function m derived from (iv) as:

$$m = 1$$

The slope at 2 is given by:

$$
\begin{aligned}
1 \times \theta &= \int_0^a Mm \, \mathrm{d}x/EI \\
&= W\int_0^a x \, \mathrm{d}x/EI \\
&= Wa^2/2EI \ldots \text{ clockwise}
\end{aligned}
$$

Example 4.2

For the simply supported beam shown in Figure 4.7, determine the deflection at the location of the applied load. The flexural rigidity of the beam is $EI = 7 \times 10^6$ kip in^2.

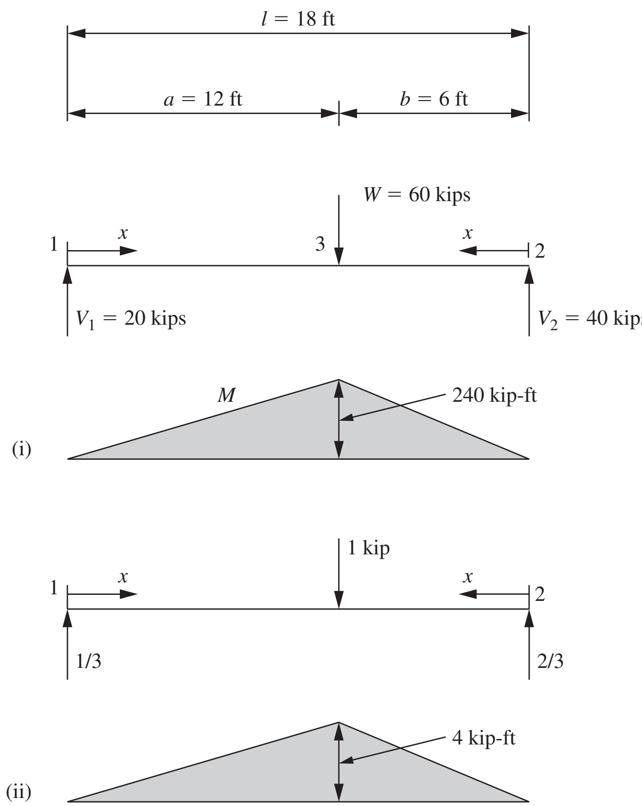


Figure 4.7

Solution

Taking the origin of coordinates at end 1 and end 2 in turn, the expressions for the bending moment due to the applied load are obtained from Figure 4.7(i) as:

$$\begin{aligned} M &= V_1 x \\ &= 20x \end{aligned}$$

and:

$$\begin{aligned} M &= V_2 x \\ &= 40x \end{aligned}$$

A unit vertical load is applied at 3 and the corresponding functions for m derived from (ii) as:

$$m = x/3$$

and:

$$m = 2x/3$$

The vertical deflection at 3 is given by:

$$\begin{aligned}
1 \times \delta &= \int_0^a Mm\,\mathrm{d}x/EI + \int_0^b Mm\,\mathrm{d}x/EI \\
&= 20\int_0^a x^2\ \mathrm{d}x/3EI + 80\int_0^b x^2\ \mathrm{d}x/3EI \\
&= 20[x^3/9EI]_0^{12} + 80[x^3/9EI]_0^6 \\
&= 3840/EI + 1920/EI \\
&= 5760 \times 144/(7 \times 10^6) \\
&= 0.118 \text{ ft} \\
&= 1.422 \text{ in} \ldots \text{ downward}
\end{aligned}$$

As an alternative to evaluating the integrals in the virtual work method, advantage may be taken of the volume integration method. For a straight prismatic member:

$$\int Mm\,\mathrm{d}x/EI = \int Mm\,\mathrm{d}x \times 1/EI.$$

The function m is always either constant along the length of the member or varies linearly. The function M may vary linearly for real concentrated loads or parabolically for real distributed loads. Thus, $\int Mm\,\mathrm{d}x$ may be regarded as the volume of a solid with a cross-section defined by the function M and a height defined by the function m. The volume of this solid is given by the area of cross-section multiplied by the height of the solid at the centroid of the cross-section.

Commonly occurring values of $\int Mm\,\mathrm{d}x$ are provided in Part 2, Chapter 2, Table 2.3 for various types of functions M and m.

Example 4.3

Determine the slope at the free end of the cantilever shown in Figure 4.8 using the volume integration method.

Solution

The functions M and m derived from Figure 4.8 (i) and (ii) and the solid defined by these functions are shown at (iii). The slope at the end of the cantilever is given by the volume of this solid $\times$ $1/EI$, which is:

$$\begin{aligned}
\theta &= Wa \times a/2 \times 1 \times 1/EI \\
&= Wa^2/2EI
\end{aligned}$$

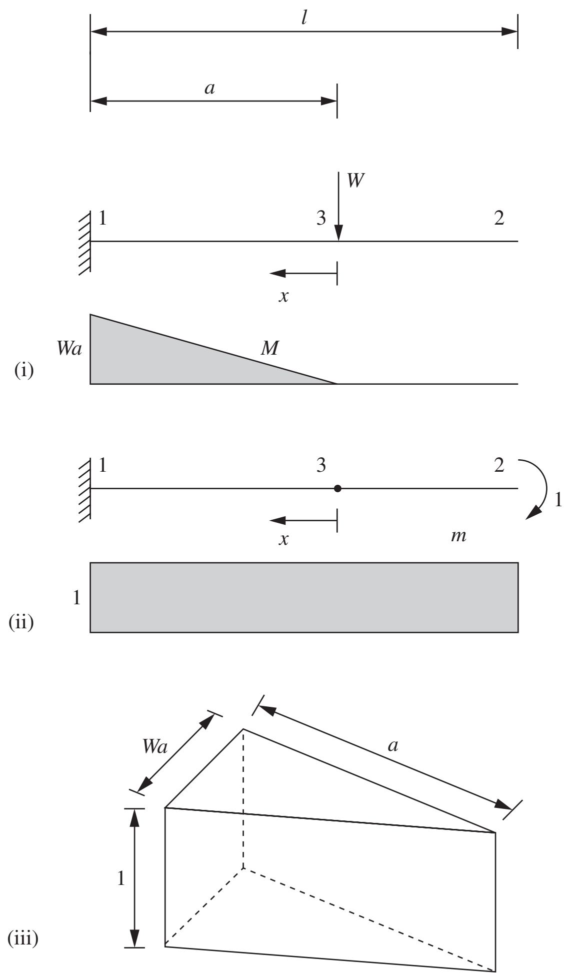


Figure 4.8

Alternatively, the value of $\int Mm\,dx/EI$ may be obtained directly from Part 2, Table 2.3, and is given by:

$$
\begin{aligned}
\theta &= a \times 1 \times Wa/2EI \\
&= Wa^2/2EI
\end{aligned}
$$

4.2 Deflection of rigid frames

(a) Virtual work method

The virtual work method may be applied to rigid frames to obtain the displacement at a specific point on the frame. Integration is carried out over all members of the frame and the values summed to provide the required displacement.

To determine the horizontal deflection of node 2 for the frame shown in Figure 4.9 (i), use is made of the bending moment diagrams produced by the applied load and by a unit virtual horizontal load at node 2. These are shown at (ii) and (iv). Member 34 has zero bending moment under both loading cases.

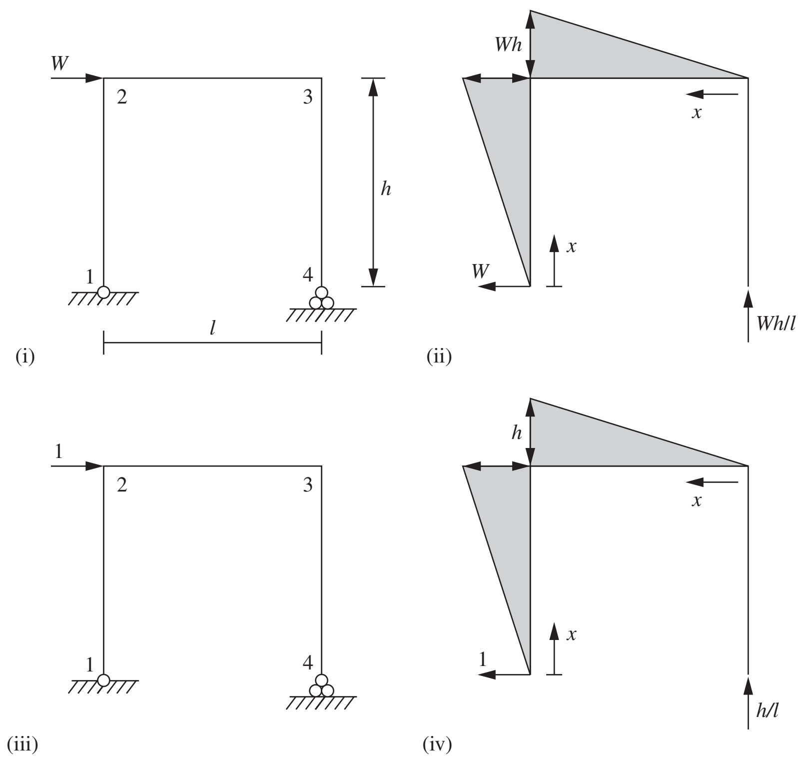


Figure 4.9

With the origin of coordinates as indicated, expressions for the bending moment produced by the applied load are obtained from Figure 4.9 (ii) as:

$$M = Wx \;\ldots\; \text{member } 12$$

and:

$$M = Whx/l \;\ldots\; \text{member } 23$$

A unit horizontal load is applied at node 2, as shown at (iii), and the corresponding functions for m derived from (iv) as:

$$m = x \;\ldots\; \text{member } 12$$

and:

$$m = hx/l \; \ldots \text{ member 23}$$

The horizontal deflection at node 2 is obtained by summing the values for members 12 and 23 to give:

$$\begin{aligned}1 \times \delta &= \int_0^h Mm\ \mathrm{d}x/EI + \int_0^l Mm\ \mathrm{d}x/EI \\ &= W \int_0^h x^2\ \mathrm{d}x/EI + W \int_0^l h^2x^2\ \mathrm{d}x/l^2EI \\ &= W[x^3/3EI]_0^h + W[h^2x^3/3l^2EI]_0^l \\ &= W(h^3/3EI + h^2l/3EI)\end{aligned}$$

Example 4.4

Determine the horizontal deflection of node 2 for the frame shown in Figure 4.10. The flexural rigidity of the members is $EI = 30 \times 10^6$ kip in^2.

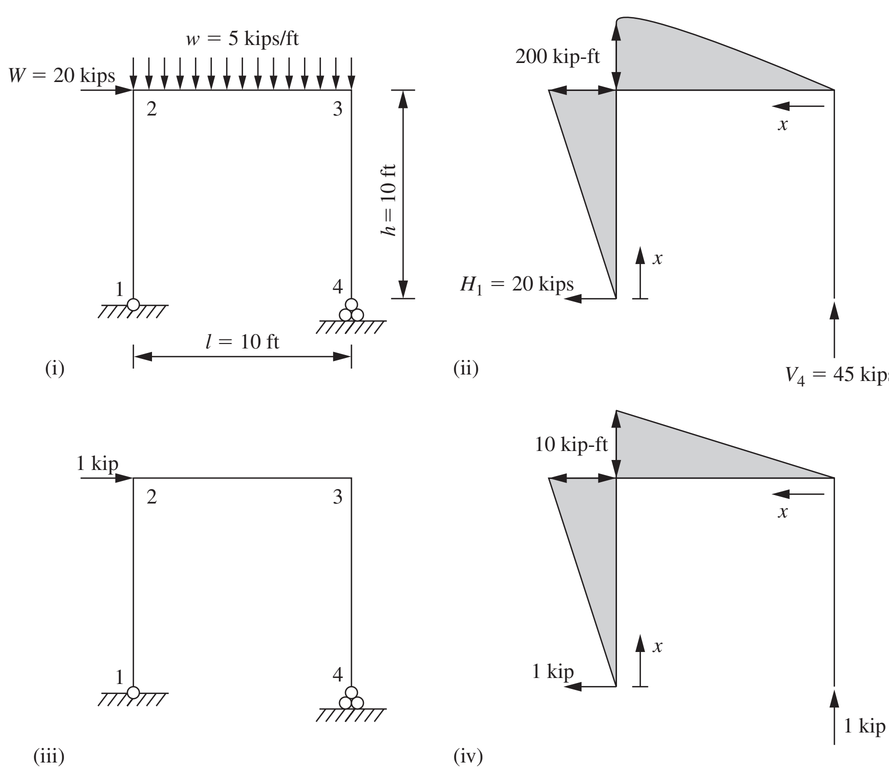


Figure 4.10

Solution

The bending moments produced in the members of the frame by the applied loads shown in Figure 4.10 (i) are obtained from (ii). With the origin of coordinates as indicated, the expressions for the moments are:

$$\begin{aligned} M &= H_1 x \;\ldots\; \text{member } 12 \\ &= 20x \end{aligned}$$

and:

$$\begin{aligned} M &= V_4 x - wx^2/2 \;\ldots\; \text{member } 23 \\ &= 45x - 5x^2/2 \end{aligned}$$

A unit horizontal load is applied at node 2, as shown at (iii), and the corresponding functions for m derived from (iv) as:

$$m = x \;\ldots\; \text{member } 12$$

and:

$$\begin{aligned} m &= hx/l \;\ldots\; \text{member } 23 \\ &= x \end{aligned}$$

No moments are produced in member 34, and the horizontal deflection at node 2 is obtained by summing the values for members 12 and 23 to give:

$$\begin{aligned} 1 \times \delta &= \int_0^h Mm\,\mathrm{d}x/EI + \int_0^l Mm\,\mathrm{d}x/EI \\ &= 20 \int_0^{10} x^2\,\mathrm{d}x/EI + 5\int_0^{10} (9x^2 - x^3/2)\mathrm{d}x/EI \\ &= 20[x^3/3EI]_0^{10} + 5[3x^3/EI - x^4/8EI]_0^{10} \\ &= 15{,}417 \times 144/(30 \times 10^6) \\ &= 0.074 \text{ ft} \\ &= 0.89 \text{ in} \end{aligned}$$

(b) Conjugate beam method

The conjugate beam method may be used to obtain an expression for the entire elastic curve over the whole of a rigid frame or beam.

The beam shown in Figure 4.11 (i) is subjected to a distributed applied load of intensity w, positive when acting upward as indicated. The shear force at any section is given by the area under the load intensity curve as:

$$Q = \int w\,\mathrm{d}x$$

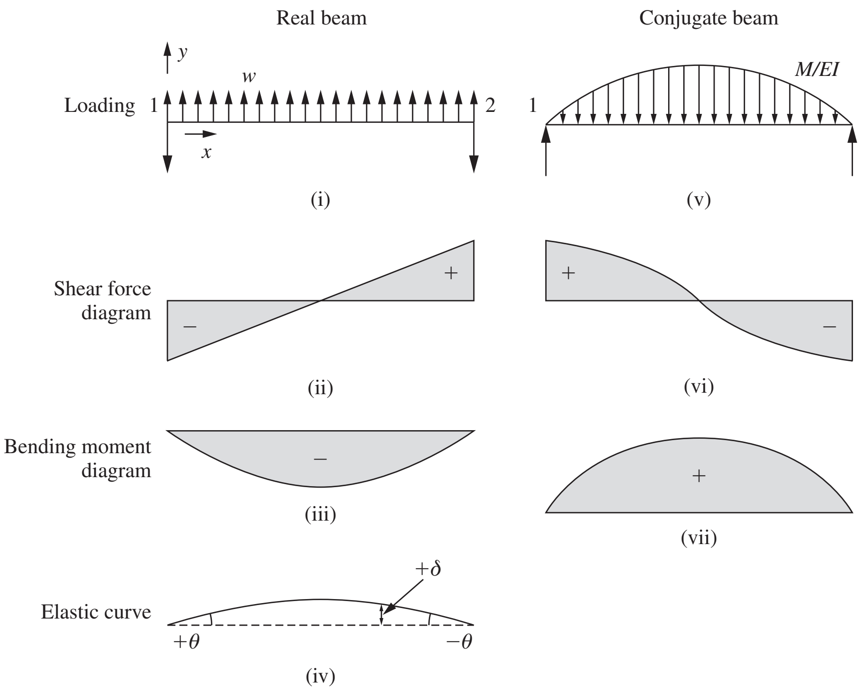


Figure 4.11

with shear force upward on the left of a section regarded as positive. The shear force diagram is shown at (ii) and is negative at end 1 and positive at end 2.

The bending moment at any section is given by the area under the shear force curve as:

$$\begin{aligned} M &= \int Q\,\mathrm{d}x \\ &= \iint w\,\mathrm{d}x \end{aligned}$$

with bending moment producing tension in the bottom fiber regarded as positive. The bending moment diagram is shown at (iii) and is negative as indicated.

In addition, the curvature of the elastic curve at any section is given by:

$$\begin{aligned} \mathrm{d}^2y/\mathrm{d}x^2 &= \mathrm{d}\theta/\mathrm{d}x \\ &= 1/R \\ &= M/EI \end{aligned}$$

and the slope and deflection at any section are given by:

$$\begin{aligned} \mathrm{d}y/\mathrm{d}x &= \theta \\ &= \int M\,\mathrm{d}x/EI \\ y &= \delta \\ &= \int \theta\,\mathrm{d}x \\ &= \iint \mathrm{M}\,\mathrm{d}x/EI \end{aligned}$$

with x positive to the right and y positive upward. The elastic curve is shown at (iv) with deflections positive (i.e., upward) as indicated and with a positive slope (i.e., counterclockwise rotation) at end 1 and a negative slope (i.e., clockwise rotation) at end 2.

An analogous beam known as the conjugate beam and of the same length as the real beam, as shown in Figure 4.11 (v), is subjected to an applied loading of intensity:

$$w' = M/EI$$

where M is the bending moment in the actual beam at any section and $\int M\,\mathrm{d}x/EI$ is known as the elastic load, W'. The elastic load acts in a positive direction (upward) when the bending moment in the real beam is positive. The loading diagram is shown at (v) and is negative as indicated.

Then, the shear at any section in the conjugate beam is given by:

$$\begin{aligned} Q' &= \int w'\,\mathrm{d}x \\ &= \int M\,\mathrm{d}x/EI \\ &= \theta \end{aligned}$$

where θ is the slope at the corresponding section in the real beam. The shear force diagram is shown at (vi) and is positive at end 1 and negative at end 2. Hence, as shown at (iv), the slope of the elastic curve is positive at end 1 and negative at end 2.

The bending moment at any section in the conjugate beam is given by:

$$\begin{aligned} M' &= \int Q'\,\mathrm{d}x \\ &= \iint M\,\mathrm{d}x/EI \\ &= \delta \end{aligned}$$

where δ is the deflection at the corresponding section in the real beam. The bending moment diagram is shown at (vii) and is positive as indicated. Hence, as shown at (iv), the deflection of the elastic curve is positive.

Thus, the slope and deflection at any section in the real beam are given by the shear and bending moment at the corresponding section in the conjugate beam, and the elastic curve of the real beam is given by the bending moment diagram of the conjugate beam. The end slope and end deflection of the real beam are given by the end reaction and end moment of the conjugate beam. The maximum deflection in the real beam occurs at the position of zero shear in the conjugate beam.

In the case of frames, the elastic load applied to the conjugate frame is positive (i.e., acts vertically upward) when the outside fiber of the real frame is in compression. Then, the displacement of the real frame at any section is perpendicular to the lever arm used to determine the moment in the conjugate frame and is outward when a positive bending moment occurs at the corresponding section of the conjugate frame.

The restraints of the conjugate structure must be consistent with the displacements of the real structure. Details of the necessary restraints in the conjugate structure are provided in Part 2, Chapter 4, Table 4.1.

At a simple end support in a real structure, there is a rotation but no deflection. Thus, the corresponding restraints in the conjugate structure must be a shear force and a zero moment, which are produced by a simple end support in the conjugate structure.

At a fixed end in a real structure, there is neither a rotation nor a deflection. Thus, there must be no restraint at the corresponding point in the conjugate structure, which must be a free end.

At a free end in a real structure, there is both a rotation and a deflection. Thus, the corresponding restraints in the conjugate structure are a shear force and a bending moment, which are produced at a fixed end.

At an interior support in a real structure, there is no deflection and a smooth change in slope. Thus, there can be no moment and no reaction at the corresponding point in the conjugate structure, which must be an unsupported hinge.

At an interior hinge in a real structure, there is a deflection and an abrupt change of slope. Thus, the corresponding restraints in the conjugate structure are a moment and a reaction, which are produced by an interior support.

Example 4.5

Determine the rotation of nodes 1 and 4 and the horizontal deflection of node 2 for the frame shown in Figure 4.12. The flexural rigidity of the members is $EI = 30 \times 10^6$ kip in^2.

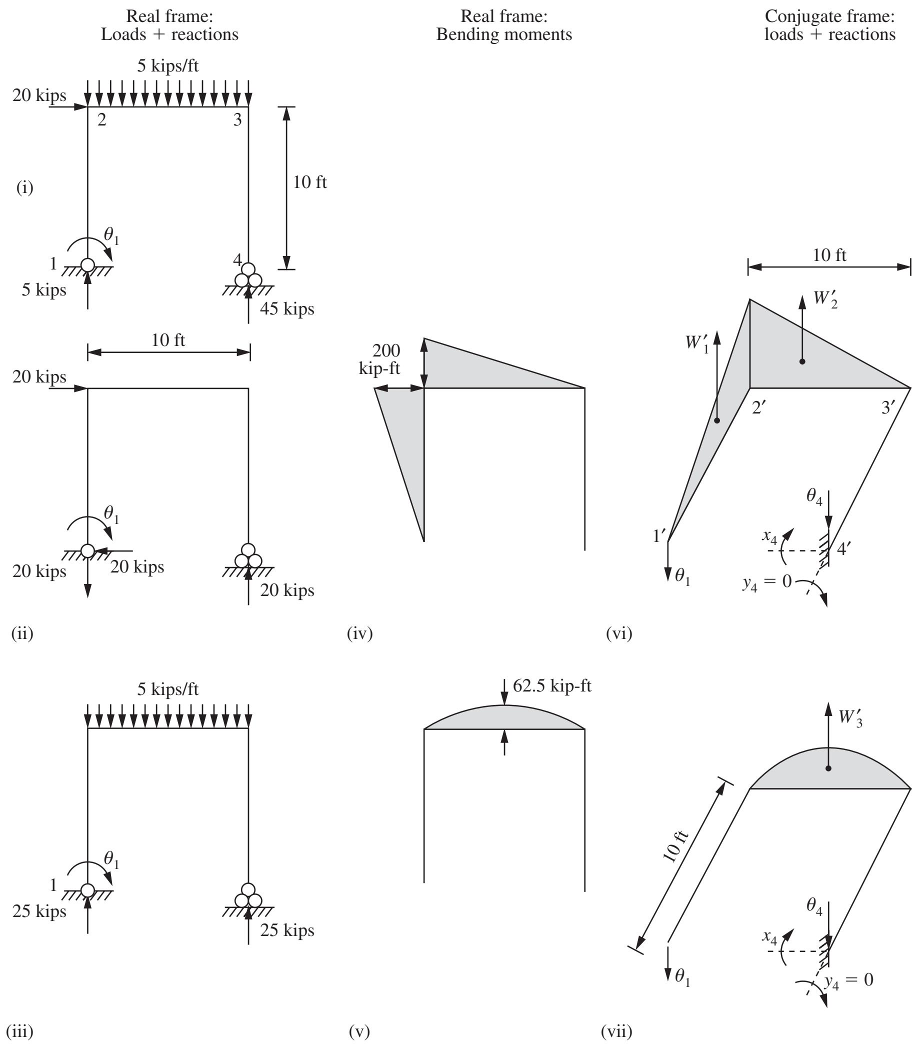


Figure 4.12

Solution

It is convenient to utilize the principle of superposition and consider the distributed load and the lateral load separately.

The lateral load is applied to the frame at (ii), which results in the bending moment diagram, drawn on the compression side of the members, shown at (iv). The elastic loads are applied to the conjugate frame at (vi) and are given by:

$$\begin{aligned} W'_1 &= W'_2 \\ &= 0.5 \times 10 \times 200/EI \\ &= 1000/EI \ \ldots \text{ upward} \end{aligned}$$

Since the vertical deflection at 4 is zero, the bending moment at 4′ about axis 3′4′ is zero. Hence, the rotation at 1 is obtained by taking moments in the conjugate frame about axis 3′4′ and is given by:

$$\begin{aligned}\theta_1 &= -(20/3 \times W'_2 + 10 \times W'_1)/10 \\ &= -1667/EI \ \ldots \text{ clockwise}\end{aligned}$$

The rotation at 4 is obtained by resolving forces vertically in the conjugate frame and is given by:

$$\begin{aligned}\theta_4 &= -W'_2 - W'_1 - \theta_1 \\ &= -333/EI \ \ldots \text{ clockwise}\end{aligned}$$

The horizontal deflection at 2 is given by the bending moment at 2′ in the conjugate frame about axis 2′3′, which is:

$$\begin{aligned}\delta_2 &= -10\theta_1 - 10/3 \times W'_1 \\ &= 13{,}336/EI \ \ldots \text{ to the right}\end{aligned}$$

The distributed load is applied to the frame at (iii), which results in the bending moment diagram, drawn on the compression side of the members, shown at (v). The elastic load is applied to the conjugate frame at (vii) and is given by:

$$\begin{aligned}W'_3 &= 0.667 \times 10 \times 62.5/EI \\ &= 416.9/EI \ \ldots \text{ upward}\end{aligned}$$

Since the vertical deflection at 4 is zero, the bending moment at 4′ about axis 3′4′ is zero. Hence, the rotation at 1 is obtained by taking moments in the conjugate frame about axis 3′4′ and is given by:

$$\begin{aligned}\theta_1 &= -(10/2 \times W'_3)/10 \\ &= -208.4/EI \ \ldots \text{ clockwise}\end{aligned}$$

The rotation at 4 is obtained by resolving forces vertically in the conjugate frame and is given by:

$$\begin{aligned}\theta_4 &= -W'_3 - \theta_1 \\ &= -208.4/EI \ \ldots \text{ clockwise}\end{aligned}$$

The horizontal deflection at 2 is given by the bending moment in the conjugate frame about axis 2′3′, which is:

$$\begin{aligned}\delta_2 &= -10\theta_1 \\ &= 2084/EI \ \ldots \text{ to the right}\end{aligned}$$

The total horizontal deflection at 2 is obtained by summing the individual values calculated for the lateral load and the distributed load and is given by:

$$\begin{aligned}\delta_2 &= 13{,}336/EI + 2084/EI \\ &= 15420 \times 144/(30 \times 10^6) \\ &= 0.074 \text{ ft} \\ &= 0.89 \text{ in} \ldots \text{ to the right}\end{aligned}$$

The total rotation at 1 is obtained by summing the individual values for the lateral load and the distributed load and is given by:

$$\begin{aligned}\theta_1 &= -1667/EI - 208.4/EI \\ &= -1875.4 \times 144/(30 \times 10^6) \\ &= -0.0090 \text{ rad} \ldots \text{ clockwise}\end{aligned}$$

The total rotation at 4 is obtained by summing the individual values for the lateral load and the distributed load and is given by:

$$\begin{aligned}\theta_4 &= -333/EI - 208.4/EI \\ &= -541.4 \times 144/(30 \times 10^6) \\ &= -0.00260 \text{ rad} \ldots \text{ clockwise}\end{aligned}$$

4.3 Deflection of pin-jointed frames

The virtual work method may be applied to pin-jointed frames to obtain the displacement at a node on the frame.

To the pin-jointed frame shown in Figure 4.13 (i), the external loads W are gradually applied. This results in the deflection of any node 3 a distance δ,

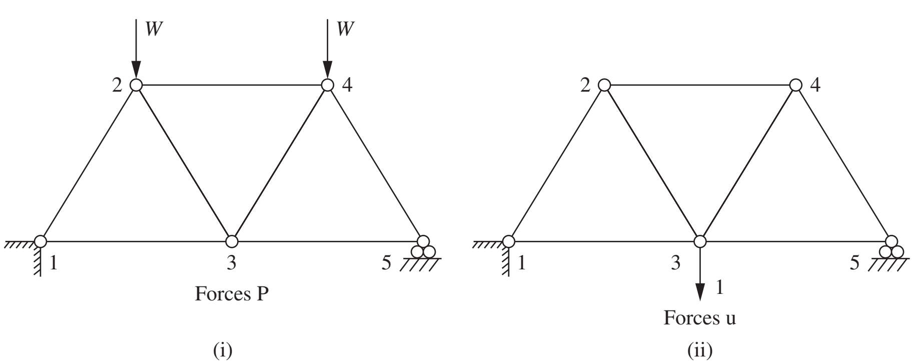


Figure 4.13

while each load moves a distance y in its line of action. The loading produces an internal force P and an extension δl in any member of the frame. The external work done during the application of the loads must equal the internal energy stored in the structure, from the principle of conservation of energy. Then:

$$\sum Wy/2 = \sum P\,\delta l/2 \;\ldots \quad (1)$$

To the unloaded structure a unit virtual load is applied at node 3 in the direction of δ, as shown in Figure 4.13 (ii). This results in a force u in any member.

Now, while the virtual load is still in position, the real loads W are gradually applied to the structure. Again equating external work and internal energy:

$$\sum Wy/2 + 1 \times \delta = \sum P\,\delta l/2 + \sum u/\delta l \;\ldots \quad (2)$$

Subtracting expression (1) from expression (2):

$$\begin{aligned} 1 \times \delta &= \sum u\,\delta l \\ &= \sum Pul/AE \end{aligned}$$

where P is the internal force in a member due to the applied loads and l, A and E are its length, area, and modulus of elasticity, and u is the internal force in a member due to the unit virtual load.

For a pin-jointed frame, tensile forces are considered positive and compressive forces negative. Increase in the length of a member is considered positive and decrease in length negative. The unit virtual load is applied to the frame in the anticipated direction of the deflection. If the assumed direction is correct, the deflection obtained will have a positive value. The deflection obtained will be negative when the unit virtual load has been applied in the opposite direction to the actual deflection.

Example 4.6

Determine the vertical deflection of node 5 for the pin-jointed frame shown in Figure 4.14. All members of the frame have a constant value for AE of 100,000 kips.

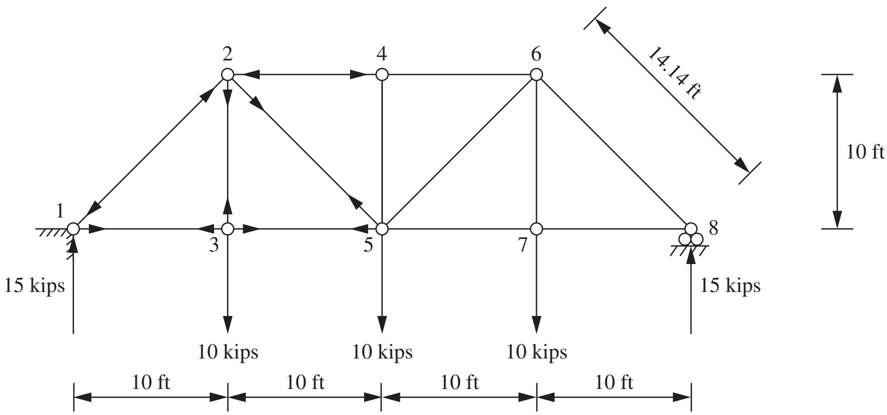


Figure 4.14

Solution

Because of the symmetry in the structure and loading, only half of the members need be considered. The member forces *P* due to the applied loads and ***u*** due to the unit virtual load are tabulated in Table 4.1.

Table 4.1 Determination of forces and displacements in Example 4.6

Member	P	l	u	Pul
12	−21.21	14.14	−0.707	212
23	10.00	10.00	0	0
13	15.00	10.00	0.500	75
45	0.00	10.00	0	0
24	−20.00	10.00	−1.000	200
25	7.07	14.14	0.707	71
35	15.00	10.00	0.500	75
Total				633

The vertical deflection is given by:

$$\begin{aligned}\delta_5 &= \sum Pul/AE \\ &= 2 \times 633 \times 12/100{,}000 \\ &= 0.15 \text{ in downward}\end{aligned}$$

Supplementary problems

S4.1 Determine the rotations at nodes 1 and 2 and the deflection at node 3 of the uniform beam shown in Figure S4.1.

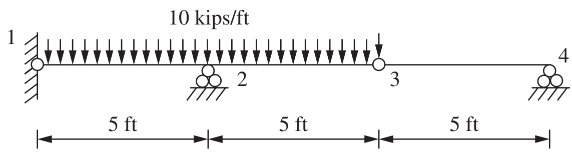


Figure S4.1

S4.2 Determine the deflection at node 2 of the uniform beam shown in Figure S4.2.

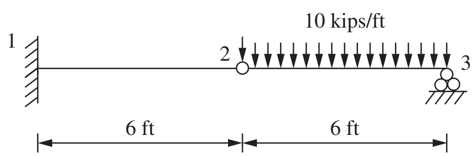


Figure S4.2

S4.3 Determine the rotation at node 1 and the deflection at node 2 of the uniform beam shown in Figure S4.3.

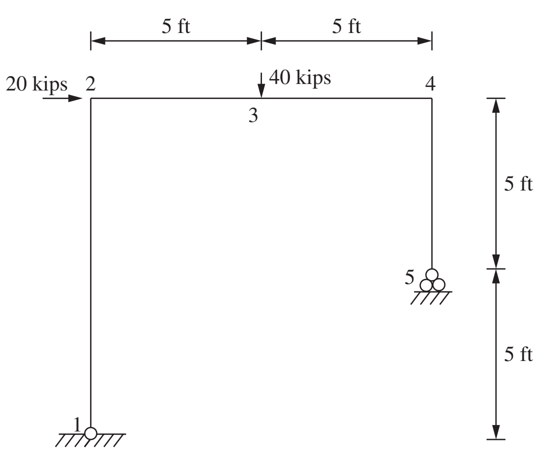


Figure S4.3

S4.4 Determine the equation of the elastic curve for the uniform beam shown in Figure S4.4. Determine the location of the maximum deflection in span 12 and the magnitude of the maximum deflection. Determine the deflection at node 3.

Figure S4.4

S4.5 Determine the deflection at node 3 of the uniform beam shown in Figure S4.5.

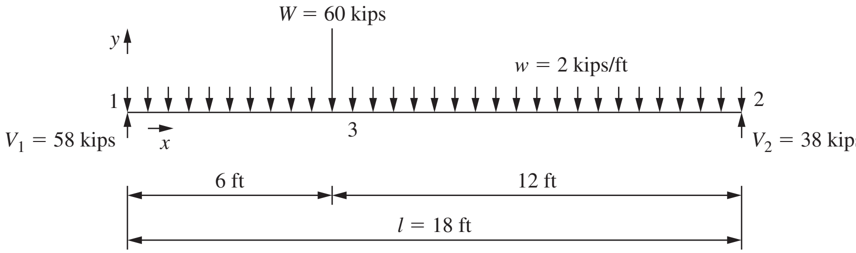


Figure S4.5

S4.6 Determine the deflection at node 3 of the pin-jointed truss shown in Figure S4.6.

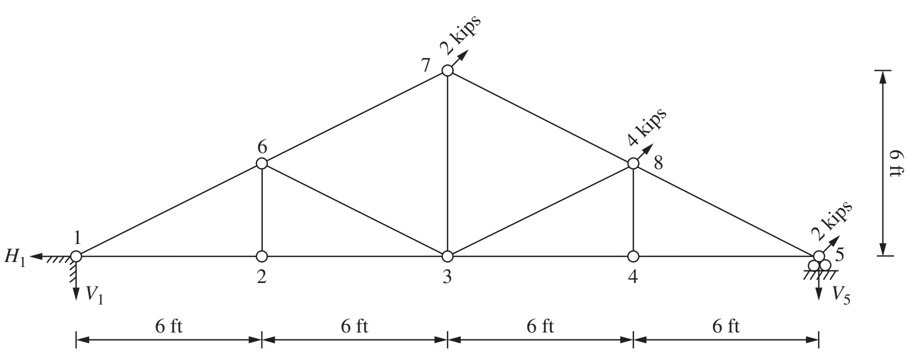


Figure S4.6

S4.7 Determine the deflection at node 4 of the pin-jointed truss shown in Figure S4.7.

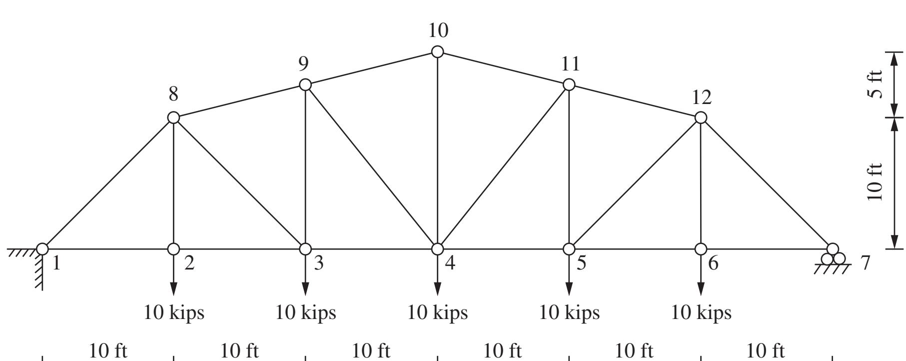


Figure S4.7

S4.8 Determine the deflection at node 2 of the pin-jointed truss shown in Figure S4.8.

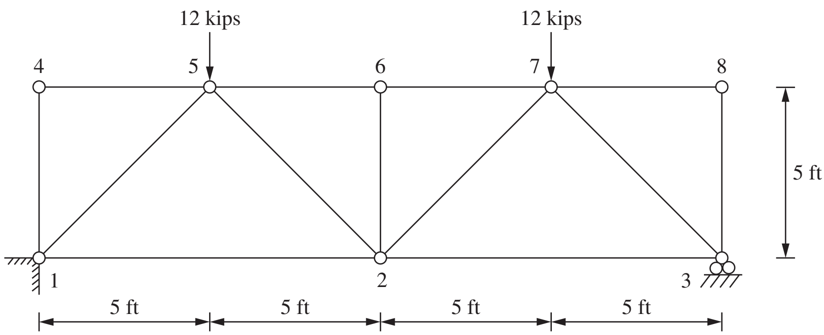


Figure S4.8

S4.9 Determine the deflection at node 3 of the pin-jointed truss shown in Figure S4.9.

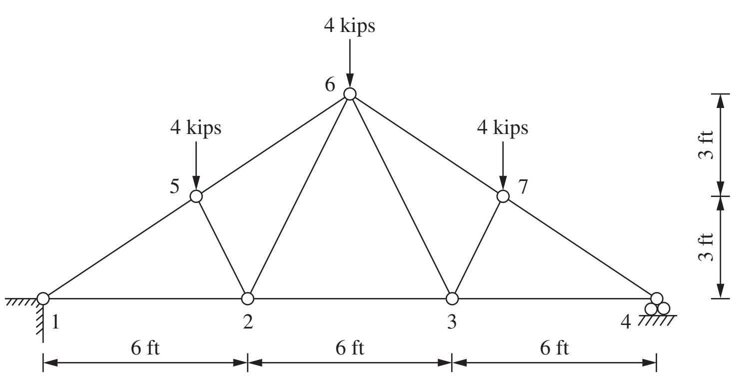


Figure S4.9

S4.10 Determine the deflection at node 2 of the pin-jointed truss shown in Figure S4.10.

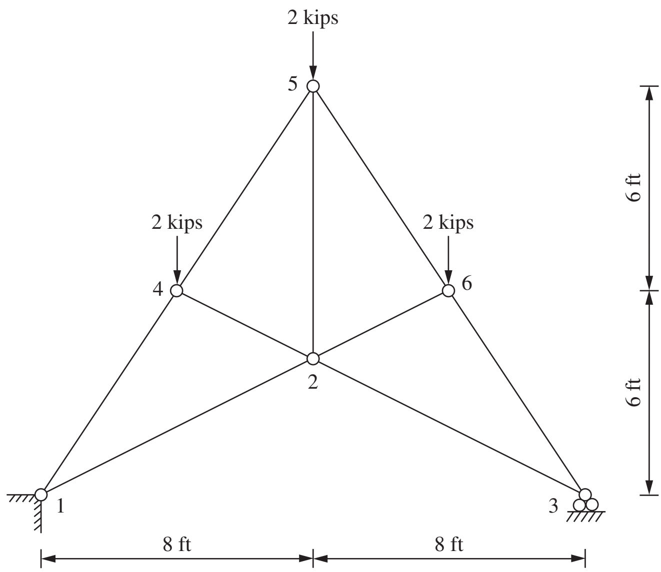


Figure S4.10

5 Influence lines

Notation

A cross-sectional area of a member
E modulus of elasticity
I second moment of area of a member
l length of a member
M bending moment in a member due to the applied loads
P axial force in a member due to the applied loads
Q shear force in a member due to the applied loads
V vertical reaction
W applied load
δ deflection
θ rotation

5.1 Introduction

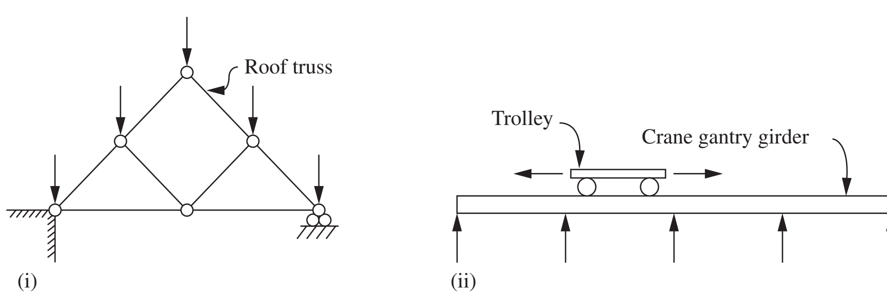


Figure 5.1

The maximum design force in a member of a structure subjected to a system of stationary loads is readily determined. The static loads are applied to the structure as shown in Figure 5.1 (i) and the member forces calculated. However, a member such as a bridge girder or a crane gantry girder are subjected to moving loads, and the maximum design force in the member depends on the location of the moving load. As shown in Figure 5.1 (ii), this involves the trial and error positioning of the crane loads on the girder to determine the most critical location. Alternatively, an influence line may be utilized to determine the location of the moving load that produces the maximum design force in a member.

5.2 Construction of influence lines

An influence line for a member is a graph representing the variation in shear, moment or force in the member due to a unit load traversing a structure. The construction of an influence line may be obtained by the application of Müller-Breslau's principle.

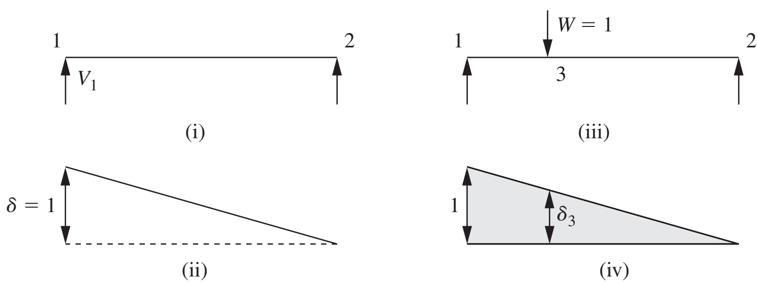


Figure 5.2

In accordance with Müller-Breslau's principle, the influence line for any constraint in a structure is the elastic curve produced by the corresponding unit virtual displacement applied at the point of application of the restraint. The term "displacement" is used in its general sense, and the displacement corresponding to a moment is a rotation and to a force is a linear deflection. The displacement is applied in the same direction as the restraint. To obtain the influence line for the support reaction at end 1 of the simply supported beam shown in Figure 5.2 (i), a unit virtual displacement is applied in the line of action of V_1. This results in the elastic curve shown at (ii). A unit load is applied to the beam at any point 3, as shown at (iii), and the unit displacement applied to end 1. The displacement produced at point 3 is δ_3, as shown at (iv). Then, applying the virtual work principle:

$$V_1 \times (\delta = 1) = (W = 1) \times \delta_3$$
$$V_1 = \delta_3$$

and the elastic curve is the influence line for V_1.

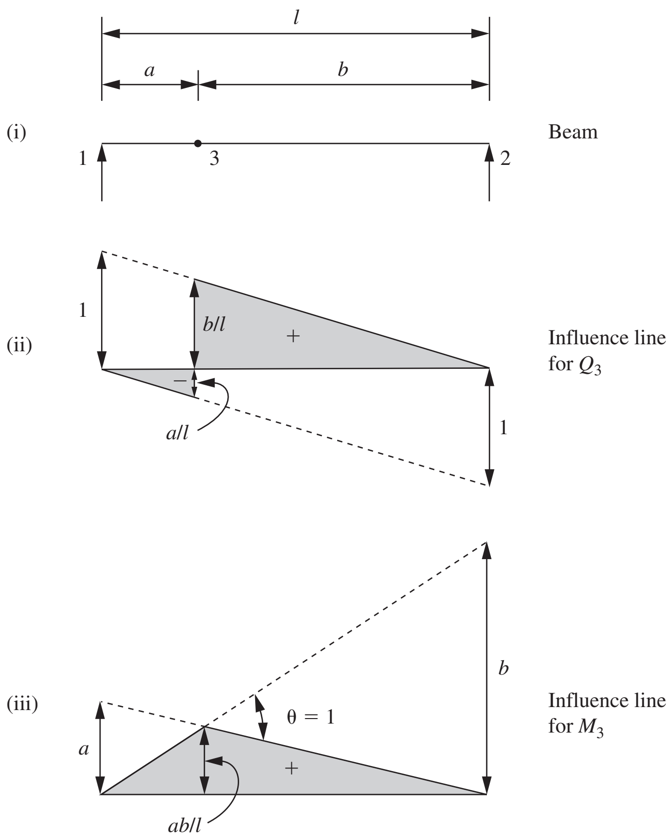


Figure 5.3

Similarly, as shown in Figure 5.3 (ii), the influence line for shear at point 3 is obtained by cutting the beam at 3 and displacing the cut ends a unit distance apart. The influence line for bending moment at 3 is produced by cutting the beam at 3 and imposing a unit virtual rotation, as shown at (iii).

5.3 Maximum effects

(a) Single concentrated load

A concentrated load W produces the maximum positive shear at point 3 in the beam shown in Figure 5.3 (i) when the load is located just to the right of 3. The shear is given by:

$$\begin{aligned} Q_{\max} &= W \times \text{influence line ordinate to the right of point 3} \\ &= Wb/l \end{aligned}$$

As shown in Figure 5.3 (ii), the maximum negative shear at point 3 occurs when the load is located just to the left of 3 and is given by:

$$\begin{aligned} Q_{min} &= W \times \text{influence line ordinate to the left of point 3} \\ &= Wa/l \end{aligned}$$

As shown in Figure 5.3 (iii), the maximum moment at point 3 occurs when the load is located at 3 and is given by:

$$\begin{aligned} M_{max} &= W \times \text{influence line ordinate at point 3} \\ &= Wab/l \end{aligned}$$

Example 5.1

Determine the maximum shear force and maximum moment at point 3 in the simply supported beam shown in Figure 5.4 due to a concentrated load of 10 kips.

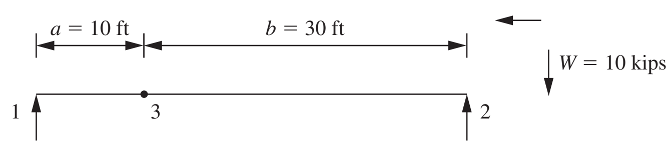


Figure 5.4

Solution

The influence line for shear is shown in Figure 5.3 (ii), and the maximum shear at point 3 occurs when the 10 kip load is immediately to the right of 3. The maximum shear is;

$$\begin{aligned} Q &= Wb/l \\ &= 10 \times 30/40 \\ &= 7.5 \text{ kips} \end{aligned}$$

The influence line for moment is shown in Figure 5.3 (iii), and the maximum moment at point 3 occurs when the 10 kip load is at 3. The maximum moment is;

$$\begin{aligned} M &= Wab/l \\ &= 10 \times 10 \times 30/40 \\ &= 75 \text{ kip-ft} \end{aligned}$$

(b) Uniformly distributed load

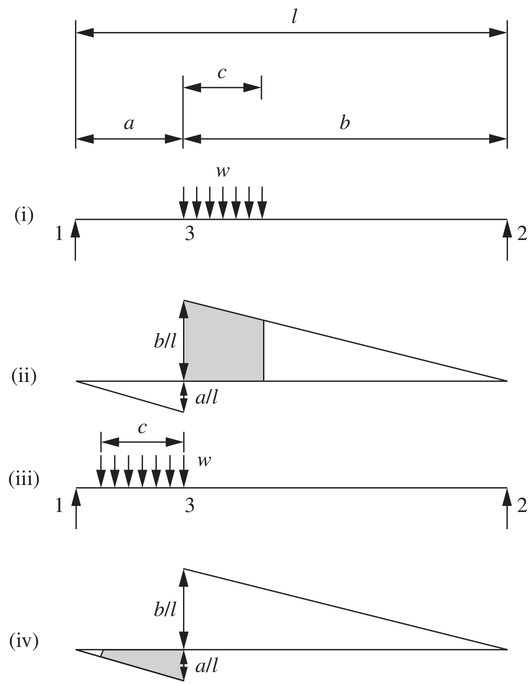


Figure 5.5

A uniformly distributed load w of length c is applied to the beam, as shown in Figure 5.5(i). This produces the maximum positive shear at point 3 when the distributed load is located just to the right of 3. As shown in (ii), the shear is given by;

$$Q_{max} = w \times \text{area under the influence line}$$

As shown at (iii) and (iv), the maximum negative shear at point 3 is given by;

$$Q_{min} = w \times \text{area under the influence line}$$

Example 5.2

Determine the maximum shear force at point 3 in the simply supported beam shown in Figure 5.6 due to a distributed load of 2 kips/ft over a length of 10 ft.

Solution

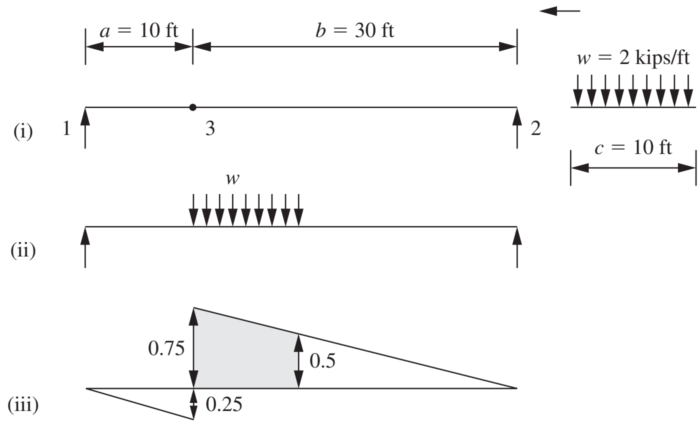


Figure 5.6

The influence line for shear is shown in Figure 5.6 (iii), and the maximum shear at point 3 occurs when the distributed load is immediately to the right of 3, as shown at (ii). The maximum shear is;

$$
\begin{aligned}
Q_{\max} &= w \times \text{area under the influence line} \\
&= 2 \times 10(0.75 + 0.5)/2 \\
&= 12.5 \text{ kips}
\end{aligned}
$$

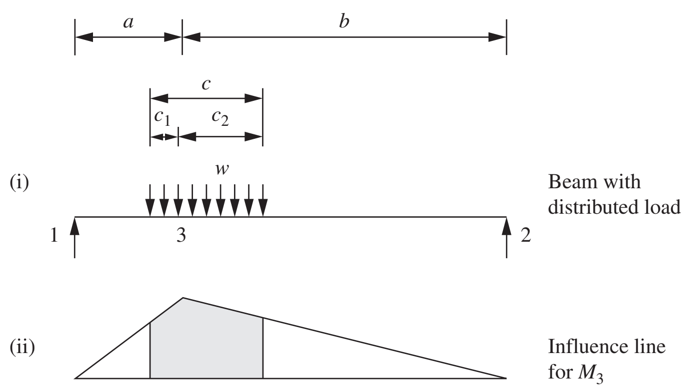


Figure 5.7

The maximum moment at point 3 is produced when point 3 divides the distributed load in the same ratio as it divides the span. As shown in Figure 5.7 (i), this occurs when:

$$c_1/c_2 = a/b$$

As shown at (ii), the maximum moment at point 3 is given by:

$$M_{max} = w \times \text{area under the influence line}$$

Example 5.3

Determine the maximum bending moment at point 3 in the simply supported beam shown in Figure 5.8 due to a distributed load of 2 kips/ft over a length of 10 ft.

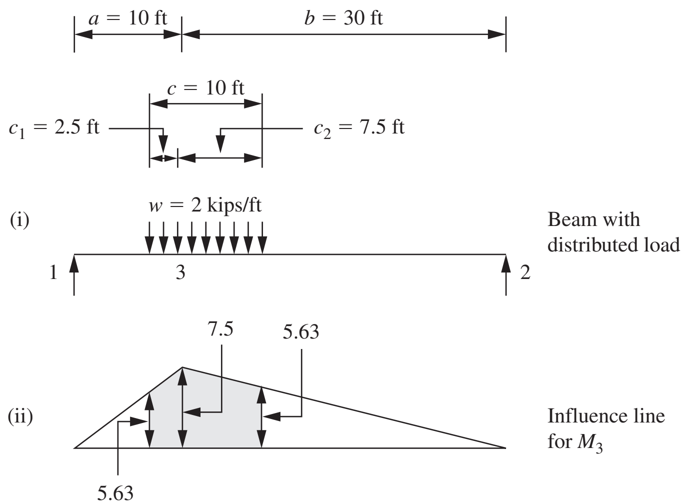


Figure 5.8

Solution

The maximum moment at point 3 is produced when point 3 divides the distributed load in the same ratio as it divides the span. As shown in Figure 5.8 (i), this occurs when:

$$\begin{aligned} c_1/c_2 &= a/b \\ &= 10/30 \end{aligned}$$

and: $c_1 = 2.5$ ft
$c_2 = 7.5$ ft

As shown at (ii), the maximum moment at point 3 is given by:

$$\begin{aligned} M_{max} &= w \times \text{area under the influence line} \\ &= 2[2.5(5.63 + 7.5)/2 + 7.5(5.63 + 7.5)/2] \\ &= 2 \times 10(5.63 + 7.5)/2 \\ &= 131 \text{ kip-ft} \end{aligned}$$

(c) Train of wheel loads

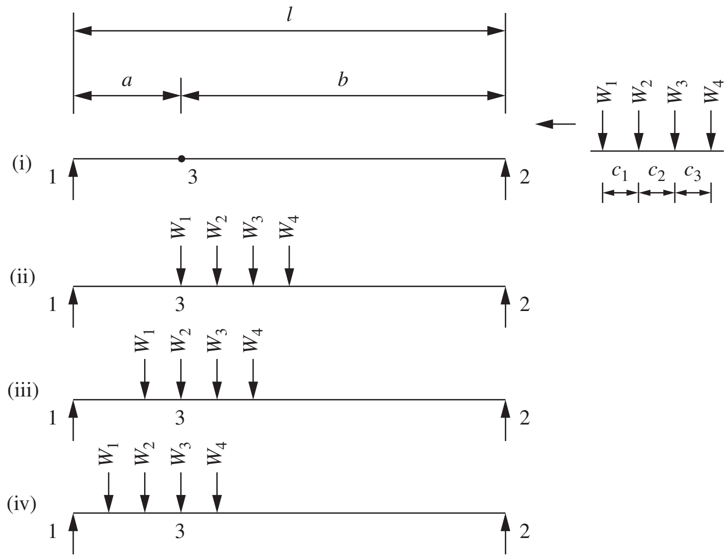


Figure 5.9

A train of wheel loads is applied to the beam, as shown in Figure 5.9 (i). This produces the maximum positive shear at point 3 when the first load is located just to the right of 3, as shown in (ii), provided that:

$$W_1/c_1 > \sum W/l$$

where: $\sum W$ = total weight of the wheel loads
l = span length

The maximum positive shear at point 3 occurs when the second load is located just to the right of 3, as shown in (iii), provided that:

$$W_2/c_2 > \sum W/l$$

The maximum positive shear at point 3 occurs when the third load is located just to the right of 3, as shown in (iv), provided that:

$$W_3/c_3 > \sum W/l$$

and so on.

Example 5.4

Determine the maximum shear at point 3 in the simply supported beam shown in Figure 5.10 due to the train of wheel loads indicated.

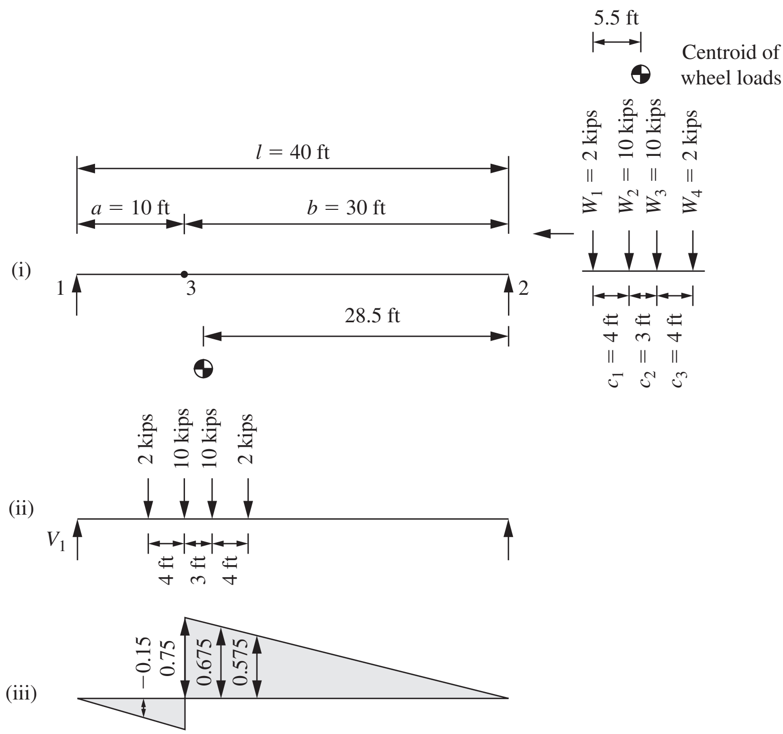


Figure 5.10

Solution

The ratio of total weight of wheel loads to span length is:

$$\begin{aligned} \sum W/l &= 24/40 \\ &= 0.6 \text{ kips/ft} \end{aligned}$$

Placing the first wheel at point 3 gives:

$$\begin{aligned} W_1/c_1 &= 2/4 \\ &= 0.5 \cdots < 0.6 \end{aligned}$$

Placing the second wheel at point 3, as shown in (ii), gives:

$$\begin{aligned} W_2/c_2 &= 10/3 \\ &= 3.3 \cdots > 0.6\text{, governs} \end{aligned}$$

Hence, the maximum shear occurs at point 3 when the second load is located immediately to the right of 3; the ordinates of the influence line diagram at the positions of the wheel loads is shown at (iii). The shear at 3 is given by:

$$\begin{aligned} Q_3 &= -2 \times 0.15 + 10 \times 0.75 + 10 \times 0.675 + 2 \times 0.575 \\ &= 15.1 \text{ kips} \end{aligned}$$

Alternatively, the shear at 3 may be determined by resolving forces vertically. The support reaction at end 1 is:

$$\begin{aligned} V_1 &= 24 \times 28.5/40 \\ &= 17.1 \text{ kips} \end{aligned}$$

The shear at 3 is given by:

$$\begin{aligned} Q_3 &= 17.1 - 2 \\ &= 15.1 \text{ kips} \end{aligned}$$

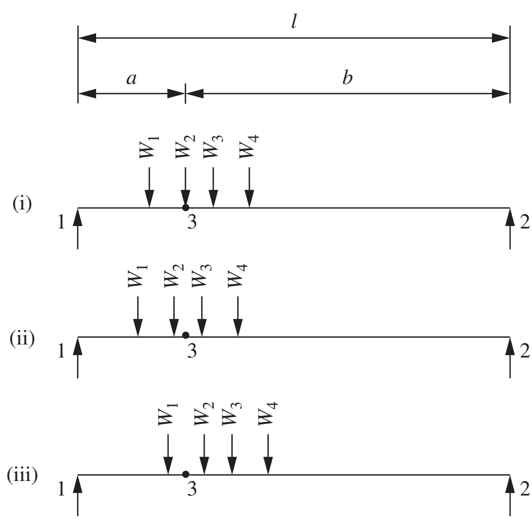


Figure 5.11

A train of wheel loads is applied to the beam, as shown in Figure 5.11 (i). This produces the maximum bending moment at a specific point 3 when a specified load is located at 3, as shown in (ii), such that if the load is moved to the left of 3, the intensity of loading on section 13 is greater than on section 23, but if it moves to the right of point 3, the intensity of loading on section 23 is greater than on section 13. The first requirement is shown at (ii) and is:

$$\sum W_L/a > \sum W_R/b$$

where: $\sum W_L = W_1 + W_2$
$\sum W_R = W_3 + W_4$

The second requirement is shown at (iii) and is:

$$\sum W_R/b > \sum W_L/a$$

where: $\sum W_L = W_1$
$\sum W_R = W_2 + W_3 + W_4$

Example 5.5

Determine the maximum bending moment at point 3 in the simply supported beam shown in Figure 5.12 due to the train of wheel loads indicated.

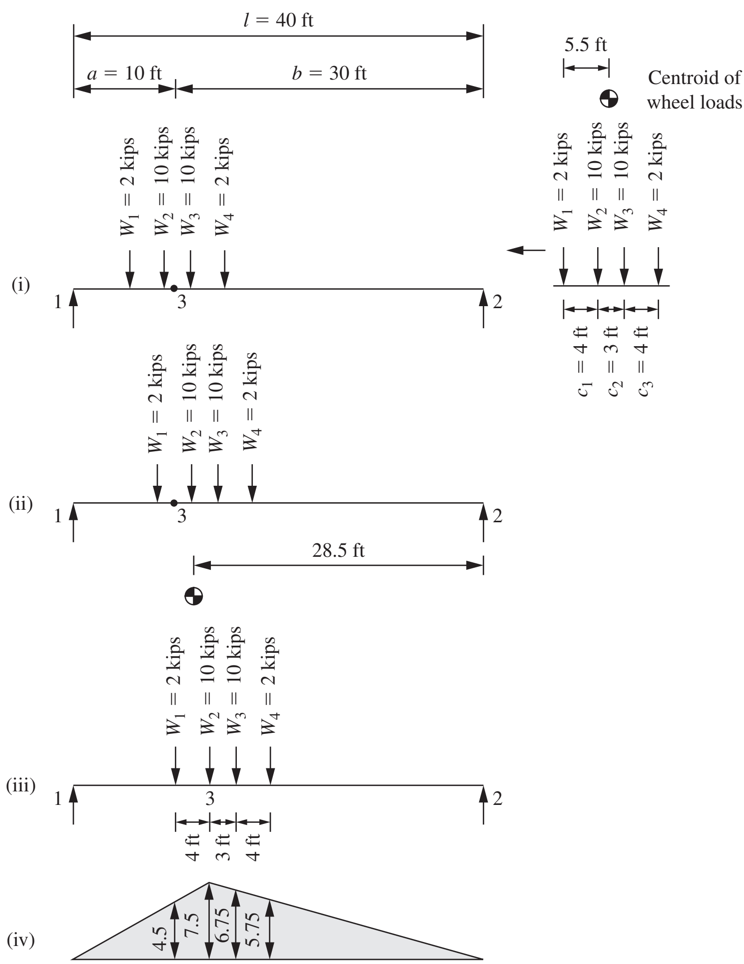


Figure 5.12

Solution

Placing the second wheel on the left of point 3, as shown at (i), gives an intensity of loading on section 23 of:

$$\begin{aligned}\sum W_R/b &= (W_3 + W_4)/b \\ &= 12/30 \\ &= 0.4\end{aligned}$$

and an intensity of loading on section 13 of:

$$\begin{aligned}\sum W_L/a &= (W_1 + W_2)/a \\ &= 12/10 \\ &= 1.2 \\ &> 0.4\end{aligned}$$

Placing the second wheel on the right of point 3, as shown at (ii), gives an intensity of loading on section 23 of:

$$\begin{aligned}\sum W_R/b &= (W_2 + W_3 + W_4)/b \\ &= 22/30 \\ &= 0.733\end{aligned}$$

and an intensity of loading on section 13 of:

$$\begin{aligned}\sum W_L/a &= (W_1)/a \\ &= 2/10 \\ &= 0.2 \\ &< 0.733\end{aligned}$$

Hence the maximum bending moment occurs at point 3 when the second wheel load is located at point 3, as shown at (iii). The influence line for bending moment at 3 is shown at (iv), and the maximum bending moment at 3 is given by:

$$\begin{aligned}M_3 &= 2 \times 4.5 + 10 \times 7.5 + 10 \times 6.75 + 2 \times 5.75 \\ &= 163 \text{ kip-ft}\end{aligned}$$

Alternatively, the bending moment at 3 may be determined by resolving forces vertically. The support reaction at end 1 is:

$$\begin{aligned}V_1 &= 24 \times 28.5/40 \\ &= 17.1 \text{ kips}\end{aligned}$$

The moment at 3 is given by:

$$\begin{aligned} M_3 &= 17.1 \times 10 - 2 \times 4 \\ &= 163 \text{ kip-ft} \end{aligned}$$

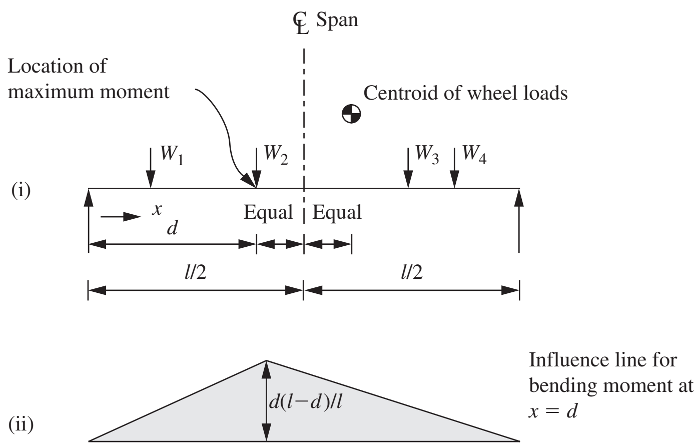


Figure 5.13

The maximum bending moment produced by a train of wheel loads in a simply supported beam always occurs under one of the wheels and does not necessarily occur at midspan. As shown in Figure 5.13 (i), a train of wheel loads produces the maximum possible bending moment in a simply supported beam under one of the wheels when the center of the span bisects the distance between this wheel and the centroid of the train. The maximum moment usually occurs under one of the wheels adjacent to the centroid of the train. The influence line for bending moment at the location of the maximum moment is shown at (ii).

Example 5.6

Determine the maximum possible bending moment in the simply supported beam shown in Figure 5.14 due to the train of wheel loads indicated.

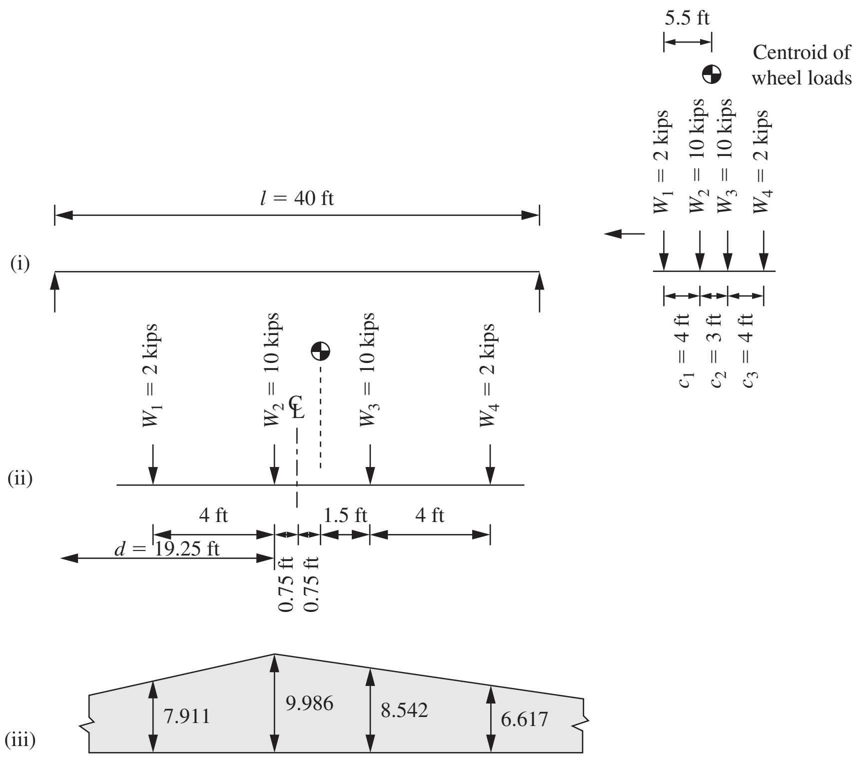


Figure 5.14

Solution

Placing the train of wheel loads as indicated in Figure 5.14 (ii) produces the maximum bending moment under the second wheel load. Hence, the maximum bending moment occurs under the second wheel load when it is located at a distance of 0.75 ft left of midspan, as shown at (ii). The influence line for bending moment at the location of the second wheel load is shown at (iii), and the maximum bending moment at this location is given by:

$$\begin{aligned} M_{\max} &= 2 \times 7.911 + 10 \times 9.986 + 10 \times 8.542 + 2 \times 6.617 \\ &= 214.336 \text{ kip-ft} \end{aligned}$$

Alternatively, the bending moment may be determined by resolving forces vertically. The support reaction at end 1 is:

$$\begin{aligned} V_1 &= 24 \times 19.25/40 \\ &= 11.55 \text{ kips} \end{aligned}$$

The maximum moment is given by:

$$\begin{aligned} M_{max} &= 11.55 \times 19.25 - 2 \times 4 \\ &= 214.338 \text{ kip-ft} \end{aligned}$$

Because of symmetry in the train of wheel loads, this moment also occurs under the third wheel load when it is located at a distance of 0.75 ft right of midspan.

When the second or third wheel load is located at midspan, the moment at midspan is:

$$M = 214.00 \text{ kip-ft}$$

and this is the maximum bending moment that occurs at midspan.

(d) Envelope of maximum effects

To design a member, it is necessary to determine the maximum bending moment and shear force that can occur at all sections of the member. Diagrams indicating maximum values are known as envelope diagrams and are determined using influence lines at selected points along the member.

Example 5.7

Construct the maximum possible bending moment envelope in the simply supported beam shown in Figure 5.15 due to a concentrated load of 100 kips.

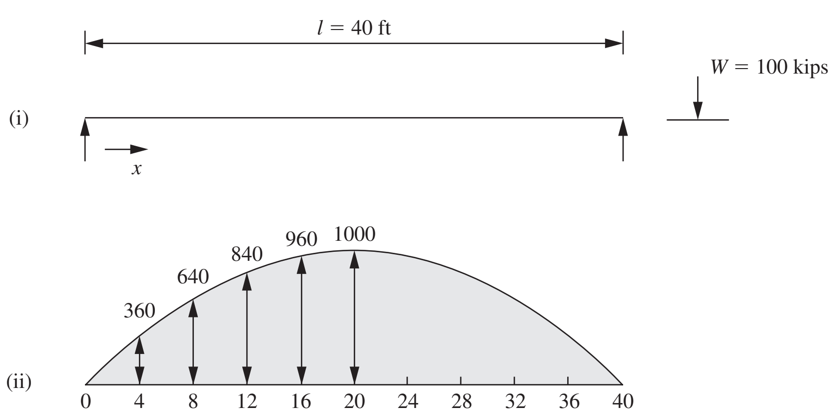


Figure 5.15

Table 5.1

x ft	Influence line ordinate	M_{max}
4	$4 \times 36/40 = 3.6$	360
8	$8 \times 32/40 = 6.4$	640
12	$12 \times 28/40 = 8.4$	840
16	$16 \times 24/40 = 9.6$	960
20	$20 \times 20/40 = 10.0$	1000

Solution

The maximum bending moment occurs at any section when the concentrated load is located at the section. Values of the moment are calculated in Table 5.1, and the bending moment envelope is shown in Figure 5.15 (ii).

5.4 Pin-jointed truss

(a) Stringers and cross beams

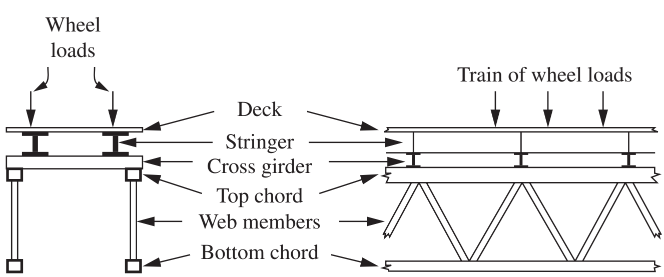


Figure 5.16

As moving loads traverse a pin-jointed truss, the loads are transferred to the truss panel points by a system of stringers and cross beams. This is shown in Figure 5.16 for a deck bridge with the loads applied to the top chord of the truss. The moving load is transferred from one panel point to the next as the load moves across the stringer. Hence, the influence line for axial force in a member is completed by connecting the influence line ordinates at the panel points on either side of a panel with a straight line.

(b) Influence lines for a Warren truss

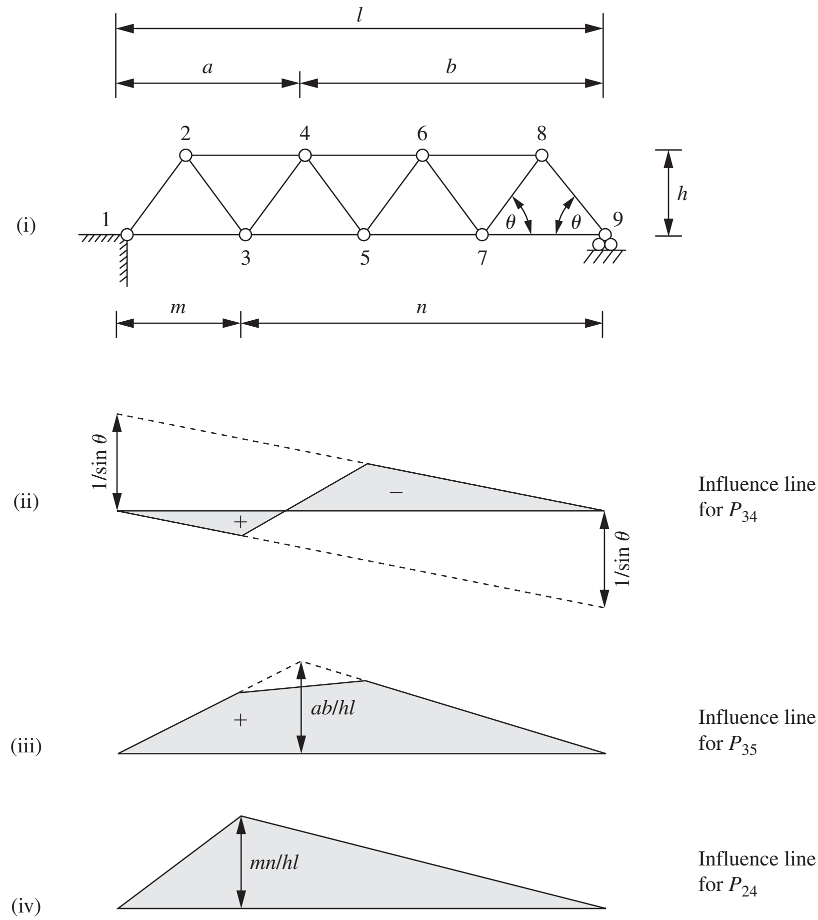


Figure 5.17

Figure 5.17 (i) shows a Warren truss with the loads from the stringers applied to the bottom panel points. The influence lines for axial force in members 34, 35, and 24 are obtained by taking a section through these three members and considering the relevant free body diagrams.

The influence line for axial force in web member 34 is obtained by multiplying the influence line for shear force in panel 34 by $1/\sin\theta$ and is shown at (ii). Positive sense of the influence line indicates tension in member 34. Because of the effect of the stringers, the influence line between nodes 3 and 5 is obtained by connecting the ordinates at 3 and 5 with a straight line.

The influence line for axial force in bottom chord member 35 is obtained by multiplying the influence line for moment at node 4 by $1/h$ and is shown at (iii). The influence line is positive, indicating tension in member 35.

The influence line for axial force in top chord member 24 is obtained by multiplying the influence line for moment at node 3 by $1/h$.

(c) Influence lines for a Pratt truss

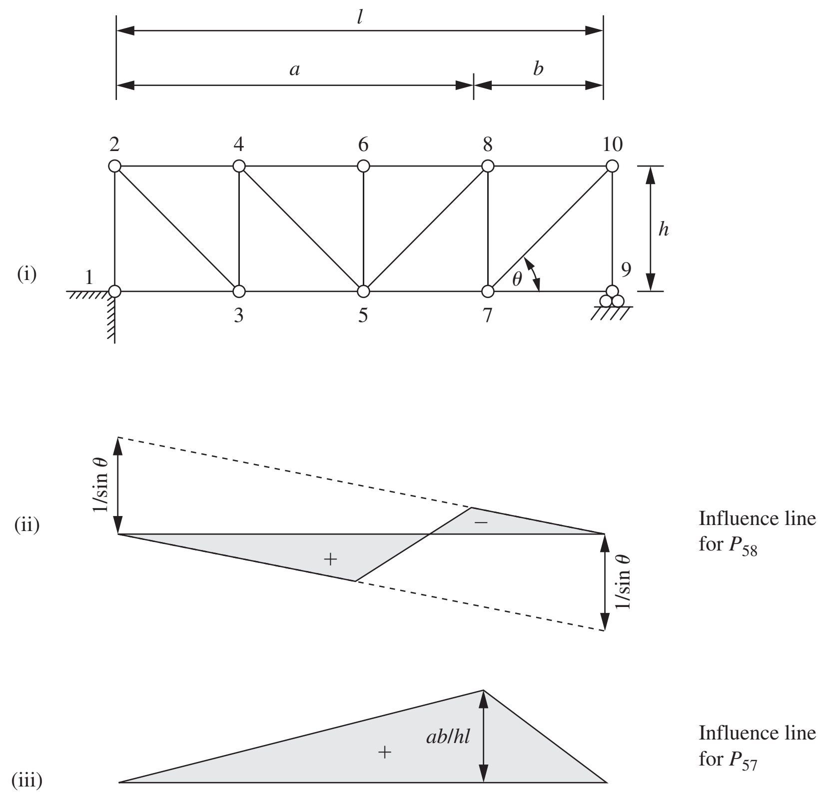


Figure 5.18

Figure 5.18 (i) shows a Pratt truss with the loads from the stringers applied to the bottom panel points. The influence line for axial force in web member 58 is obtained by multiplying the influence line for shear force in panel 58 by $1/\sin\theta$ and is shown at (ii). Positive sense of the influence line indicates tension in member 58. The influence line for axial force in member 78 is identical in shape and of opposite sign.

The influence line for axial force in bottom chord member 57 is obtained by multiplying the influence line for moment at node 8 by $1/h$ and is shown at (iii). The influence line is positive, indicating tension in member 57.

The influence line for axial force in top chord member 810 is obtained by multiplying the influence line for moment at node 7 by $1/h$. This influence line is identical in shape to the influence line for P_{57} and of opposite sign.

(d) Influence lines for a bowstring truss

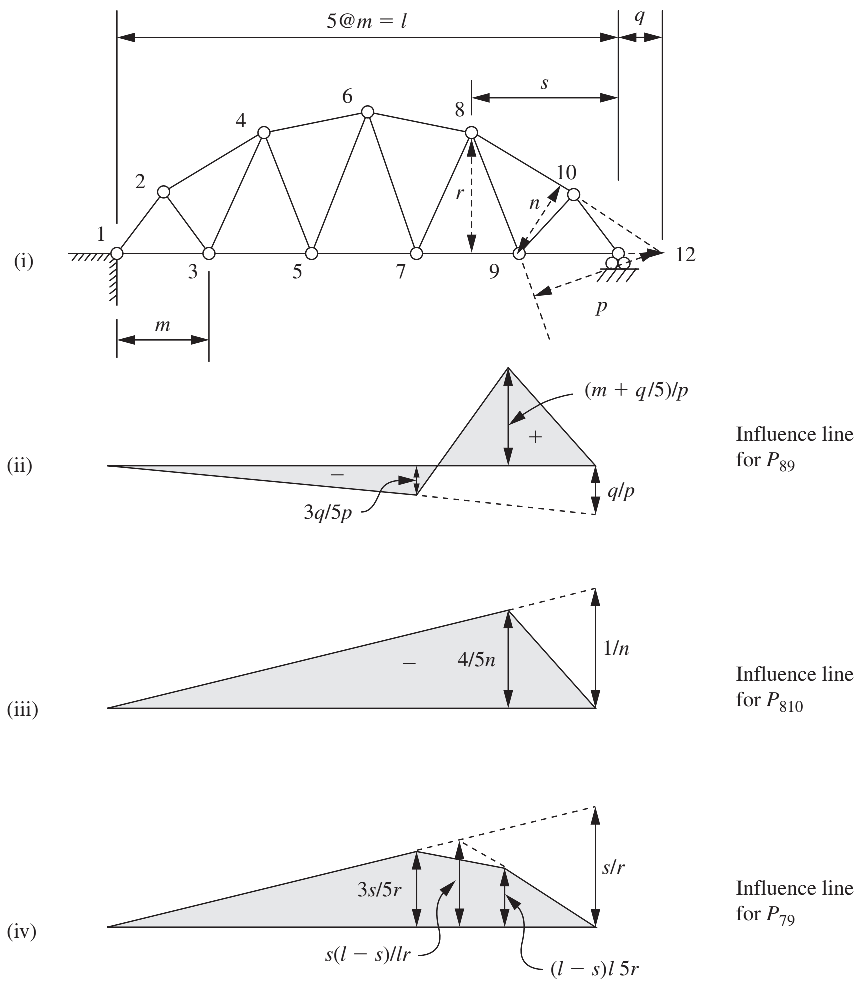


Figure 5.19

Figure 5.19 (i) shows a bowstring truss with the loads from the stringers applied to the bottom panel points. The influence lines for axial force in members 89, 810, and 79 are obtained by taking a section through these three members and considering the relevant free body diagrams.

The influence line for axial force in web member 89 is obtained by multiplying the influence line for moment at point 12 by $1/p$, where p is the perpendicular from point 12 to the line of action of member 89. The influence line for P_{89} is shown at (ii).

The influence line for axial force in top chord member 810 is obtained by multiplying the influence line for moment at node 9 by $1/n$, where n is the perpendicular from node 9 to member 810. The influence line for P_{810} is shown at (iii).

The influence line for axial force in bottom chord member 79 is obtained by multiplying the influence line for moment at node 8 by $1/r$, where r is the perpendicular from node 8 to member 79. The influence line for P_{79} is shown at (iv).

5.5 Three-hinged arch

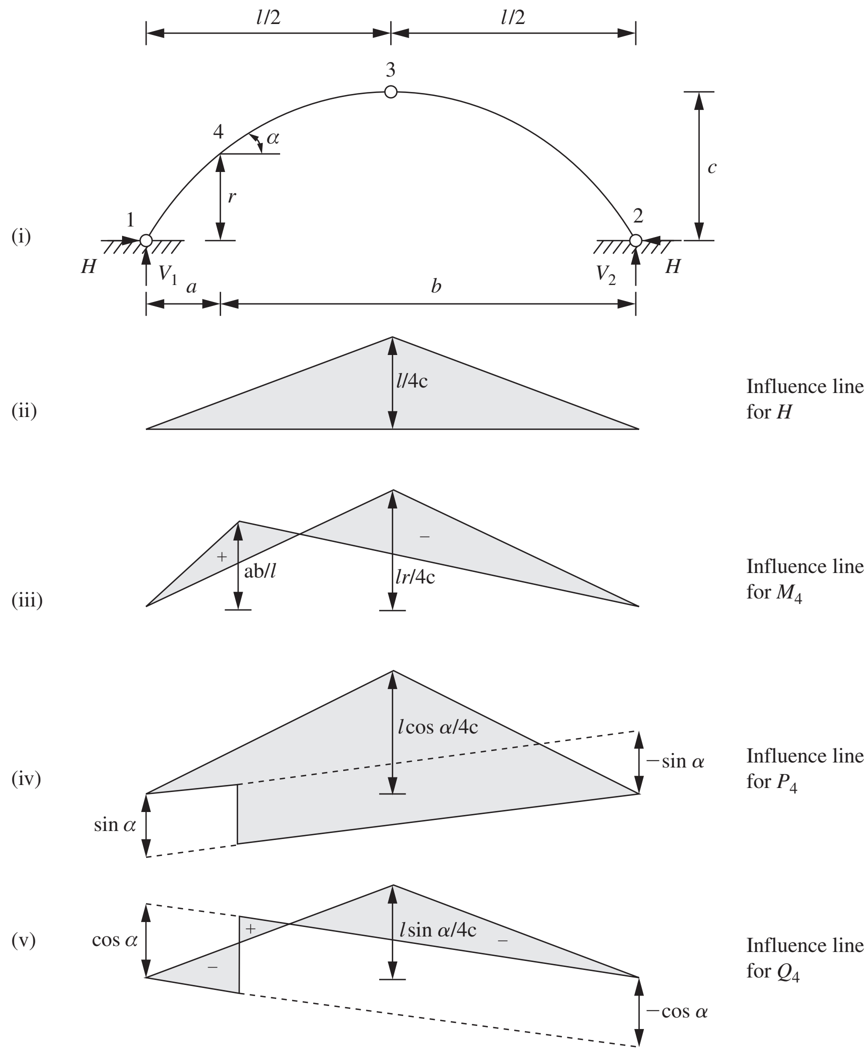


Figure 5.20

The horizontal thrust at the springings of a three-hinged arch is equal to the bending moment at the center of a simply supported beam of the same length multiplied by $1/c$. Hence, the influence line for horizontal thrust is the influence line for the free bending moment multiplied by $1/c$ and is shown in Figure 5.20 (ii).

The influence line for bending moment at point 4 is the influence line for free bending moment at 4 minus the horizontal thrust multiplied by r and is given by:

$$M_4 = (M_s)_4 - Hr$$

The influence line is shown at (iii).
The influence line for thrust at point 4 is given by:

$$\begin{aligned} P_4 &= H\cos\alpha - V_2 \sin\alpha \ldots \text{ unit load from 1 to 4} \\ &= H\cos\alpha + V_2 \sin\alpha \ldots \text{ unit load from 4 to 2} \end{aligned}$$

The influence line is shown at (iv).
The influence line for shear at point 4 is given by:

$$\begin{aligned} Q_4 &= -H\sin\alpha - V_2 \cos\alpha \ldots \text{ unit load from 1 to 4} \\ &= -H\sin\alpha + V_1 \cos\alpha \ldots \text{ unit load from 4 to 2} \end{aligned}$$

The influence line is shown at (v).

Supplementary problems

S5.1 Construct the influence lines for V_2 and M_2 for the beam shown in Figure S5.1. Determine the maximum value of M_2 due to a distributed load of 2 kips/ft over a length of 60 ft.

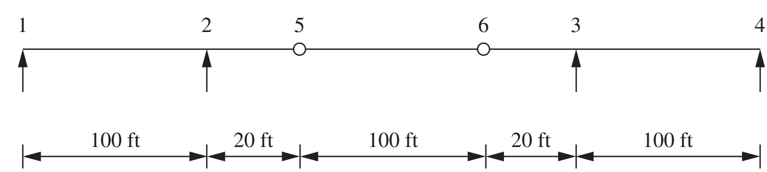


Figure S5.1

S5.2 Construct the influence line for V_2 for the beam shown in Figure S5.2. Determine the maximum value of V_2 due to distributed load of 10 kips/ft over a length of 6 ft.

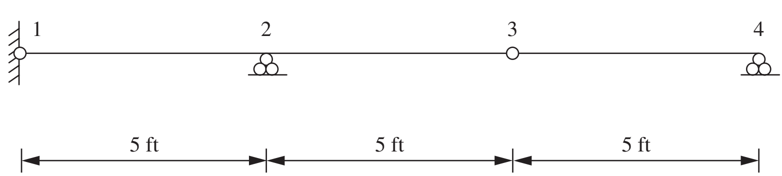


Figure S5.2

S5.3 Construct the influence line for V_4 for the beam shown in Figure S5.3. Determine the maximum value of V_4 due to a train of three wheel loads each of 3 kips at 2 ft on center.

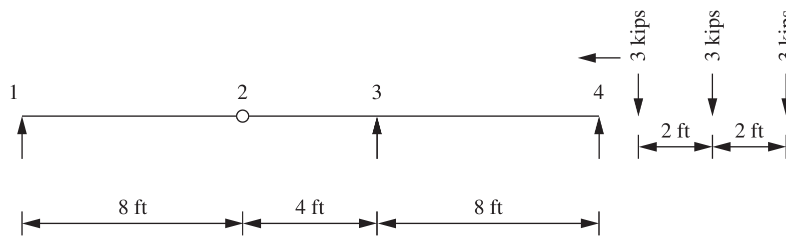


Figure S5.3

S5.4 Figure S5.4 shows a truss with the loads from the stringers applied to the bottom panel points. Construct the influence line for axial force in member 24. Determine the maximum value of P_{24} due to a concentrated load of 5 kips.

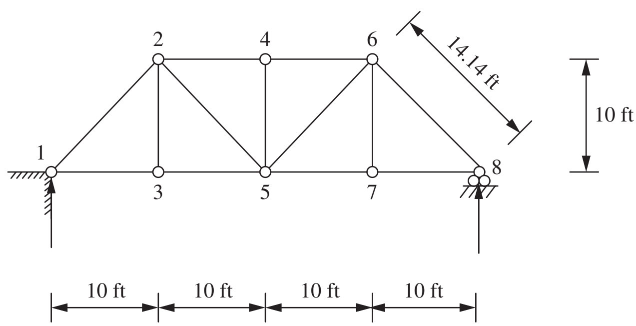


Figure S5.4

S5.5 Figure S5.5 shows a truss with the loads from the stringers applied to the bottom panel points. Construct the influence line for axial force in member 45. Determine the maximum value of P_{45} due to concentrated load of 10 kips.

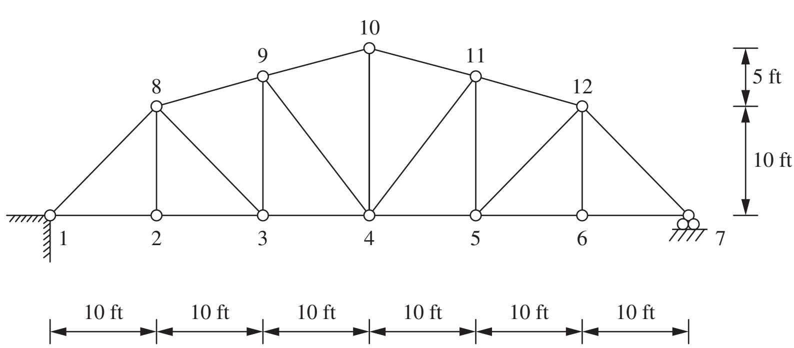


Figure S5.5

S5.6 Figure S5.6 shows a truss with the loads from the stringers applied to the bottom panel points. Construct the influence line for axial force in member 1718. Determine the maximum value of P_{1718} due to concentrated load of 20 kips.

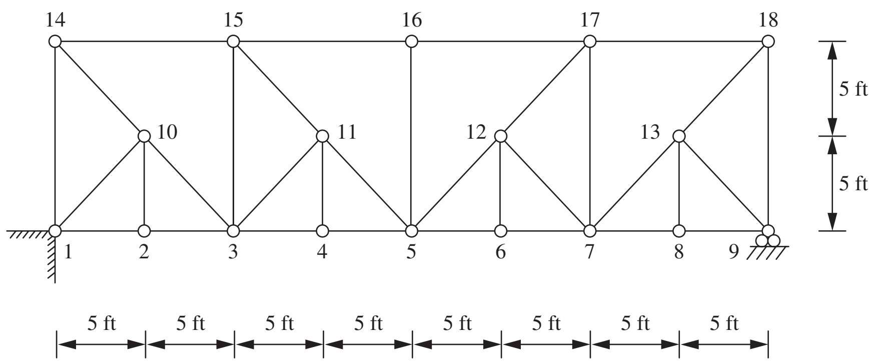


Figure S5.6

6 Space frames

Notation

F	force
F_x	force component along the x-axis
F_y	force component along the y-axis
F_z	force component along the z-axis
f_i	angle in a triangle opposite side F_i
H	horizontal force
l	length of member
M	bending moment
M_x	bending moment about the x-axis
M_y	bending moment about the y-axis
M_z	bending moment about the z-axis
P	axial force in a member
R	support reaction
V	vertical force
W_{LL}	concentrated live load
w_{DL}	distributed dead load
θ	angle of inclination

6.1 Introduction

The design of a building is generally accomplished by considering the structure as an assemblage of planar frames, each of which is designed as an independent two-dimensional frame. In some instances, however, it is necessary to consider the building as a whole and design it as a three-dimensional structure.

The sign convention shown in Figure 6.1 may be adopted for a three-dimensional structure acted on by a generalized system of forces. A space structure is illustrated in Figure 6.2 (i). The plan view of the structure is in the xz plane, as shown in Figure 6.2 (ii), and the elevation of the structure is in the xy plane, as shown in Figure 6.2 (iii).

Displacements in a space structure may occur in six directions, a displacement in the x, y, and z directions and a rotation about the x-, y-, and z-axes. The sign convention for displacements is shown in Figure 6.3 (i). A total of six displacement components define the restraint conditions at support 1 of the

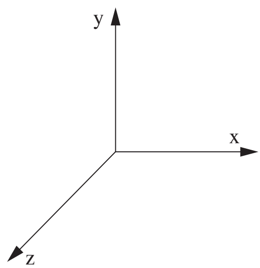


Figure 6.1

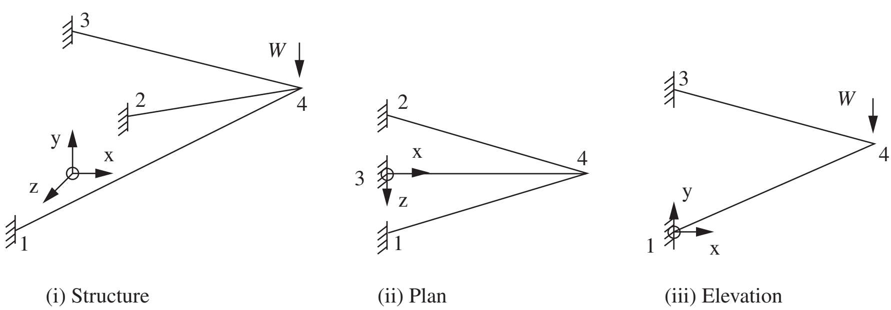


Figure 6.2

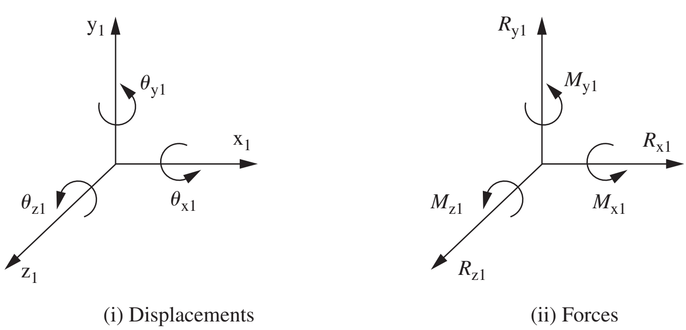


Figure 6.3

structure shown in Figure 6.2. The arrows indicate the positive directions of the displacement components, and, using the right-hand screw system, rotations are considered positive when acting clockwise as viewed from the origin. Similarly, a total of six force components, as shown in Figure 6.3 (ii), define the support reactions at support 1 of the structure in Figure 6.2.

6.2 Conditions of equilibrium

For a three-dimensional structure, six conditions of static equilibrium may be obtained at any point in the structure and at each support. These six equations of statics are:

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum F_z = 0$$
$$\sum M_x = 0,\ \sum M_y = 0,\ \sum M_z = 0$$

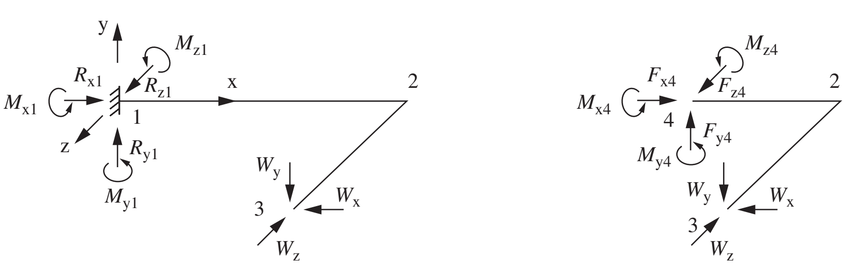
A three-dimensional structure is externally determinate when six external restraints are applied to the structure, since these restraints may be determined by the available six equations of static equilibrium.

The cranked cantilever of Figure 6.4 (i) has an applied load W at the free end that has the components W_x, W_y, and W_z as shown. The magnitude of W is given by:

$$W = (W_x^2 + W_y^2 + W_z^2)^{0.5}$$

and the three direction cosines of the applied load are given by:

$$\cos\theta_x = W_x/W$$
$$\cos\theta_y = W_y/W$$
$$\cos\theta_z = W_z/W$$



(i) Structure

(ii) Free-body diagram

Figure 6.4

As shown in Figure 6.4 (i), the cranked cantilever is statically determinate since six restraints are provided at the fixed end.

Similarly, six member stresses may be determined at a section cut through the structure, at any point 4, as shown by the free-body diagram shown in Figure 6.4 (ii).

Example 6.1

The cranked cantilever shown in Figure 6.4 (i) has an applied load at the free end with components $W_x = -10\,\text{kips}$, $W_y = -15\,\text{kips}$, and $W_z = -20\,\text{kips}$. The relevant lengths of the cantilever are $l_{12} = 12$ feet and $l_{23} = 6$ feet. Determine the magnitude of the reactions at support 1.

Solution

Applying the equilibrium equations, with the origin of the coordinates at support 1, gives:

Resolving along the x-axis:

$$\begin{aligned} R_{x1} + W_x &= 0 \\ R_{x1} &= -W_x \\ &= 10 \text{ kips} \end{aligned}$$

Resolving along the y-axis:

$$\begin{aligned} R_{y1} + W_y &= 0 \\ R_{y1} &= -W_y \\ &= 15 \text{ kips} \end{aligned}$$

Resolving along the z-axis:

$$\begin{aligned} R_{z1} + W_z &= 0 \\ R_{z1} &= -W_z \\ &= 20 \text{ kips} \end{aligned}$$

Taking moments about the x-axis:

$$\begin{aligned} M_{x1} + W_y l_{23} &= 0 \\ \mathrm{M}_{x1} &= -W_y l_{23} \\ &= -15 \times 6 \\ &= -90 \text{ kip-ft} \end{aligned}$$

Taking moments about the y-axis:

$$\begin{aligned} M_{y1} + W_x l_{23} + W_z l_{12} &= 0 \\ M_{y1} &= \ - W_x l_{23} - W_z l_{12} \\ &= \ 10 \times 6 - 20 \times 12 \\ &= -180 \text{ kip-ft} \end{aligned}$$

Taking moments about the z-axis:

$$\begin{aligned} M_{z1} + W_y l_{12} &= 0 \\ M_{z1} &= -W_y l_{12} \\ &= 15 \times 12 \\ &= 180 \text{ kip-ft} \end{aligned}$$

6.3 Pin-jointed space frames

In a pin-jointed three-dimensional space frame with j nodes, including the supports, $3j$ equations of equilibrium may be derived, since each node provides the relationships:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

If the frame has n members and r external restraints, the number of unknowns is $(n + r)$. In a pin-jointed three-dimensional space frame the frame is statically determinate when:

$$(n + r) = 3j$$

The frame is indeterminate when:

$$(n + r) > 3j$$

Example 6.2

The pin-jointed space frame shown in Figure 6.5 consists of nine members. The supports consist of a fixed pin at node 1, providing three restraints as shown, and rollers at nodes 2, 3, and 4, providing only vertical restraint. Determine if the structure is statically determinate.

Solution

The total number of external restraints is:

$$\begin{aligned} r &= 3 + 3 \times 1 \\ &= 6 \end{aligned}$$

Hence, the structure is stable and determinate externally.
The total number of members is:

$$n = 9$$

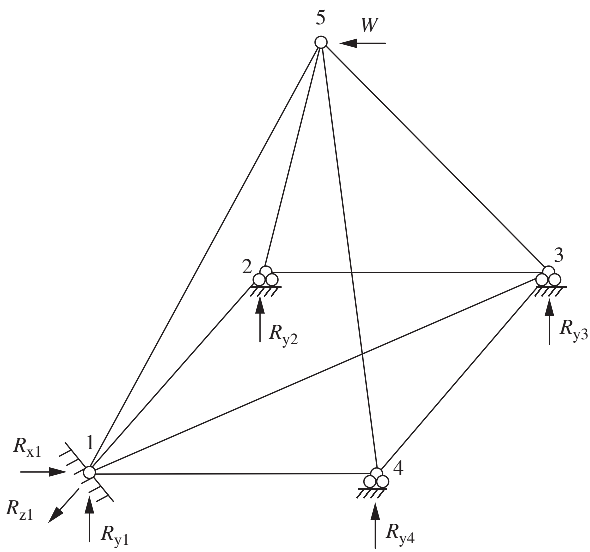


Figure 6.5

The total number of nodes is

$$\begin{aligned} j &= 5 \\ (n + r) &= 9 + 6 \\ &= 15 \\ 3j &= 3 \times 5 \\ &= 15 \end{aligned}$$

Hence:

$$(n + r) = 3j \ldots$$

and the structure is statically determinate.

6.4 Member forces

The member forces in a pin-jointed space frame may be obtained by resolution of forces at the nodes. Figure 6.6 shows a total of i members, 01, 02 … $0i$ with a common node 0. The force in member $0i$ is P_{0i}, and the three direction cosines of member $0i$ are $\cos\theta_{x0i}$, $\cos\theta_{y0i}$, and $\cos\theta_{z0i}$. The force components of member $0i$ along the three coordinate axes are:

$$\begin{aligned} P_{x0i} &= P_{0i} \cos\theta_{x0i} \\ P_{y0i} &= P_{0i} \cos\theta_{y0i} \\ P_{z0i} &= P_{0i} \cos\theta_{z0i} \end{aligned}$$

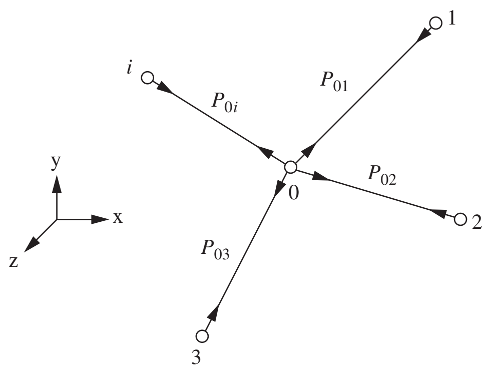


Figure 6.6

The length of member $0i$ is l_{0i}, and the projections of member $0i$ along the three coordinate axes are:

$$\begin{aligned} x_{0i} &= l_{0i}\cos\theta_{x0i} \\ y_{0i} &= l_{0i}\cos\theta_{y0i} \\ z_{0i} &= l_{0i}\cos\theta_{z0i} \end{aligned}$$

Assuming that no external force is applied at node 0, resolving along the three coordinate axes gives:

$$\begin{aligned} &\sum P_{x0i} = 0 \\ &\sum P_{y0i} = 0 \\ &\sum P_{z0i} = 0 \end{aligned}$$

and

$$P_{0i}/l_{0i} = P_{x0i}/x_{0i} = P_{y0i}/y_{0i} = P_{z0i}/z_0 i$$

Example 6.3

The pin-jointed space frame shown in Figure 6.7 consists of three members. The supports consist of fixed pins at nodes 1, 2, and 3, each providing three restraints as shown. Determine the member forces produced by the 100 kip vertical load applied at node 4.

Solution

The total number of external restraints is:

$$\begin{aligned} r &= 3 \times 3 \\ &= 9 \end{aligned}$$

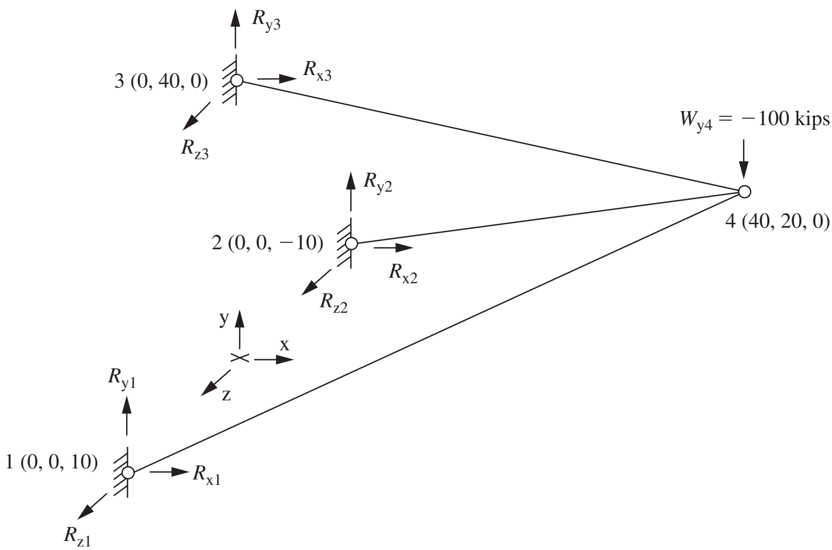


Figure 6.7

Hence, the structure is stable.
The total number of members is:

$$n = 3$$

The total number of nodes is:

$$\begin{aligned} j &= 4 \\ (n + r) &= 3 + 9 \\ &= 12 \\ 3j &= 3 \times 4 \\ &= 12 \end{aligned}$$

Hence:

$(n + r) = 3j$ ... the structure is statically determinate

And the structure is statically determinate.
The lengths of the members are:

$$\begin{aligned} l_{14} &= (x_{14}^2 + y_{14}^2 + z_{14}^2)^{0.5} \\ &= (40^2 + 20^2 + 10^2)^{0.5} \\ &= 45.83 \text{ ft} \\ l_{24} &= 45.83 \text{ ft} \\ l_{34} &= (x_{34}^2 + y_{34}^2 + z_{34}^2)^{0.5} \\ &= (40^2 + 20^2 + 0^2)^{0.5} \\ &= 44.72 \text{ ft} \end{aligned}$$

The direction cosines are:

$$
\begin{aligned}
\cos\theta_{x14} &= x_{14}/l_{14} \\
&= 40/45.83 \\
&= 0.873 \\
\cos\theta_{y14} &= y_{14}/l_{14} \\
&= 20/45.83 \\
&= 0.436 \\
\cos\theta_{z14} &= z_{14}/l_{14} \\
&= -10/45.83 \\
&= -0.218 \\
\cos\theta_{x34} &= x_{34}/l_{34} \\
&= 40/44.72 \\
&= 0.894 \\
\cos\theta_{y34} &= y_{34}/l_{34} \\
&= -20/44.72 \\
&= -0.447 \\
\cos\theta_{z34} &= z_{34}/l_{34} \\
&= 0/44.72 \\
&= 0
\end{aligned}
$$

Because of the symmetry of the structure and the loading, the forces in members 14 and 24 are identical. Hence:

$$P_{14} = P_{24}$$

Resolving along the x-axis at node 4 gives:

$$
\begin{aligned}
2P_{x14} + P_{x34} &= 0 \\
2P_{14}\cos\theta_{x14} + P_{34}\cos\theta_{x34} &= 0 \\
1.746P_{14} + 0.894P_{34} &= 0
\end{aligned}
$$

Resolving along the y-axis at node 4 gives:

$$
\begin{aligned}
2P_{y14} + P_{y34} &= -W_{y4} \\
2P_{14}\cos\theta_{y14} + P_{34}\cos\theta_{y34} &= -W_{y4} \\
0.872P_{14} - 0.447P_{34} &= 100 \text{ kips}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
P_{34} &= -111.86 \text{ kips ... tension} \\
P_{14} &= +57.21 \text{ kips ... compression}
\end{aligned}
$$

Supplementary problems

S6.1 The cranked cantilever shown in Figure S6.1 has a load of 100 kips applied at the free end. Determine the magnitude of the reactions at support 1.

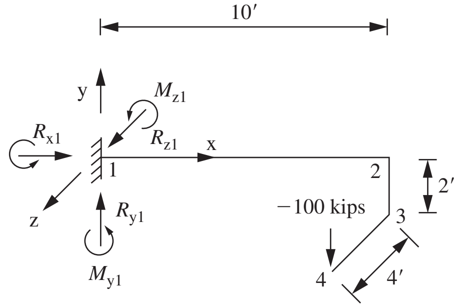


Figure S6.1

S6.2 The pin-jointed space frame shown in Figure S6.2 consists of three members. The supports consist of fixed pins at nodes 1, 2, and 3, each providing three restraints as shown. Determine the member forces produced by the 100 kip vertical load applied at node 4.

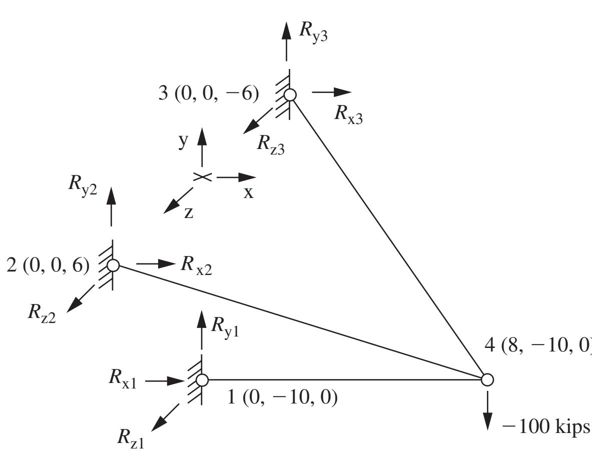


Figure S6.2

S6.3 The pin-jointed space frame shown in Figure S6.3 consists of three members. The supports consist of fixed pins at nodes 1, 2, and 3, each providing three restraints. Determine the member forces produced by the 100 kip vertical load applied at node 4.

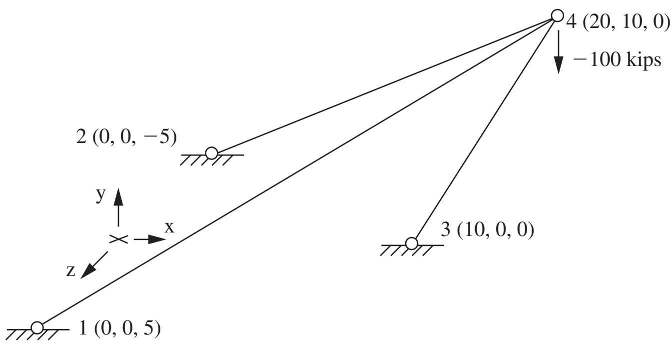


Figure S6.3

S6.4 The pin-jointed space frame shown in Figure S6.4 consists of three members. The supports consist of fixed pins at nodes 1, 2, and 3, each providing three restraints. Determine the member forces produced by the 100 kip vertical load applied at node 4.

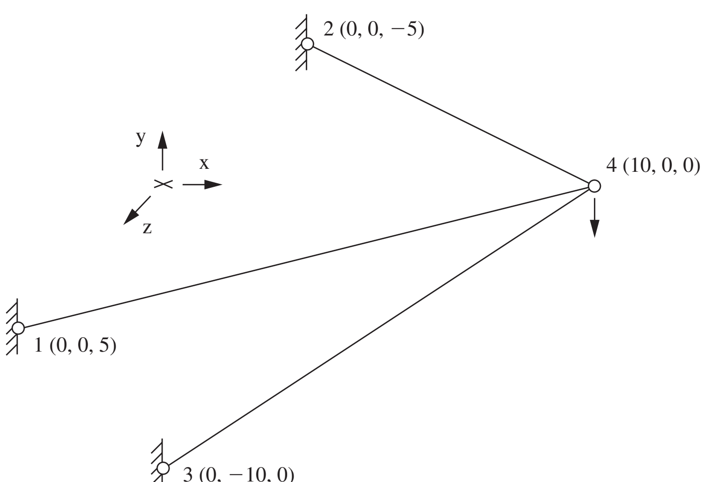


Figure S6.4

Answers to supplementary problems part 1

Chapter 1

S1.1 $V_1 = 14$ kips
$H_1 = 10$ kips
$M_1 = 100$ kip-ft
$V_4 = 6$ kips
$H_4 = 0$ kips

S1.2 $V_1 = -4.8$ kips … downward
$V_2 = 28.8$ kips … upward
$M_2 = 480$ kip-ft … producing tension in the top fiber of the girder

S1.3 $V_1 = 5$ kips
$H_1 = 5$ kips
$M_{32} = 100$ kip-ft … producing tension in the top fiber of the member
$V_7 = 5$ kips
$H_7 = 5$ kips

S1.4 $V_1 = 25$ kips
$H_1 = 25$ kips
$V_2 = 75$ kips
$H_2 = 25$ kips
$P_{13} = 35.33$ kips … tension
$P_{23} = 79.04$ kips … compression

S1.5 $V_1 = 10$ kips
$H_1 = 10$ kips
$V_4 = 10$ kips
$H_4 = 0$ kips
$P_{13} = 14.14$ kips … tension

S1.6 $V_1 = 11.67$ kips
$H_1 = 10$ kips
$V_4 = 8.33$ kips
$H_4 = 0$ kips
$M_{34} = 166.6$ kips … producing tension in the bottom fiber of the member

S1.7 $V_1 = 30\,\text{kips}$
$H_1 = 10\,\text{kips}$
$V_6 = 50\,\text{kips}$
$H_6 = 0\,\text{kips}$

S1.8 $V_1 = 10\,\text{kips}$
$H_1 = 10\,\text{kips}$
$V_3 = 10\,\text{kips}$
$H_3 = 10\,\text{kips}$
$P_{12} = 14.14\,\text{kips}$ … compression
$P_{23} = 10\,\text{kips}$ … compression

S1.9 $V_1 = 0\,\text{kips}$
$H_1 = 10.5\,\text{kips}$
$M_1 = 147\,\text{kip-ft}$

S1.10 $V_1 = 4\,\text{kips}$
$H_1 = 0\,\text{kips}$
$M_1 = 24\,\text{kip-ft}$
$P_{34} = 7.2\,\text{kips}$ … tension
$P_{24} = 6.0\,\text{kips}$ … compression

Chapter 2

S2.1 $V_1 = 2.24\,\text{kips}$
$H_1 = 3.57\,\text{kips}$
$V_5 = 4.92\,\text{kips}$
$P_{34} = 5.37\,\text{kips}$ … compression
$P_{38} = 5.0\,\text{kips}$ … tension
$P_{78} = 4.0\,\text{kips}$ … tension

S2.2 $P_{45} = 32\,\text{kips}$ … tension
$P_{411} = 3.2\,\text{kips}$ … compression
$P_{1011} = 30.93\,\text{kips}$ … compression

S2.3 $P_{23} = 17.78\,\text{kips}$ … tension
$P_{27} = 21.89\,\text{kips}$ … tension
$P_{67} = 33.33\,\text{kips}$ … compression

S2.4 $P_{12} = 12\,\text{kips}$ … tension
$P_{14} = 0\,\text{kips}$
$P_{15} = 16.97\,\text{kips}$ … compression

$P_{45} = 0\,\text{kips}$
$P_{56} = 12\,\text{kips}$ … compression
$P_{25} = 0\,\text{kips}$
$P_{26} = 0\,\text{kips}$

S2.5 $P_{12} = 9\,\text{kips}$ … tension
$P_{15} = 10.82\,\text{kips}$ … compression
$P_{23} = 6\,\text{kips}$ … tension
$P_{56} = 9.02\,\text{kips}$ … compression
$P_{25} = 3.35\,\text{kips}$ … compression
$P_{26} = 3.35\,\text{kips}$ … tension

S2.6 $P_{12} = 5\,\text{kips}$ … tension
$P_{114} = 30\,\text{kips}$ … compression
$P_{110} = 7.07\,\text{kips}$ … compression
$P_{23} = 5\,\text{kips}$ … tension
$P_{310} = 35.35\,\text{kips}$ … tension
$P_{315} = 10\,\text{kips}$ … compression
$P_{1014} = 42.43\,\text{kips}$ … tension
$P_{1415} = 30\,\text{kips}$ … compression

S2.7 $P_{23} = 6.67\,\text{kips}$ … tension
$P_{27} = 2.10\,\text{kips}$ … compression
$P_{37} = 0\,\text{kips}$
$P_{78} = 6.67\,\text{kips}$ … compression
$P_{67} = 6.72\,\text{kips}$ … compression

S2.8 $P_{12} = 3.35\,\text{kips}$ … tension
$P_{14} = 5.41\,\text{kips}$ … compression
$P_{24} = 1.12\,\text{kips}$ … compression
$P_{45} = 3.60\,\text{kips}$ … compression
$P_{25} = 4\,\text{kips}$ … tension

S2.9 $P_{23} = 13.34\,\text{kips}$ … tension
$P_{26} = 2.5\,\text{kips}$ … compression
$P_{27} = 3\,\text{kips}$ … tension
$P_{67} = 16.77\,\text{kips}$ … compression
$P_{37} = 0\,\text{kips}$

S2.10 $P_{49} = 0\,\text{kips}$
$P_{59} = 28.28\,\text{kips}$ … compression
$P_{89} = 22.36\,\text{kips}$ … compression
$P_{78} = 20\,\text{kips}$ … compression

Chapter 3

S3.1 (i) Determinate
(ii) Indeterminate
(iii) Unstable
(iv) Indeterminate

S3.2 (i) Indeterminate
(ii) Indeterminate
(iii) Indeterminate
(iv) Determinate
(v) Determinate

S3.3 (i) Indeterminate
(ii) Unstable
(iii) Determinate
(iv) Indeterminate

S3.4 Support reactions:

$V_1 = 73.03\,\text{kips}$
$H_1 = -70.71\,\text{kips}$
$V_4 = 77.68\,\text{kips}$

Shears:

$Q_{21} = 73.03\,\text{kips}$
$Q_{23} = 2.32\,\text{kips}$
$Q_{32} = 2.32\,\text{kips}$
$Q_{34} = 2.32\,\text{kips}$
$Q_{43} = -77.68\,\text{kips}$

Moments:

$M_{21} = 292.12\,\text{kip-ft}$ … compression in top fiber
$M_{32} = 301.40\,\text{kip-ft}$ … compression in top fiber
$M_{max} = 301.67\,\text{kip-ft}$ … at $x = 8.23$ ft

S3.5 Support reactions:

$V_1 = 0\,\text{kips}$
$H_1 = 0\,\text{kips}$
$V_2 = 100\,\text{kips}$
$V_4 = 0\,\text{kips}$

Shears:

$Q_{21} = -50\,\text{kips}$
$Q_{23} = 50\,\text{kips}$
$Q_{32} = 0\,\text{kips}$

$Q_{34} = 0\,\text{kips}$
$Q_{43} = 0\,\text{kips}$

Moments:

$M_{21} = 125\,\text{kip-ft}$ ... compression in bottom fiber
$M_{32} = 0\,\text{kip-ft}$
$M_{34} = 0\,\text{kip-ft}$

S3.6 Support reactions:

$V_1 = 30\,\text{kips}$
$H_1 = 0\,\text{kips}$
$M_1 = 180\,\text{kip-ft}$
$V_3 = 30\,\text{kips}$

Shears:

$Q_{21} = 30\,\text{kips}$
$Q_{23} = 30\,\text{kips}$
$Q_{32} = -30\,\text{kips}$

Moments:

$M_{12} = 180\,\text{kip-ft}$ ... compression in bottom fiber
$M_{21} = 0\,\text{kip-ft}$
$M_{32} = 0\,\text{kip-ft}$
$M_{max} = 45\,\text{kip-ft}$ ... at $x = 9$ ft, compression in top fiber

S3.7 Support reactions:

$V_1 = 20\,\text{kips}$
$H_1 = -20\,\text{kips}$
$M_1 = 0\,\text{kip-ft}$
$V_3 = 60\,\text{kips}$

Shears:

$Q_{12} = 20\,\text{kips}$
$Q_{21} = 20\,\text{kips}$
$Q_{23} = 20\,\text{kips}$
$Q_{32} = -60\,\text{kips}$

Moments:

$M_{21} = 160\,\text{kip-ft}$ ... compression in outer fiber
$M_{23} = 160\,\text{kip-ft}$ ... compression in top fiber
$M_{32} = 0\,\text{kip-ft}$
$M_{max} = 180\,\text{kip-ft}$ ... at $x = 2$ ft, compression in top fiber

S3.8 Support reactions:

$V_1 = 0\,\text{kips}$
$H_1 = -20\,\text{kips}$
$M_1 = 0\,\text{kip-ft}$

$V_5 = 40\,\text{kips}$
$H_5 = 0\,\text{kips}$

Shears:

$Q_{12} = 20\,\text{kips}$
$Q_{21} = 20\,\text{kips}$
$Q_{23} = 0\,\text{kips}$
$Q_{32} = 0\,\text{kips}$
$Q_{34} = -40\,\text{kips}$
$Q_{43} = -40\,\text{kips}$
$Q_{45} = 0\,\text{kips}$
$Q_{54} = 0\,\text{kips}$

Moments:

$M_{21} = 200\,\text{kip-ft}$ … compression in outer fiber
$M_{23} = 200\,\text{kip-ft}$ … compression in top fiber
$M_{32} = 200\,\text{kip-ft}$ … compression in top fiber
$M_{34} = 200\,\text{kip-ft}$ … compression in top fiber
$M_{43} = 0\,\text{kip-ft}$
$M_{45} = 0\,\text{kip-ft}$
$M_{54} = 0\,\text{kip-ft}$

S3.9 Support reactions:

$V_1 = -10\,\text{kips}$
$H_1 = 3.33\,\text{kips}$
$M_1 = 0\,\text{kip-ft}$
$V_7 = -10\,\text{kips}$
$H_7 = -3.33\,\text{kips}$

Shears:

$Q_{12} = -3.33\,\text{kips}$
$Q_{21} = -3.33\,\text{kips}$
$Q_{23} = 7.45\,\text{kips}$
$Q_{32} = 7.45\,\text{kips}$
$Q_{34} = -1.49\,\text{kips}$
$Q_{43} = -1.49\,\text{kips}$

Moments:

$M_{21} = 26.64\,\text{kip-ft}$ … compression in inner fiber
$M_{23} = 26.64\,\text{kip-ft}$ … compression in bottom fiber
$M_{32} = 6.70\,\text{kip-ft}$ … compression in top fiber
$M_{34} = 6.70\,\text{kip-ft}$ … compression in top fiber
$M_{43} = 0\,\text{kip-ft}$

S3.10 Support reactions:

$V_1 = -160\,\text{kips}$
$H_1 = 160\,\text{kips}$

$M_1 = 0\,\text{kip-ft}$
$V_3 = -160\,\text{kips}$
$H_3 = -160\,\text{kips}$
$M_2 = 0\,\text{kip-ft}$
At $x = 8$ ft the bending moment in the arch rib is
$M = 0\,\text{kip-ft}$

Chapter 4

S4.1 $\theta_1 = 52/EI$ rad
$\theta_2 = 156/EI$ rad
$y_3 = 1560/EI$ ft

S4.2 $y_2 = 2160/EI$ ft

S4.3 $\theta_1 = 1917/EI$ rad
$x_2 = 15837/EI$ ft

S4.4 The equation of the elastic curve is
$y = -2.5x^3/3EI + 2.5[x - 12]^3 + 120x$
The location of the maximum deflection in span 12 is
$x = 6.93$ ft
The maximum deflection in span 12 is
$y = 554/EI$ ft
The deflection at node 3 is
$y_3 = -2158/EI$

S4.5 $y_3 = 8136/EI$ ft

S4.6 $y_3 = 247/EA$ ft

S4.7 $y_4 = 3062/EA$ ft

S4.8 $y_2 = 410/EA$ ft

S4.9 $y_3 = 317/EA$ ft

S4.10 $y_2 = 199/EA$ ft

Chapter 5

S5.1 $M_2 = 1800\,\text{kip-ft}$

S5.2 $V_2 = 96\,\text{kips}$

S5.3 $V_3 = 6.75\,\text{kips}$

S5.4 $P_{24} = 5\,\text{kips}$ … compression

S5.5 $P_{45} = 10.67\,\text{kips}$ … tension

S5.6 $P_{1718} = 15\,\text{kips}$ … compression

Chapter 6

S6.1 $M_{x1} = 400\,\text{kip-ft}$
$M_{z1} = 1000\,\text{kip-ft}$
$R_{y1} = 100\,\text{kips}$

S6.2 $P_{14} = 80\,\text{kips}$
$P_{24} = 70.71\,\text{kips}$
$P_{34} = 70.71\,\text{kips}$

S6.3 $P_{14} = 114.55\,\text{kips}$
$P_{24} = 114.55\,\text{kips}$
$P_{34} = 282.80\,\text{kips}$

S6.4 $P_{14} = 55.90\,\text{kips}$
$P_{24} = 55.90\,\text{kips}$
$P_{34} = 141.40\,\text{kips}$

Part Two

Analysis of Indeterminate Structures

1 Statical indeterminacy

Notation

c number of releases introduced in a structure
d degree of indeterminacy
h number of internal hinges introduced in a structure
H horizontal reaction
j number of joints
M bending moment
n number of members
r number of external restraints
s number of internal rollers introduced in a structure
V vertical reaction

1.1 Introduction

A structure is in equilibrium with a system of applied loads when the resultant force in any direction and the resultant moment about any point are zero. For a two-dimensional plane structure, three equations of static equilibrium may be obtained:

$$\Sigma H = 0$$
$$\Sigma V = 0$$
$$\Sigma M = 0$$

where H and V are the resolved components in the horizontal and vertical directions of all forces and M is the resultant moment about any point.

A statically determinate structure is one in which all member forces and external reactions may be determined by applying the equations of equilibrium.

An indeterminate or redundant structure is one that possesses more unknown member forces and reactions than available equations of equilibrium. To determine the member forces and reactions, additional equations must be obtained from conditions of geometrical compatibility. The number of unknowns, in excess of the available equations of equilibrium, is the degree of indeterminacy, and the unknown forces and reactions are the redundants. The redundants may be removed from the structure, leaving a stable, determinate structure, which is known as the cut-back structure. External redundants are redundants that exist among the external reactions. Internal redundants are redundants that exist among the member forces.

Several methods have been proposed[1–5] for evaluating the indeterminacy of a structure.

1.2 Indeterminacy in pin-jointed frames

In a pin-jointed frame, external reactions are provided by either roller supports or hinge supports, as shown in Figure 1.1 (i) and (ii). The roller support provides only one degree of restraint in the vertical direction, and both horizontal and rotational displacements can occur. The hinge support provides two degrees of restraint in the vertical and horizontal directions, and only rotational displacement can occur. The magnitudes of the external restraints may be obtained from the three equations of equilibrium. Thus, a structure is externally indeterminate when it possesses more than three external restraints and unstable when it possesses fewer than three.

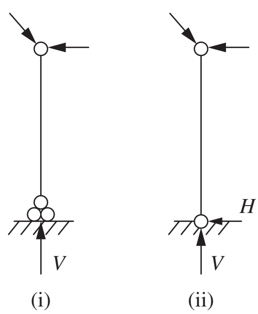


Figure 1.1

Figure 1.2 (i) and (ii) shows pin-jointed frames that have three degrees of restraint and are stable and determinate. Figure 1.3 (i) shows a pin-jointed frame that has four degrees of restraint and is one degree indeterminate. The

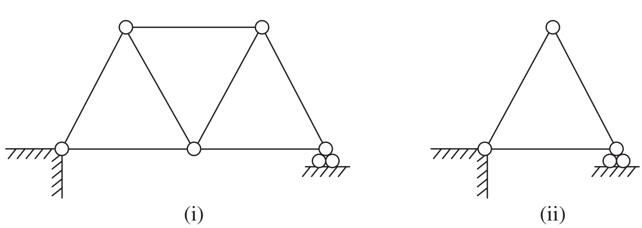


Figure 1.2

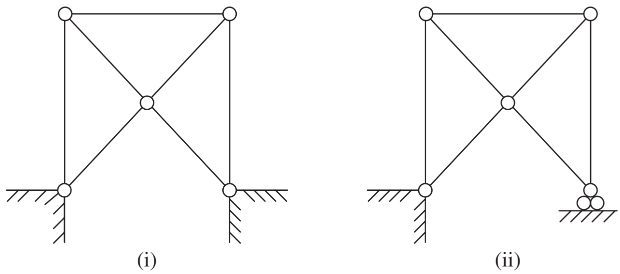


Figure 1.3

cut-back structure is shown in Figure 1.3 (ii). Figure 1.4 (i) shows a pin-jointed frame that is two degrees indeterminate; the cut-back structure is shown in Figure 1.4 (ii).

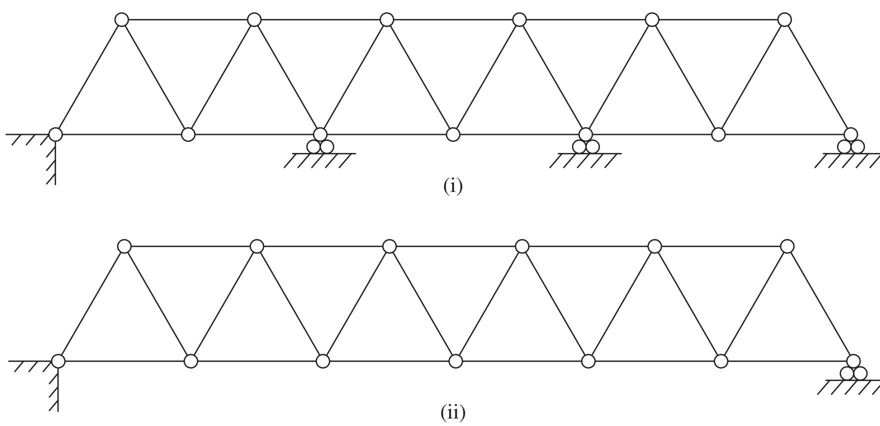


Figure 1.4

In a pin-jointed frame with j joints, including the supports, $2j$ equations of equilibrium may be obtained, since at each joint:

$$\Sigma H = 0 \quad \text{and} \quad \Sigma V = 0$$

Each member of the frame is subjected to an axial force, and if the frame has n members and r external restraints, the number of unknowns is $(n + r)$. Thus, the degree of indeterminacy is:

$$D = n + r - 2j$$

Figure 1.5 (i) and (ii) shows pin-jointed frames that are determinate. For frame (i):

$$D = 5 + 3 - (2 \times 4) = 0$$

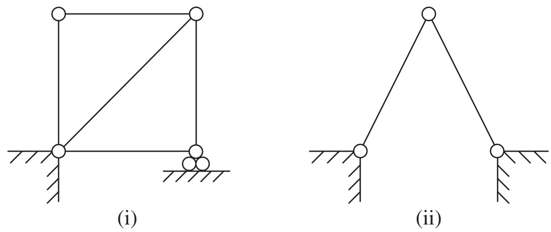


Figure 1.5

and for (ii):

$$D = 2 + 4 - (2 \times 3) = 0$$

Figure 1.6 (i) and (ii) shows pin-jointed frames that are indeterminate. For frame (i)

$$D = 10 + 3 - (2 \times 6) = 1$$

and for (ii):

$$D = 11 + 4 - (2 \times 6) = 3$$

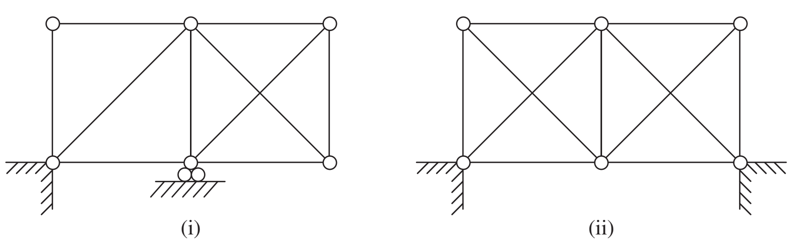


Figure 1.6

1.3 Indeterminacy in rigid frames

In addition to roller and hinge supports a rigid frame may be provided with a rigid support, shown in Figure 1.7, which provides three degrees of restraint.

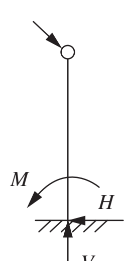


Figure 1.7

In a rigid frame with j joints, including the supports, $3j$ equations of equilibrium may be obtained, since at each joint:

$$\Sigma H = 0, \quad \Sigma V = 0, \quad \Sigma M = 0$$

Each member of the frame is subjected to three forces, axial and shear forces and a moment, and the number of unknowns is $(3n + r)$. Thus, the degree of indeterminacy is:

$$D = 3n + r - 3j$$

The degree of indeterminacy of the arches shown in Figure 1.8 is

arch (i) $D = 3 + 4 - (3 \times 2) = 1$
arch (ii) $D = 3 + 4 - (3 \times 2) = 1$
arch (iii) $D = 3 + 5 - (3 \times 2) = 2$
arch (iv) $D = 3 + 6 - (3 \times 2) = 3$
arch (v) $D = 3 + 3 - (3 \times 2) = 0$

and arch (v) is the cut-back structure for (i), (ii), (iii), and (iv).

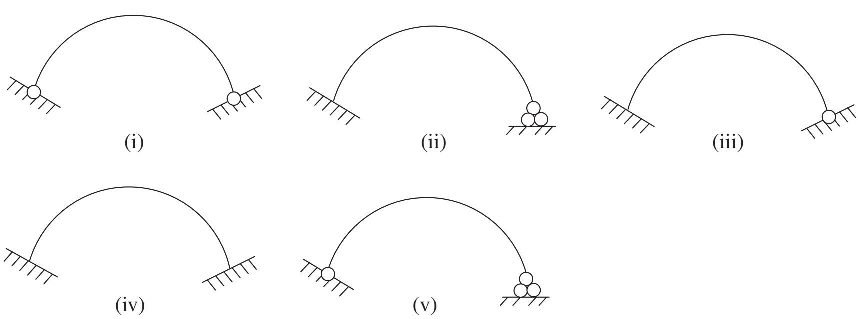


Figure 1.8

The degree of indeterminacy of the portal frame shown in Figure 1.9 (i) is:

$$D = (3 \times 3) + 6 - (3 \times 4) = 3$$

and the cut-back structure is obtained by introducing three releases, as at (ii), (iii), or (iv).

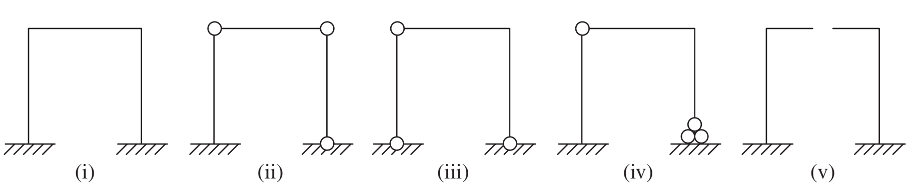


Figure 1.9

The redundants in the portal frame may also be regarded as the axial and shear forces and the moment in the beam, and the cut-back structure is obtained by cutting the beam as shown at (v). Similarly in a multibay, multistory frame the degree of indeterminacy equals 3 × the number of beams.

For the rigid frame shown in Figure 1.10, the degree of indeterminacy is:

$$\begin{aligned} D &= 3 \times 6 \\ &= 18 \end{aligned}$$

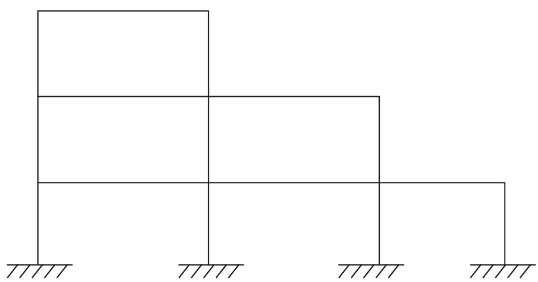


Figure 1.10

1.4 Indeterminacy in rigid frames with internal hinges

(a) Hinges within a member

The introduction of an internal hinge in a rigid frame provides an additional equation of equilibrium at the hinge of $M = 0$. In effect, a moment release has been introduced in the member.

The introduction of a horizontal, internal roller provides two additional equations of equilibrium at the roller of $M = 0$ and $H = 0$. In effect, a moment release and a release of horizontal restraint have been introduced in the member. Thus, the degree of indeterminacy is:

$$D = 3n + r - 3j - h - 2s$$

where n is the number of members, j is the number of joints in the rigid frame before the introduction of hinges, r is the number of external restraints, h is the number of internal hinges, and s is the number of rollers introduced.

The degree of indeterminacy of the frames shown in Figure 1.11 is:

(i) $D = (3 \times 3) + 4 - (3 \times 4) - 1 = 0$
(ii) $D = (3 \times 1) + 4 - (3 \times 2) - 1 = 0$
(iii) $D = (3 \times 7) + 12 - (3 \times 8) - 3 = 6$

The degree of indeterminacy of beam 15, which has a hinge and a roller introduced in span 34, as shown in Figure 1.12, is:

$$D = (3 \times 4) + 6 - (3 \times 5) - (2 \times 1) - 1 = 0$$

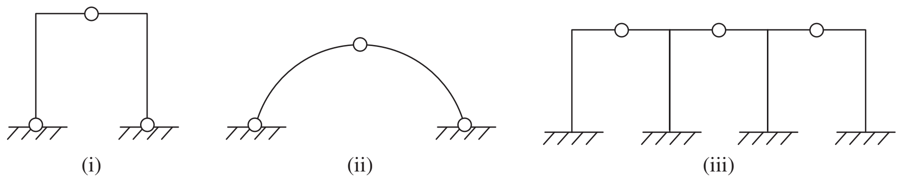


Figure 1.11

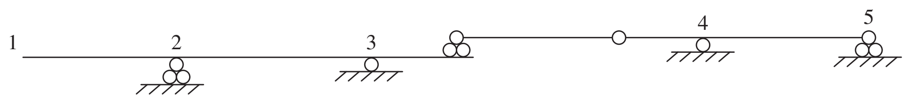


Figure 1.12

(b) Hinges at a joint

For two members meeting at a rigid joint there is one unknown moment, as shown in Figure 1.13, and the introduction of a hinge is equivalent to producing one release.

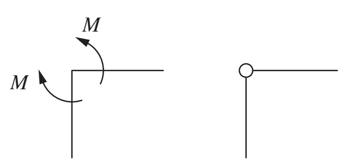


Figure 1.13

For three members meeting at a rigid joint there are two unknown moments, as shown in Figure 1.14, and the introduction of a hinge into one of the members produces one release; the introduction of a hinge into all three members produces two releases.

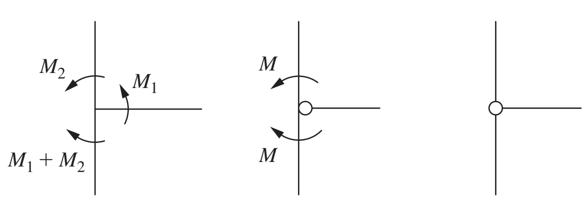


Figure 1.14

For four members meeting at a rigid joint there are three unknown moments, as shown in Figure 1.15; the introduction of a hinge into one of the members produces one release, the introduction of a hinge into two members produces

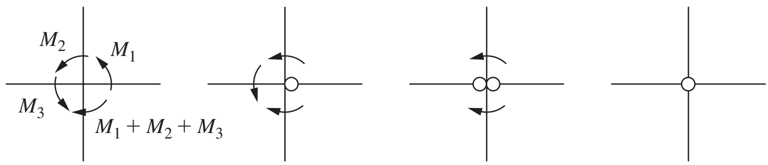


Figure 1.15

two releases, and the introduction of a hinge into all four members produces three releases.

In general, the introduction of hinges into i of the n members meeting at a rigid joint produces i releases. The introduction of a hinge into all n members produces $(n - 1)$ releases.

Thus, the degree of indeterminacy is given by:

$$D = 3n + r - 3j - c$$

where c is the number of releases introduced.

The degree of indeterminacy of the frames shown in Figure 1.16 is:

(i) $D = (3 \times 5) + 3 - (3 \times 4) - 5 = 1$
(ii) $D = (3 \times 6) + 3 - (3 \times 6) - 2 = 1$
(iii) $D = (3 \times 2) + 3 - (3 \times 2) - 2 = 1$
(iv) $D = (3 \times 5) + 9 - (3 \times 6) - 1 = 5$
(v) $D = (3 \times 4) + 5 - (3 \times 4) - 4 = 1$

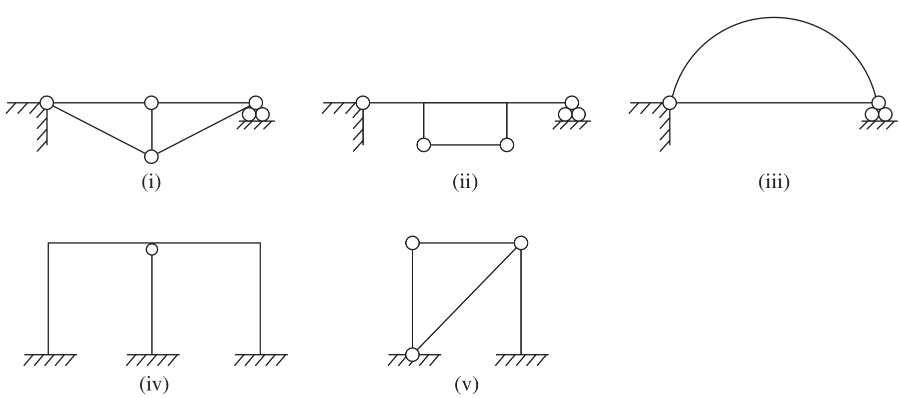


Figure 1.16

Supplementary problems

S1.1 Determine the degree of indeterminacy of the braced beam shown in Figure S1.1.

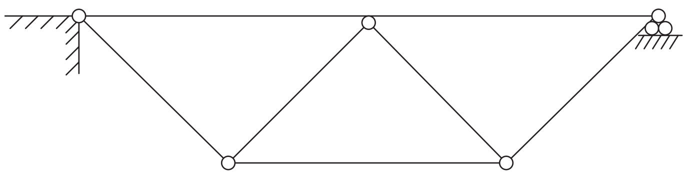


Figure S1.1

S1.2 Determine the degree of indeterminacy of the tied arch shown in Figure S1.2.

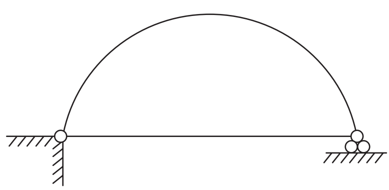


Figure S1.2

S1.3 Determine the degree of indeterminacy of the rigid-jointed frame shown in Figure S1.3.

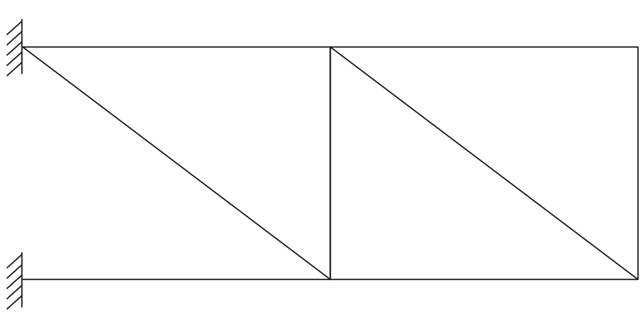


Figure S1.3

S1.4 Determine the degree of indeterminacy of the open spandrel arch shown in Figure S1.4.

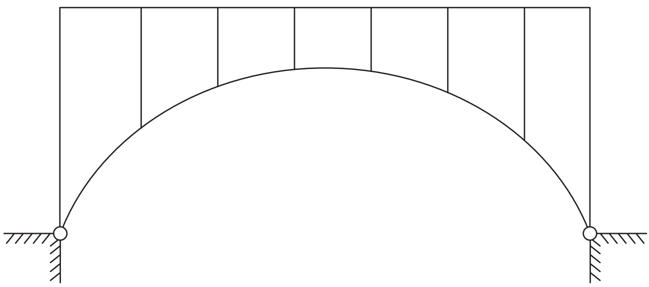


Figure S1.4

S1.5 Determine the degree of indeterminacy of the frames shown in Figure S1.5.

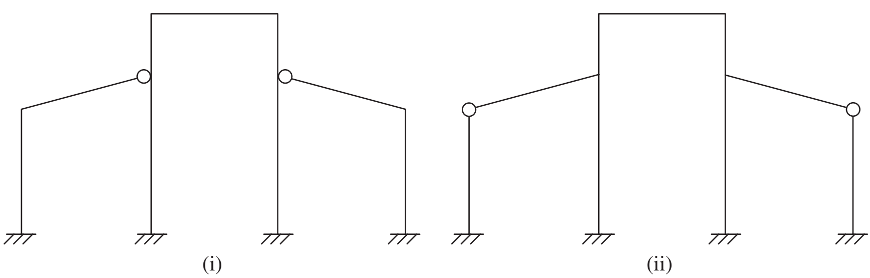


Figure S1.5

S1.6 Determine the degree of indeterminacy of the gable frames shown in Figure S1.6.

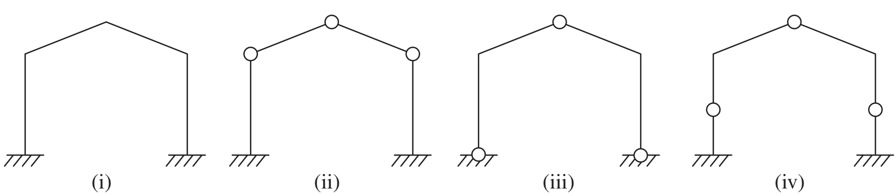


Figure S1.6

S1.7 Determine the degree of indeterminacy of the bridge structures shown in Figure S1.7.

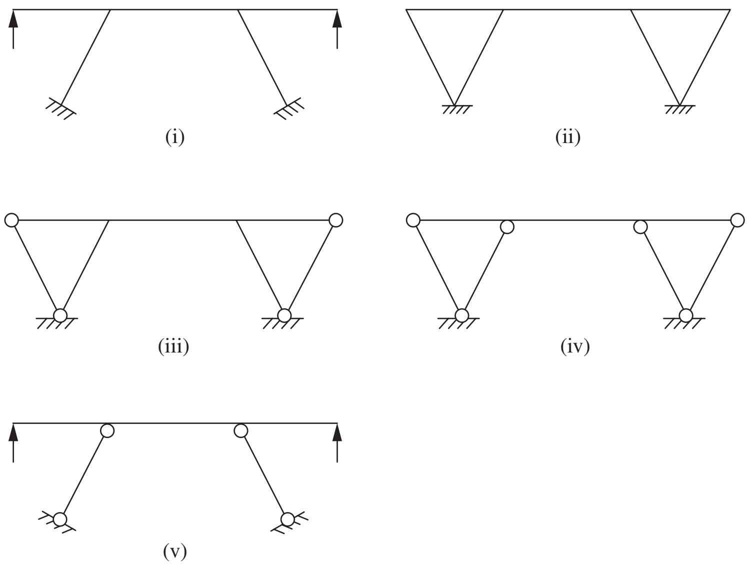


Figure S1.7

References

1. Matheson, J. A. L. Degree of redundancy of plane frameworks. Civil Eng. and Pub. Works. Rev. 52. June 1957. pp. 655–656.
2. Henderson, J. C. De C. and Bickley, W. G. Statical indeterminacy of a structure. Aircraft Engineering. 27. December 1955. pp. 400–402.
3. Rockey, K. C. and Preece, B. W. The degree of redundancy of structures. Civil Eng. and Pub. Works. Rev. 56. December 1961. pp. 1593–1596.
4. Di Maggio, F. C. Statical indeterminacy and stability of structures. Proc. Am. Soc. Civil Eng. 89 (ST3). June 1963. pp. 63–75.
5. Rangasami, K. S. and Mallick, S. K. Degrees of freedom of plane and space frames. The Structural Engineer. 44. March 1966. pp. 109–111 and September 1966. pp. 310–312.

2 Virtual work methods

Notation

A cross–sectional area of a member
E Young's modulus
G modulus of torsional rigidity
H horizontal reaction
I second moment of area of a member
I_o second moment of area of an arch at its crown
l length of a member
m bending moment in a member due to a unit virtual load
M bending moment in a member due to the applied loads
P axial force in a member due to the applied loads
q shear force in a member due to a unit virtual load
Q shear force in a member due to the applied loads
u axial force in a member due to a unit virtual load
V vertical reaction
W applied load
x horizontal deflection
y displacement of an applied load in its line of action; vertical deflection
δ deflection due to the applied load
δl extension of a member, lack of fit of a member
δx element of length of a member
$\delta\theta$ relative rotation between two sections in a member due to the applied loads
θ rotation due to the applied loads
μ form factor in shear
ϕ shear deformation due to the applied loads

2.1 Introduction

The principle of virtual work provides the most useful means of obtaining the displacement of a single point in a structure. In conjunction with the principles of superposition and geometrical compatibility, the values of the redundants in indeterminate structures may then be evaluated.

The principle may be defined as follows: if a structure in equilibrium under a system of applied forces is subjected to a system of displacements compatible with the external restraints and the geometry of the structure, the total work done by the applied forces during these external displacements equals the work done by the internal forces, corresponding to the applied forces, during the internal deformations, corresponding to the external displacements. The expression "virtual work" signifies that the work done is the product of a real loading system and imaginary displacements or an imaginary loading system and real displacements. Thus, in Chapters 2 and 3 and Sections 6.2, 6.4, 6.5, 9.8, and 10.3 displacements are obtained by considering virtual forces undergoing real displacements, while in Sections 7.8–7.14, 9.5–9.8, and 11.3 equilibrium relationships are obtained by considering real forces undergoing virtual displacements.

A rigorous proof of the principle based on equations of equilibrium has been given by Di Maggio[1]. A derivation of the virtual-work expressions for linear structures is given in the following section.

2.2 Virtual-work relationships

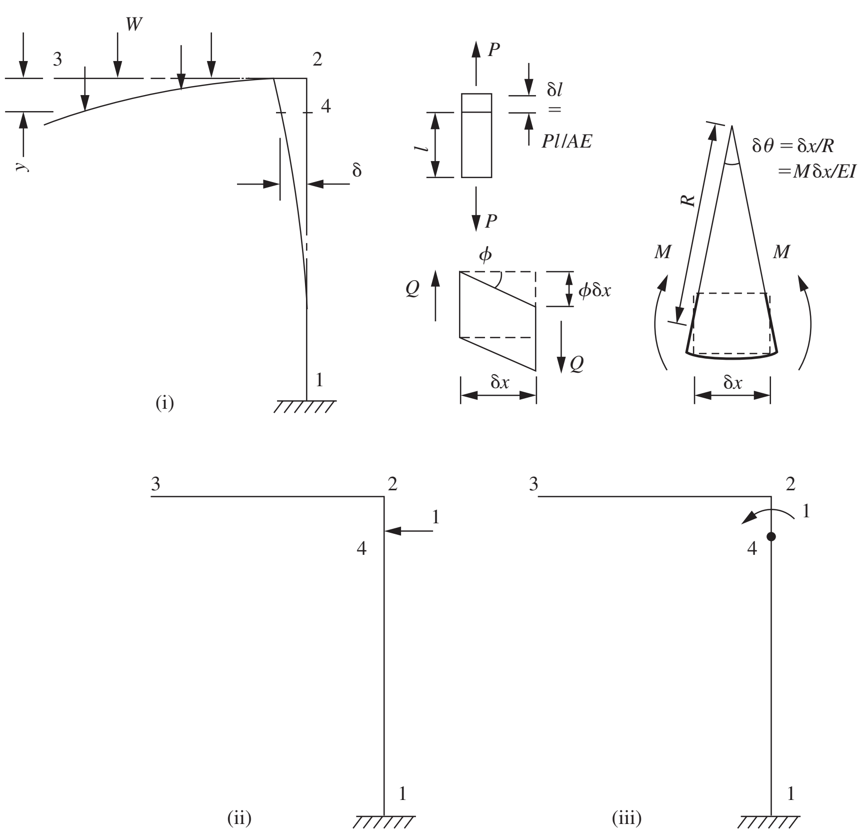


Figure 2.1

To the structure 123 shown in Figure 2.1 (i), the external loads W are gradually applied. This results in the deflection of any point 4 a distance δ while each load moves a distance y in its line of action. The loading produces an internal force P and an extension δl in any element of the structure, a bending moment M and a relative rotation $\delta\theta$ to the ends of any element, a shear force Q, and a shear strain ϕ. The external work done during the application of the loads must equal the internal energy stored in the structure from the principle of conservation of energy.

Then:

$$\sum Wy/2 = \sum P\delta l/2 + \sum M\delta\theta/2 + \sum Q\phi\ \delta x/2 \tag{1}$$

To the unloaded structure a unit virtual load is applied at 4 in the direction of δ as shown in Figure 2.1 (ii). This results in a force u, a bending moment m, and a shear force q in any element.

Now, while the virtual load is still in position, the real loads W are gradually applied to the structure. Again, equating external work and internal energy:

$$\begin{aligned}\sum Wy/2 + 1\times\delta = &\sum P\ \delta l/2 + \sum M\ \delta\theta/2 + \sum Q\phi\ \delta x/2 + \sum u/\delta l\\ &+ \sum m\ \delta\theta + \sum q\phi\ \delta x\end{aligned} \tag{2}$$

Subtracting expression (1) from expression (2):

$$1\times\delta = \sum u\ \delta l + \sum m\ \delta\theta + \sum q\phi\ \delta x$$

For pin-jointed frameworks, with the loading applied at the joints, only the first term on the right-hand side of the expression is applicable.

Then:

$$\begin{aligned}1\times\delta &= \sum u\ \delta l\\ &= \sum Pul/AE\end{aligned}$$

where P is the internal force in a member due to the applied loads and l; A and E are its length, area, and modulus of elasticity; and u is the internal force in a member due to the unit virtual load.

For rigid frames, only the last two terms on the right-hand side of the expression are significant.

Then:

$$\begin{aligned}1\times\delta &= \sum m\ \delta\theta + \sum q\phi\ \delta x\\ &= \int Mm\ \mathrm{d}x/EI + \int Qq\ \mathrm{d}x/\mu AG\end{aligned}$$

where M and Q are the bending moment and shear force at any section due to the applied loads and I, G and A are the second moment of area, the rigidity modulus, and the area of the section; μ is the form factor; and m and q are the bending moment and shear force at any section due to the unit virtual load.

Usually the term representing the deflection due to shear can be neglected, and the expression reduces to:

$$1 \times \delta = \int Mm \, \mathrm{d}x/EI$$

In a similar manner, the rotation θ of any point 4 of the structure may be obtained by applying a unit virtual bending moment at 4 in the direction of θ.

Then:

$$1 \times \theta = \int Mm \, \mathrm{d}x/EI + \int Qq \, \mathrm{d}x/\mu AG$$

where m and q are the bending moment and shear force at any section due to the unit virtual moment.

2.3 Sign convention

For a pin-jointed frame, tensile forces are considered positive and compressive forces negative. Increase in the length of a member is considered positive and decrease in length negative. The unit virtual load is applied to the frame in the anticipated direction of the deflection. If the assumed direction is correct, the deflection obtained will have a positive value. The deflection obtained will be negative when the unit virtual load has been applied in the opposite direction to the actual deflection.

For a rigid frame, moments produced by the virtual load or moment are considered positive, and moments produced by the applied loads, which are of opposite sense, are considered negative. A positive value for the displacement indicates that the displacement is in the same direction as the virtual force or moment.

2.4 Illustrative examples

Example 2.1

Determine the horizontal and vertical deflection of point 4 of the pin-jointed frame shown in Figure 2.2. All members have a cross-sectional area of 8 in^2 and a modulus of elasticity of 29,000 kips/in^2.

Considering the equilibrium of the longitudinal beam:

$$2V_1 + 2V_2 + V_3 = W$$

Solving these three equations simultaneously, we obtain:

$$V_1 = 0.067W$$
$$V_2 = 0.253W$$
$$V_3 = 0.36W$$

and the bending moment diagram may be drawn as shown at (ii) in Figure 7.13.

Example 7.6

Determine the bending moments for all the members of the two-story frame shown in Figure 7.14. The second moments of area of the beams are 160 in^4 and of the columns 100 in^4.

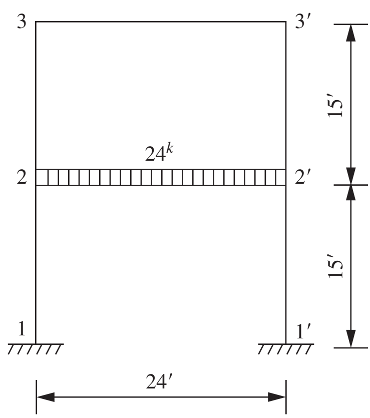


Figure 7.14

Solution

The fixed-end moments are:

$$M^F_{22'} = 24 \times 24/12$$
$$= 48 \text{ kip-ft}$$

The distribution procedure for the left half of the frame is shown in Table 7.7, and the final moments in the right half of the frame are equal and of opposite sense to those obtained for the left half. Because of the symmetry of the structure and the loading, no relative displacements occur between the ends of the members and only joint rotations need be considered.

Table 7.7 Distribution of moments in Example 7.6

Joint	1		2			3
Member	12	21	22′	23	32	33′
I/l	100/15	100/15	160/24	100/15	100/15	160/24
Relative EI/l	1	1	1	1	1	1
Modified stiffness	4	4	2	4	4	2
Distribution factor	0	⅖	⅕	⅖	⅔	⅓
Carry-over factor	0	→		←	½	
	←	½		½	→	
Fixed-end moments			−48			
Distribution and carry-over	9	19	10	19	9	
Distribution and carry-over				−3	−6	−3
Distribution and carry-over	1	1	1	1	1	−1
Final moments, kip-ft	10	20	−37	17	4	−4

(d) Skew symmetry

A symmetrical structure subjected to loading that is of opposite sense at corresponding points undergoes skew symmetrical deformation. The bending moments and rotations at corresponding points in the structure are equal and of the same sense, and a point of contraflexure, which is equivalent to a hinge, occurs at the center of the structure.

For a continuous beam with an even number of spans, both members at the central support can be considered hinged, and the modified stiffnesses are as shown at (i) in Figure 7.15. For a continuous beam with an odd number of

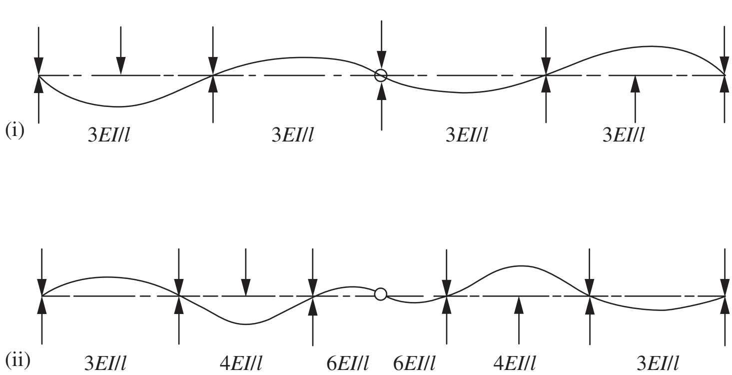


Figure 7.15

spans, the central member can be considered as two hinged members each of length $l/2$, which results in the modified stiffness shown at (ii) in Figure 7.15.

As in the case of symmetrical loading, distribution is required in only half the structure, there is no carry-over between halves, and the usual values of the fixed-end moments apply in all members.

7.5 Illustrative examples

Example 7.7

Determine the reactions at the support 3 of the rigid frame shown in Figure 7.16. The relative *EI* values are shown ringed.

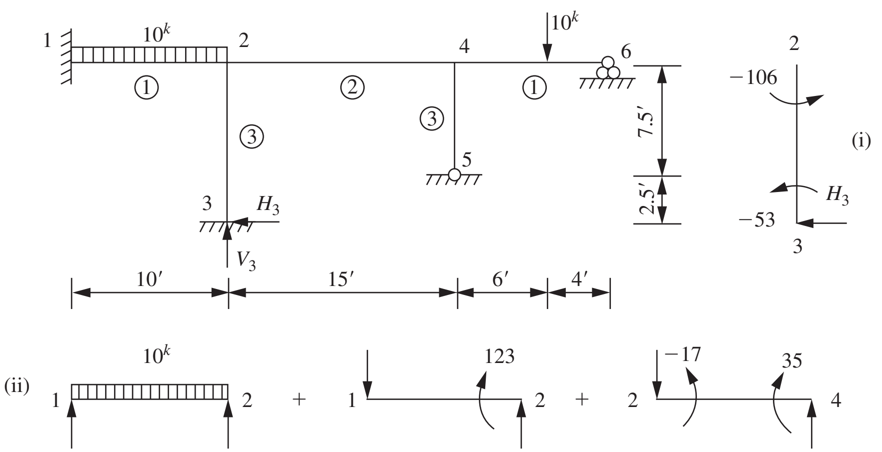


Figure 7.16

Solution

The fixed-end moments are:

$$\begin{aligned} M^F_{21} &= -M^F_{12} = 10 \times 120/12 \\ &= 100 \text{ kip-in} \\ M^F_{64} &= 10 \times 4 \times 36 \times 12/100 \\ &= 173 \text{ kip-in} \\ M^F_{46} &= -10 \times 6 \times 16 \times 12/100 \\ &= -115 \text{ kip-in} \end{aligned}$$

and the distribution procedure is shown in Table 7.8. Distribution to the tops of the columns is unnecessary, as the moments there may be obtained after completion of the distribution by considering the algebraic sum of the moments at joints 2 and 4. The moment at the foot of column 23 is half the

Table 7.8 Distribution of moments in Example 7.7

Joint	1		2			4		6
Member	12	21	23	24	42	45	46	64
Relative *EI/l*	1/10	1/10	3/10	2/15	2/15	6/15	1/10	1/10
Modified stiffness	4/10	4/10	12/10	8/15	8/15	18/15	3/10	4/10
Distribution factor	1	9/61	36/61	16/61	16/61	36/61	9/61	1
Carry-over factor	½	→		½	→		←	½
	←	0		←	½		0	→
Fixed-end moments	−100	100					−115	173
Distribution	100	−15		−26	30		17	−173
Carry-over		50		15	−13		−86	
Distribution		−10		−17	26		15	
Carry-over				13	−9			
Distribution		−2		−3	2		1	
Carry-over and distribution				1	−1			
Final moments, kip-in	0	123	−106	−17	35	133	−168	0

moment at the top since the carry-over factor is ½ and there are no sway or initial fixed-end moments.

The reaction H_3 is obtained by taking moments about 2 for member 23, as shown at (i) in Figure 7.16. Then:

$$\begin{aligned} H_3 &= (106 + 53)/120 \\ &= 1.33 \text{ kips} \end{aligned}$$

The reaction V_3 is obtained by considering the vertical reaction at 2 due to the fixing moments on spans 12 and 24, due to the distributed load on span 12 treated as a simply supported beam. Then, referring to (ii) in Figure 7.16, the vertical reaction is given by:

$$\begin{aligned} V_3 &= 5 + 123/120 - 18/180 \\ &= 5.92 \text{ kips} \end{aligned}$$

Example 7.8

Determine the slope of the beam at 2 and the final moments in the continuous beam shown in Figure 7.17 when the support 2 sinks by ½ in and the support 3 sinks by 1 in. The second moment of area of the beam is 120 in^4, and the modulus of elasticity is 29,000 kips/in^2.

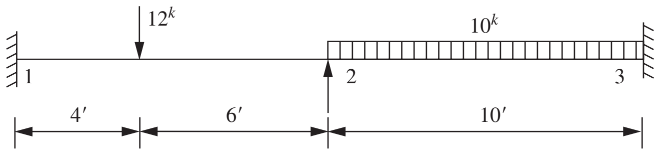


Figure 7.17

Solution

Due to the applied loading:

$$\begin{aligned}
M^F_{21} &= 12 \times 6 \times 16 \times 12/100 \\
&= 138 \text{ kip-in} \\
M^F_{12} &= -12 \times 4 \times 36 \times 12/100 \\
&= -207 \text{ kip-in} \\
M^F_{32} &= -M^F_{23} \\
&= 10 \times 10 \times 12/12 \\
&= 100 \text{ kip-in}
\end{aligned}$$

Due to the sinking of the supports:

$$\begin{aligned}
M^F_{12} &= M^F_{21} \\
&= M^F_{23} \\
&= M^F_{32} \\
&= -6 \times 29{,}000 \times 120 \times 1/2 \times 1/14{,}400 \\
&= -725 \text{ kip-in}
\end{aligned}$$

The distribution procedure is shown in Table 7.9.

Table 7.9 Distribution of moments in Example 7.8

Joint	1	2		3
Member	12	21	23	32
Relative *EI*/*l*	1	1	1	1
Distribution factor	0	½	½	0
Carry-over factor	←	½	½	→
Fixed-end moments	−932	−587	−825	−625
Distribution and carry-over	353	706	706	353
Final moments, kip-in	−579	119	−119	−272

Since there is no rotation at the fixed ends, we have:

$$M_{21} = M_{21}^F + s\theta_2$$
$$119 = -587 + \theta_2 \times 4 \times 29{,}000 \times 120/120$$

and $\theta_2 = 0.00609$ radians, clockwise

7.6 Secondary effects

Triangulated trusses are normally analyzed as if they are pin-jointed, although the members are invariably rigidly connected. The direct stresses obtained in this analysis are referred to as the primary stresses. These direct stresses cause axial deformations in the members, which produce relative lateral displacements of the ends of each member. The bending moments caused by these lateral displacements produce additional or secondary stresses in the members. It is customary to neglect the secondary stresses in light trusses with flexible members, but in heavy trusses the secondary stresses may be considerable.

The procedure of allowing for secondary effects consists of determining the primary stresses and, from a Williot–Mohr diagram, the lateral displacements of the ends of each member. The fixed-end moments due to these displacements are then distributed in the normal way, and the secondary stresses are obtained from the final moments.

The reactions due to the final moments may be obtained at the ends of each member. In general, these reactions will not be in equilibrium at a particular joint, and the algebraic sum of the reactions is equivalent to a constraint required at the joint to prevent its displacement. Applying a force equal and opposite to the constraint produces equilibrium. This causes additional axial forces and deformations in the members and additional secondary moments. The magnitude of the constraints is usually small compared with the applied loads and may be neglected.

Example 7.9

Determine the secondary bending moments in the members of the rigidly jointed truss shown in Figure 7.18. The second moment of area of members 14, 34, and 24 is 30 in^4, and that of members 12 and 23 is 40 in^4. The cross-sectional area of members 14, 34, and 24 is 4 in^2, and that of members 12 and 23 is 5 in^2.

Solution

The primary structure is shown at (i) in Figure 7.18, and the axial forces and deformations are listed in Table 7.10. The Williot–Mohr diagram is shown at (iii) in Figure 7.18, and the relative lateral displacement of the ends of members 12 and 14 is indicated; no lateral displacement is produced in member 24 due to the symmetry of the structure.

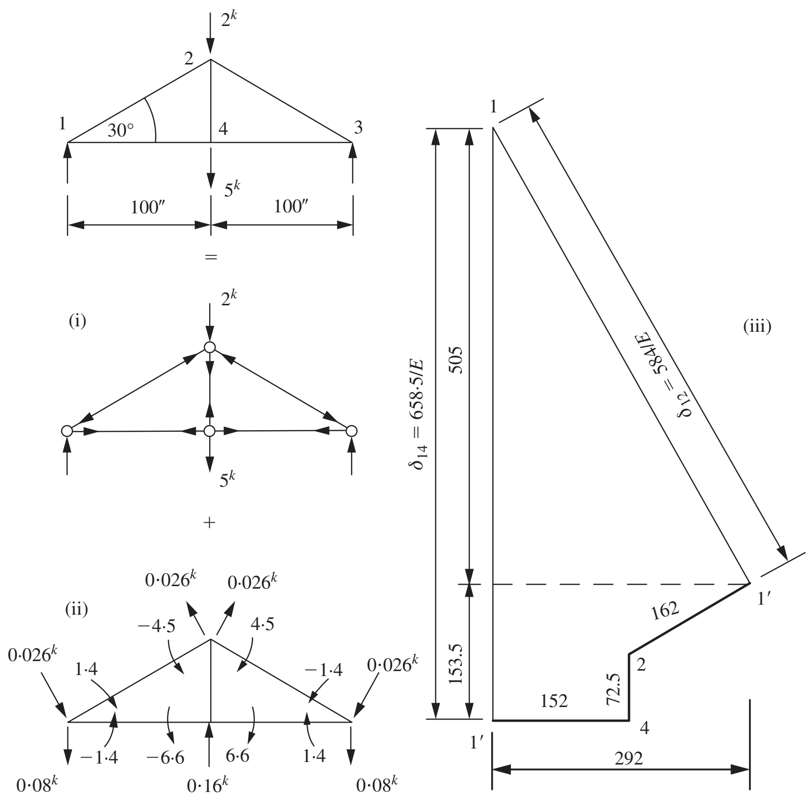


Figure 7.18

Table 7.10 Determination of forces and displacements in Example 7.9

Member	*I*	*A*	*l*	*P*	*Pl/A*	$\delta \times E$
12	40	5	116	7	162	584
14	30	4	100	6.07	152	658.5
24	30	4	58	5	72.5	0

Due to these displacements, the fixed-end moments are:

$$
\begin{aligned}
M_{12}^F &= M_{21}^F \\
&= -6EI\delta/l^2 \\
&= -6 \times 40 \times 584/13{,}300 \\
&= -10.5 \text{ kip-in} \\
M_{14}^F &= M_{41}^F \\
&= -6 \times 30 \times 658.5/10{,}000 \\
&= -11.8 \text{ kip-in}
\end{aligned}
$$

Due to symmetry, member pairs 21 and 23 and 41 and 43 can be considered fixed-ended at 2 and 4, respectively; member 24 carries no moment; and the distribution procedure is shown in Table 7.11.

Table 7.11 Distribution of moments in Example 7.9

Joint	2	1		4
Member	21	12	14	41
Relative *EI*/*l*		346	300	
Distribution factor	0	0.535	0.465	0
Carry-over factor	←	½	½	→
Fixed-end moments	−10.5	−10.5	−11.8	−11.8
Distribution and carry-over	6	12.1	10.4	5.2
Final moments, kip-in	−4.5	1.4	−1.4	−6.6

The final moments, together with the forces required to maintain equilibrium at the joints, are shown at (ii) in Figure 7.18. These forces are approximately 3% of the applied loads and may be neglected.

7.7 Non-prismatic members

The methods of obtaining the stiffness, carry-over factors, and fixed-end moments for non-prismatic members were given in Sections 6.6 and 6.7. In addition, tabulated functions are available for a large range of non-prismatic members[2,3,4,11,12].

Example 7.10

Determine the bending moments in the frame shown in Figure 7.19. The second moments of area of the members are shown ringed.

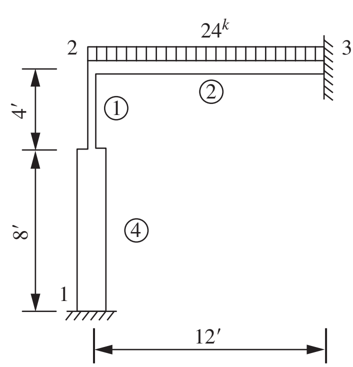


Figure 7.19

Solution

The stiffness and carry-over factors for member 21 have been obtained in Example 6.7 and are:

$$s'_{21} = 5EI/12$$
$$c'_{21} = 4/5$$

The fixed-end moments are:

$$\begin{aligned} M^F_{32} &= -M^F_{23} \\ &= 24 \times 12/12 \\ &= 24 \text{ kip-ft} \end{aligned}$$

The distribution procedure is shown in Table 7.12.

Table 7.12 Distribution of moments in Example 7.10

Joint	1	2		3
Member	12	21	23	32
Stiffness		$5EI/12$	$8EI/12$	
Distribution factor	0	5/13	8/13	0
Carry-over factor	←	⅘	½	→
Fixed-end moments			−24	24
Distribution and carry-over	7.4	9.25	14.75	7.38
Final moments, kip-ft	7.4	9.25	−9.25	31.38

7.8 Distribution procedure for structures subjected to unspecified joint translation

(a) Introduction

All frames subjected to lateral loads and frames unsymmetrical in shape or loading will deflect laterally, as shown in Figure 7.20. The horizontal displacement, denoted by x in the figure, is termed the side sway. In a similar manner, vertical sway displacements, y, are produced in the structures shown in Figure 7.21.

The sway produces relative lateral displacement of the ends of some of the frame members, which causes moments in addition to those due to joint rotations. The final moments are obtained by superposition, as shown in Figure 7.22, the magnitude of the sway being obtained from equations of static equilibrium known as the sway equations.

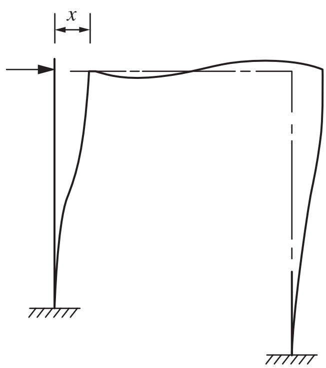


Figure 7.20

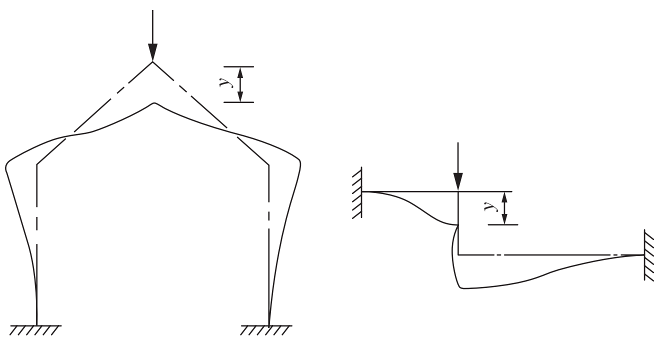


Figure 7.21

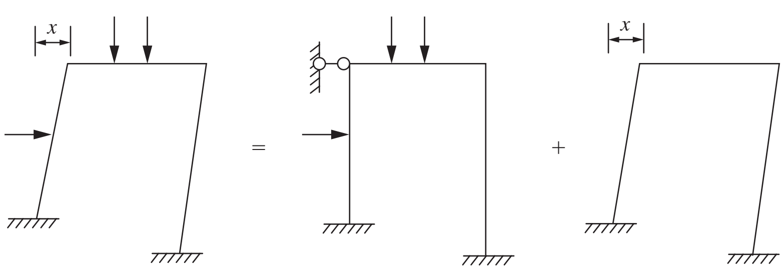


Figure 7.22

(b) The sway equations

The frame shown at (i) in Figure 7.23 may be considered to deform as the mechanism indicated under the action of the applied loads. Applying the equation of virtual work to the small displacements involved, the external work done by the applied loads equals the internal work done by the terminal moments in the columns rotating through an angle ϕ. Due to the sign convention adopted (clockwise moments positive) these terminal moments are all

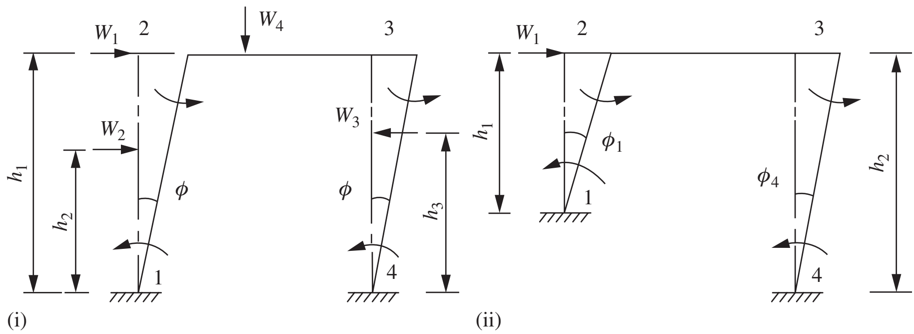


Figure 7.23

negative, and a change of sign must be introduced to satisfy the virtual work equation. Then:

$$-(M_{12} + M_{21})\phi - (M_{34} + M_{43})\phi = W_1h_1\phi + W_2h_2\phi - W_3h_3\phi$$
$$-M_{12} - M_{21} - M_{34} - M_{43} = W_1h_1 + W_2h_2 - W_3h_3$$

and the final terminal moments in the frame must satisfy this sway equation. Similarly, for the frame shown at (ii) in Figure 7.23, the sway equation is:

$$-(M_{12} + M_{21})\phi_1 - (M_{34} + M_{43})\phi_4 = W_1h_1\phi_1$$

Since the sway displacements at the tops of the columns are equal:

$$h_1 \times \phi_1 = h_2 \times \phi_4$$

and:

$$-(M_{12} + M_{21}) - h_1(M_{34} + M_{43})/h_2 = W_1h_1$$

In the case of a frame with inclined columns, as shown in Figure 7.24, the beam also rotates, and it is necessary to construct the displacement diagram, shown at (i), to establish the sway equation. Since axial deformations in the members of the frame are negligible, the points 1, 2, 3, and 4 coincide. A unit horizontal displacement is imposed on 2, and, as 2 must move perpendicularly to the original direction of 12, the point 2′ is obtained. Similarly, 3 must move perpendicularly to 23 and 34, and the point 3′ is obtained. The member rotations are obtained from the diagram as:

$$\phi_1 = 22'/l_{12}$$
$$\phi_2 = 2'3'/l_{23}$$
$$\phi_4 = 33'/l_{34}$$

where ϕ_2 is anticlockwise and produces clockwise moments M_{23} and M_{32}.

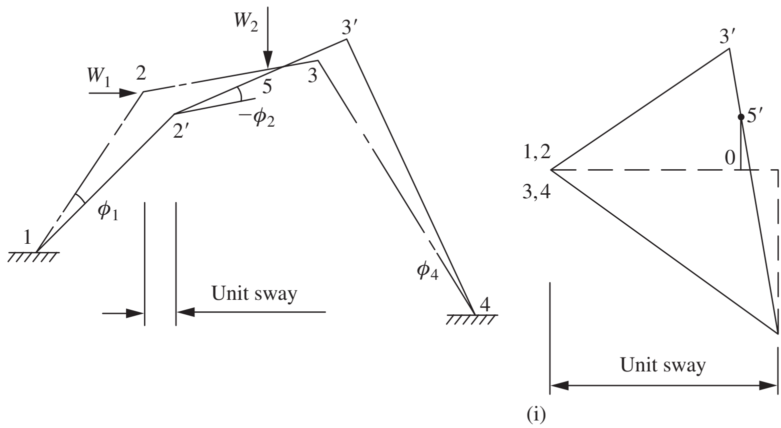


Figure 7.24

The vertical displacement of 5 is 05′ where 5′ divides 2′3′ in the ratio 5 divides 23. The sway equation is:

$$-(M_{12} + M_{21})\phi_1 + (M_{23} + M_{32})\phi_2 - (M_{34} + M_{43})\phi_4 = W_1 \times 1 - W_2 \times (05')$$

and the rotations may be eliminated by using the expressions obtained from the displacement diagram.

The two-story frame shown in Figure 7.25 has two degrees of sway freedom, as shown at (i) and (ii), corresponding to the different translations possible at the beam levels. Considering sway (1) and (2) in turn, the sway equations are:

$$-(M_{32} + M_{23}) - (M_{45} + M_{54}) = W_2 h_2$$
$$-(M_{21} + M_{12}) - (M_{56} + M_{65}) = (W_1 + W_2) h_1$$

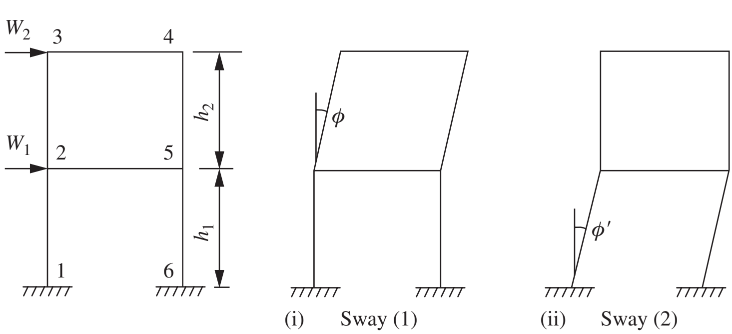


Figure 7.25

and the terminal moments in the frame must satisfy both these equations.

The two-bay frame shown in Figure 7.26 has the two degrees of sway freedom shown at (i) and (ii). The sway equations are:

$$-(M_{12} + M_{21})\phi_1 - (M_{34} + M_{43})\phi_2 = Wl_{12}\phi_1$$
$$-(M_{45} + M_{54})\phi_4 - (M_{34} + M_{43})\phi_5 - (M_{67} + M_{76})\phi_3 = 0$$

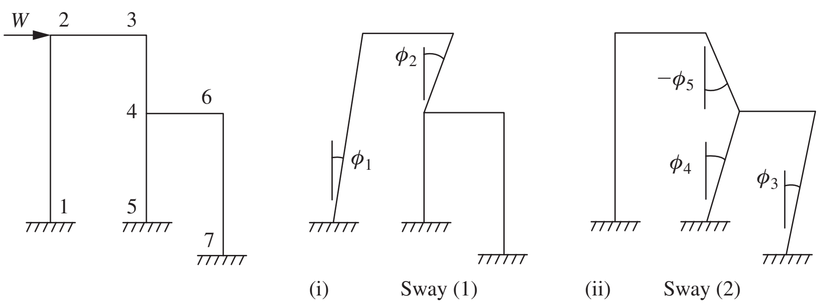


FIGURE 7.26

and the rotations may be eliminated by considering the geometry of the structure.

The ridged portal shown in Figure 7.27 has the two degrees of sway freedom shown at (i) and (ii). Considering sway (1) and (2) in turn, the sway equations are:

$$-(M_{12} + M_{21}) - (M_{54} + M_{45}) = W_1 l_{12}$$
$$-(M_{12} + M_{21})\phi_1 - (M_{54} + M_{45})\phi_1 - (M_{23} + M_{32})\phi_2$$
$$-(M_{34} + M_{43})\phi_2 - (M_{43} + M_{34})\phi_2 = W_2 \times 1 - W_1 l_{12}\phi_1$$

and the rotations in the second equation may be eliminated using expressions obtained from a displacement diagram.

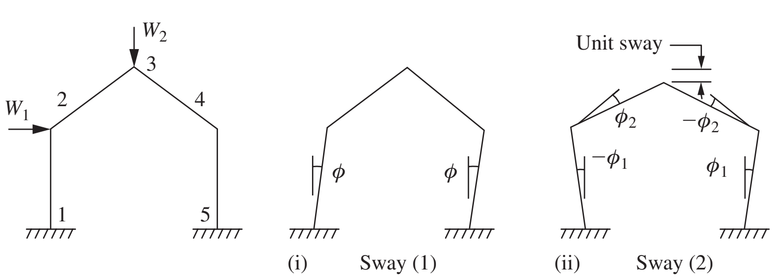


Figure 7.27

In all cases, the sway equations can be considered as consisting of a left-hand side involving moments in the structure and a right-hand side involving the loads applied to the structure.

(c) Sway procedure

The analysis of structures subjected to sway proceeds in two stages as shown at (i) and (ii) in Figure 7.28. The first stage consists of a non-sway distribution with the external loads applied to the structure, which is prevented from displacing laterally. The fixed-end moments are obtained, and the distribution proceeds normally to give the final moments M^W. The second stage consists of determining the moments produced by a unit sway displacement. The fixed-end moments due to the unit displacement are obtained, and the distribution proceeds normally to give the final moments M^S. The actual moments in the structure are:

$$M = M^W + xM^S$$

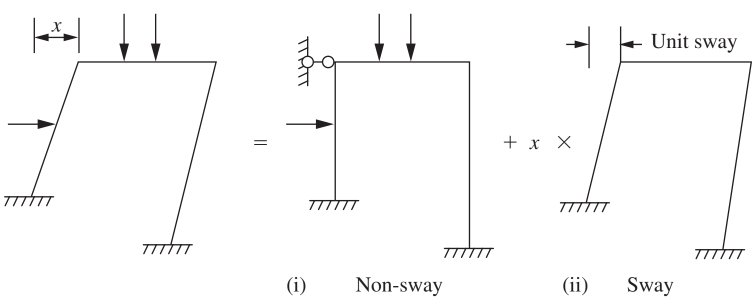


Figure 7.28

and these moments satisfy the sway equation. Hence, if substituting the moments M^W and M^S in the left-hand side of the sway equation produces the values C^W and C^S, respectively, then:

$$C^W + xC^S = \text{right-hand side of sway equation}$$

and the value of x may be obtained.

The analysis of a structure with two degrees of sway freedom proceeds in three stages as shown at (i), (ii), and (iii) of Figure 7.29. The moments produced by the non-sway, sway (1), and sway (2) stages are M^W, M^{S1}, and M^{S2}, and these, when substituted in turn in the left-hand side of sway equation (1) and sway equation (2), give the values $C_1^W, C_1^{S1}, C_1^{S2}$ and $C_2^W, C_2^{S1}, C_2^{S2}$ respectively.

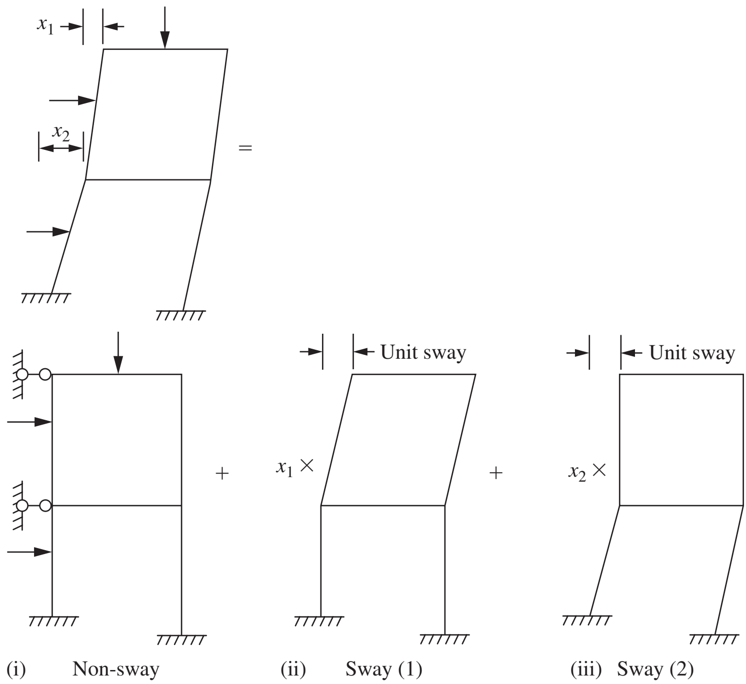


Figure 7.29

Then $\quad C_1^W + x_1 C_1^{S1} + x_2 C_1^{S2} =$ right-hand side of sway equation (1)

and $\quad C_2^W + x_1 C_2^{S1} + x_2 C_2^{S2} =$ right-hand side of sway equation (2)

The values of x_1 and x_2 are obtained by solving these two equations simultaneously, and the actual moments in the structure are:

$$M = M^W + x_1 M^{S1} + x_2 M^{S2}$$

In a similar fashion, structures having more than two degrees of freedom may be analyzed.

In practice, it is not necessary to impose unit displacement on the structure; any arbitrary displacement may be imposed that produces convenient values for the initial fixed-end moments. The value obtained for x will thus not be the actual displacement of the structure, but this in any case is generally not required.

When the external loads are applied to the joints of the structure, the non-sway distribution is not required, since the loading produces no fixed-end moments.

(d) Illustrative examples

Example 7.11

Determine the bending moments in the frame shown in Figure 7.30. The second moments of area of the members are shown ringed.

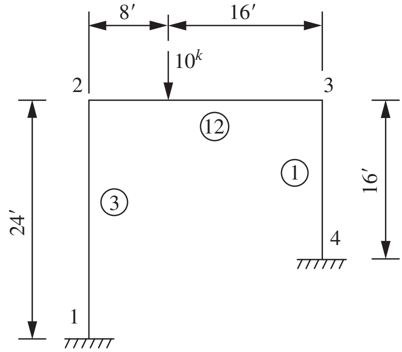


Figure 7.30

Solution

The sway equation is derived in a manner similar to that used for the frame shown at (ii) in Figure 7.23 and is:

$$M_{12} + M_{21} + 3(M_{34} + M_{43})/2 = 0$$

The fixed-end moments due to the applied loads are:

$$\begin{aligned} M^F_{23} &= -10 \times 8 \times 16 \times 16 \times 12/(24 \times 24) \\ &= -427 \text{ kip-in} \\ M^F_{32} &= 10 \times 16 \times 8 \times 8 \times 12/(24 \times 24) \\ &= 213 \text{ kip-in} \end{aligned}$$

The fixed-end moments due to an arbitrary sway displacement are:

$$\begin{aligned} M^F_{12} &= M^F_{21} \\ &= -6E\delta \times 48/24^2 \\ &= -6E\delta/12 \\ &= -400x \\ M^F_{34} &= M^F_{43} \\ &= -6E\delta \times 16/16^2 \\ &= -6E\delta/16 \\ &= -300x \end{aligned}$$

The distribution procedure for the sway and non-sway stages is shown in Table 7.13.

Table 7.13 Distribution of moments in Example 7.11

Joint	1	2		3		4
Member	12	21	23	32	34	43
Relative EI/l	2	2	8	8	1	1
Distribution factor	0	1/5	4/5	8/9	1/9	0
Carry-over factor	←	½	← ½	½ →	½	→
M^F, sway	−400	−400			−300	−300
Distribution		80	320	267	33	
Carry-over	40		134	160		16
Distribution		−27	−107	−142	−18	
Carry-over	−14		−71	−54		−9
Distribution		14	57	48	6	
Carry-over	7		24	28		3
Distribution		−5	−19	−25	−3	
Carry-over	−2		−12	−9		−1
Distribution		2	10	8	1	
Carry-over	1		4	5		
Distribution		−1	−3	−4	−1	
M^S	−368	−337	337	282	−282	−291
M^F, non-sway			−427	213		
Distribution		85	342	−189	−24	
Carry-over	42		−95	171		−12
Distribution		19	76	−152	−19	
Carry-over	10		−76	38		−10
Distribution		15	61	−34	−4	
Carry-over	8		−17	30		−2
Distribution		3	14	−27	−3	
Carry-over	2		−13	7		−2
Distribution		3	10	−6	−1	
Carry-over	1		−3	5		
Distribution		1	2	−4	−1	
M^W	63	126	−126	52	−52	−26
$0.046 \times M^S$	−17	−16	16	13	−13	−13
M kip-in	46	110	−110	65	−65	−39

Substituting the final non-sway moments in the left-hand side of the sway equation gives:

$$189 - 3 \times 78/2 = 72$$

Substituting the final sway moments in the left-hand side of the sway equation gives:

$$-705 - 3 \times 573/2 = -1565$$

Then: $72 - 1565x = 0$
And: $x = 0.046$

The actual moments are obtained by adding the final non-sway moments to $x \times$ the final sway moments.

Example 7.12

Determine the position on the beam, for the frame shown in Figure 7.31, at which a unit load may be applied without causing sway. All the members are of uniform cross-section.

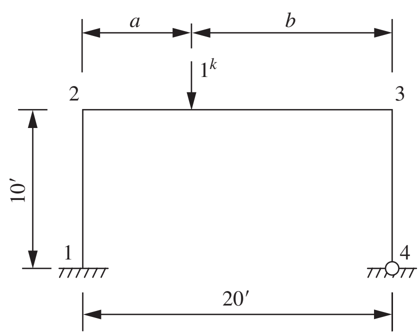


Figure 7.31

Solution

The sway equation is:

$$M_{12} + M_{21} + M_{34} = 0$$

and a sway distribution is not required since there is no sway.

The fixed-end moments due to the applied loads are:

$$M^F_{23} = -ab^2/400$$
$$M^F_{32} = ba^2/400$$

and these are in the ratio −300b:300a.

The distribution procedure is shown in Table 7.14; substituting the final moments in the sway equation gives:

$$279b - 125a = 0$$

Table 7.14 Distribution of moments in Example 7.12

Joint	1	2		3		4
Member	12	21	23	32	34	43
Relative EI/l	2	2	1	1	2	2
Modified stiffness		4	2	2	3	
Distribution factor	0	2/3	1/3	2/5	3/5	1
Carry-over factor	←	½	←	½		
			½	→	0	→
M^F, non-sway			−300b	300a		
Distribution		200b	100b	−120b	−180a	
Carry-over	100b		−60a	50b		
Distribution		40a	20a	−20b	−30b	
Carry-over	20a		−10b	10a		
Distribution		7b	3b	−4a	−6a	
Carry-over	3b		−2a	2b		
Distribution		a	a	−b	−b	
M^W	20a +103b	41a +207b	−41a −207b	186a +31b	−186a −31b	0

Also:

$$b + a = 20$$

Solving these equations simultaneously, we obtain:

$$a = 13.8\,\text{ft}$$

Example 7.13

Determine the bending moments in the frame shown in Figure 7.32. All the members of the frame have the same second moment of area.

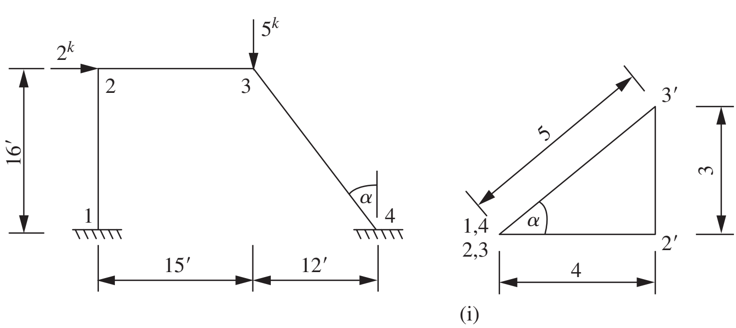


Figure 7.32

Solution

The displacement diagram, for a horizontal displacement of 4 ft units imposed at 2 is shown at (i), and the sway equation is:

$$
\begin{aligned}
&-(M_{12} + M_{21}) \times 22'/16 + (M_{23} + M_{32}) \\
&\quad \times 2'3'/15 - (M_{34} + M_{43}) \times 33'/20 = 2 \times 22' - 5 \times 2'3' \\
&\quad - (M_{12} + M_{21}) \times 4/16 + (M_{23} + M_{32}) \times 3/15 \\
&\quad - (M_{34} + M_{43}) \times 5/20 = 2 \times 4 - 5 \times 3 \\
&\quad - 5(M_{12} + M_{21}) - 4(M_{23} + M_{32}) + 5(M_{34} + M_{43}) = 140
\end{aligned}
$$

The fixed-end moments due to the sway displacements are:

$$
\begin{aligned}
M_{12} &= M_{21} \\
&= -6EI \times (22')/16^2 \\
&= -6EI \times 4/16^2 = -750x \\
M_{23} &= M_{32} \\
&= 6EI \times (2'3')/15^2 \\
&= 6EI \times 3/15^2 \\
&= 640x \\
M_{34} &= M_{43} \\
&= -6EI \times (32')/20^2 \\
&= -6EI \times 5/20^2 \\
&= -600x
\end{aligned}
$$

Table 7.15 Distribution of moments in Example 7.13

Joint	1	2		3		4
Member	12	21	23	32	34	43
Relative EI/l	1/16	1/16	1/15	1/15	1/20	1/20
Distribution factor	0	15/31	16/31	4/7	3/7	0
Carry-over factor	←	½	←	½		
			½	→	½	→
M^S, sway	−750	−750	640	640	−600	−600
Distribution		53	57	−24	−16	
Carry-over	26		−12	28		−8
Distribution		6	6	−16	−12	
Carry-over	3		−8	3		−6
Distribution and carry-over	2	4	4	−2	−1	0
M^S	−719	−687	687	629	−629	−614
$-0.00757 \times M^S$	5.4	5.2	−5.2	−4.8	4.8	4.7

The distribution procedure is shown in Table 7.15, and substituting the final sway moments in the sway equation gives:

$$(-7030 - 5264 - 6215)x = 140$$
$$x = -0.00757$$

The final moments are shown in the table.

Example 7.14

Determine the bending moments in the symmetrical ridged portal frame shown in Figure 7.33, which carries a uniformly distributed load of 1 kip/ft on plan. All the members have the same cross-section.

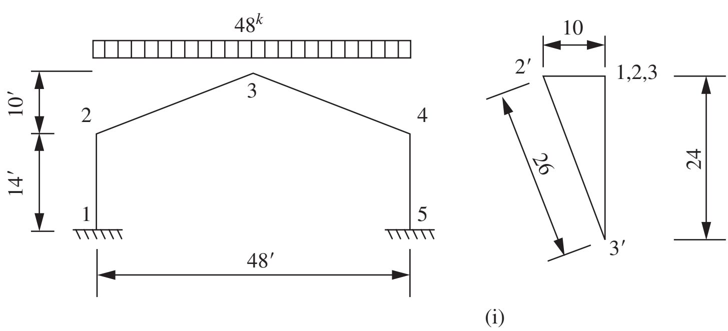


Figure 7.33

Solution

Due to symmetry of the structure and the loading only one mode of sway, shown at (i) in Figure 7.27, is possible. The displacement diagram for a vertical displacement of 24 ft units imposed at 3 is shown at (i) in Figure 7.33, and the sway equation is:

$$(M_{12} + M_{21}) \times 10/14 - (M_{23} + M_{32}) \times 26/26 + (M_{34} + M_{43}) \times 26/26$$
$$- (M_{45} + M_{54}) \times 10/14 = 48 \times 12 \times 24/2$$

Since the bending moments at corresponding points of the symmetrical frame are equal and of opposite sense, this reduces to:

$$-5(M_{12} + M_{21}) - 7(M_{23} + M_{32}) = 7 \times 24 \times 12 \times 24/2 = 24{,}190 \text{ kip-in}$$

The fixed-end moments due to the applied loads are:

$$\begin{aligned} M^F_{32} &= -M^F_{23} \\ &= 24 \times 24 \times 12/12 \\ &= 576 \text{ kip-in} \end{aligned}$$

The fixed-end moments due to the sway displacement are:

$$\begin{aligned} M^F_{12} &= M^F_{21} \\ &= 6EI \times 10/14^2 \\ &= 6EI \times 10/196 \\ &= 260y \\ M^F_{32} &= M^F_{23} \\ &= -6EI \times 26/26^2 \\ &= -6EI \times 1/26 \\ &= -196y \end{aligned}$$

The distribution procedure for the sway and non-sway stages is shown in Table 7.16, where joint 3 is considered fixed and there is no carry-over between the two halves of the frame. Substituting the final moments for these two stages in the left-hand side of the sway equation gives:

$$5 \times 457 + 7 \times 425 = 5260$$

Table 7.16 Distribution of moments in Example 7.14

Joint	1	2		3
Member	12	21	23	32
Relative EI/l	1/14	1/14	1/26	1/26
Distribution factor	0	13/20	7/20	0
Carry-over factor	←	½	½	→
M^F, sway	260	260	−196	−196
Distribution and carry-over	−21	−42	−22	−11
M^S	239	218	−218	−207
M^F, non-sway			−576	576
Distribution and carry-over	187	374	202	101
M^W	187	374	−374	677
$4.48 \times M^S$	1065	975	−975	−930
M kip-in	1252	1349	−1349	−253

and: $5 \times 561 - 7 \times 303 = 684$

Then: $684 + 5260y = 24{,}190$

$$y = 4.48$$

and the final moments are shown in the table.

Example 7.15

Determine the bending moments in the two bay frame shown in Figure 7.34. The relative EI/l values are shown ringed.

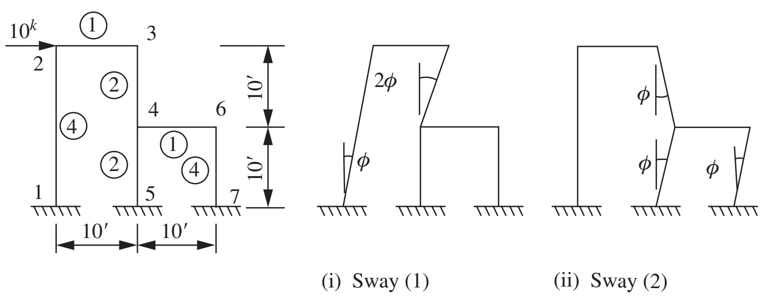


Figure 7.34

Solution

The two sway equations are:

$$\begin{aligned}
&\text{sway (1)}, \quad -(M_{12} + M_{21}) - 2(M_{34} + M_{43}) = 200 \text{ kip-ft} \\
&\text{sway (2)}, \quad -(M_{34} + M_{43}) - (M_{45} + M_{54}) - (M_{67} + M_{76}) = 0
\end{aligned}$$

The fixed-end moments due to sway (1) are:

$$\begin{aligned}
M^F_{12} &= M^F_{21} \\
&= 6\delta \times 4/20 \\
&= -100x_1 \\
M^F_{43} &= M^F_{34} \\
&= -6\delta \times 2/10 \\
&= -100x_1
\end{aligned}$$

The fixed-end moments due to sway (2) are:

$$\begin{aligned}
M^F_{43} &= M^F_{34} \\
&= -M^F_{45} \\
&= -M^F_{54} \\
&= -6\delta \times 2/10 \\
&= -100x_2
\end{aligned}$$