University of Salahaddin-Hawler
College of Engineering
Software and Informatics Engineering Department
Second Year Class

Numerical Analysis and Probability

Solution of Linear Simultaneous Equations

Gauss – Jacobi Iteration Method

Gauss–Seidel Method

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2023-2024

Exams Schedule

2- Practical 1 28th Feb 2024 %10

All studied practical lectures

3- Quiz 4 Theory 6th March 2024 %5 ………

Solution of a System of n Linear Equations in n Variables.

- There are two iterative methods:

  1- Gauss – Jacobi Iteration method

  2- Gauss – Seidal Iteration method

The Gauss-Jacobi Iterative Methods

- Solution steps as an algorithm:

1. Rearrange the equation so that the variable is put on the left side
2. Assume (guess) an initial value of the variable to start the first iteration.

The Gauss-Jacobi Iterative Methods

3. Substitute the value of the variable in the right side of the equation and calculate a new value for the variable.

4. If the new value of the variable is not equal to the previous value, consider the new one as the value of the variable.

The Jacobi Iterative Methods

5. Repeat steps 3 and 4 until the new value is equal to the old value of the variable. Then output the value of the variable and stop.

6. In case the new value does not approach the old value (the difference increases at each iteration) stop calculation and try another initial value or another rearrangement of the given equations.

The Jacobi Iterative Methods

- The Jacobi method makes a assumption:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

- The coefficient matrix has no zeros on its main diagonal, namely a_{11},a_{22} ,a_{33} are non zeros.

The Jacobi Iterative Methods

- If any of the diagonal entries $a_{11}, a_{22}, a_{33}, \ldots \ldots . . a_{nm}$ are zero , then rows or columns must be interchanged to obtain an coefficient matrix that has non zero entries on the main diagonal.

e.g.

$$\begin{aligned} 3x_1 + 7x_2 + 13x_3 &= 76 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 12x_1 + 3x_2 - 0x_3 &= 1 \end{aligned} \longrightarrow \begin{aligned} 12x_1 + 3x_2 - 0x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \end{aligned}$$

The sufficient condition for convergence in the Gauss methods is that the system of equation must be strictly diagonally dominant.

Diagonally Dominant

In mathematics, a square matrix is said to be diagonally dominant if for every row of the matrix, the absolute of the coefficient of the diagonal entry in a row is larger than or equal to the sum of the absolute of the coefficients of all the other (non-diagonal) entries in that row.

Diagonally Dominant

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Diagonally Dominant

- Is diagonally dominant because:

$|a_{11}| \geq |a_{12}| + |a_{13}|$ since $|+3| \geq |-2| + |+1|$

$|a_{22}| \geq |a_{21}| + |a_{23}|$since $|-3| \geq |+1| + |+2|$

$|a_{33}| \geq |a_{31}| + |a_{32}|$ since $|+4| \geq |-1| + |+2|$

The Jacobi Iterative Methods

- Main idea of Jacobi to begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

The Jacobi Iterative Methods

- Then make an initial guess of the solution

$$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)}).$$

- Substitute these values into the right hand side the of the rewritten equations to obtain the first approximation $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$. ,this accomplishes one iteration.

The Jacobi Iterative Methods

- In the same way, the second approximation $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots x_n^{(2)}\right)$ is computed by substituting the first approximation's x vales into the right hand side of the rewritten equations.

The Jacobi Iterative Methods

- **E.X:** Apply the Jacobi method to solve, considered three decimal digits

$$5x_1 - 2x_2 + 3x_n = -1$$
$$-3x_1 + 9x_2 + x_n = 2$$
$$2x_1 - x_2 - 7x_n = 3$$

- Continue iterations until two successive approximations are identical when rounded to three decimal digits.

The Jacobi Iterative Methods

- To begin, rewrite the system:

$$x_1 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$
$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

$$x_1 = 0, x_2 = 0, x_3 = 0$$

- Choose the initial guess:

The Jacobi Iterative Methods

- The 1st approximation is:

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3^{(1)} = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

The Jacobi Iterative Methods

- **So, the iterations process continue till convergences is secure**
- The 2nd approximation is:

$$x_1^{(2)} = \frac{-1}{5} + \frac{2}{5} * (0.222) + \frac{3}{5} * 0.429 = 0.146$$

$$x_2^{(2)} = \frac{2}{9} + \frac{3}{9} * (-0.200) + \frac{1}{9} * 0.429 = 0.203$$

$$x_3^{(2)} = \frac{-3}{7} + \frac{2}{7} * (-0.200) - \frac{1}{7} * 0.222 = -0.517$$

The Jacobi Iterative Methods

- The 3rd approximation is:

$$x_1^{(3)} = \frac{-1}{5} + \frac{2}{5} *(0.203) - \frac{3}{5} * -0.518 = 0.192$$

$$x_2^{(3)} = \frac{2}{9} + \frac{3}{9} *(0.145) - \frac{1}{9} * -0.518 = 0.328$$

$$x_3^{(3)} = \frac{-3}{7} + \frac{2}{7} *(0.145) - \frac{1}{7} * 0.203 = -0.416$$

The Jacobi Iterative Methods

- The 4th approximation is:

$$x_1^{(4)} = \frac{-1}{5} + \frac{2}{5}*(0.328) - \frac{3}{5}* -0.416 = 0.181$$

$$x_2^{(4)} = \frac{2}{9} + \frac{3}{9}*(0.192) - \frac{1}{9}* - 0.416 = 0.332$$

$$x_3^{(4)} = \frac{-3}{7} + \frac{2}{7}*(0.192) - \frac{1}{7}* 0.328 = -0.421$$

The Jacobi Iterative Methods

- The 5th approximation is:

$$x_1^{(5)} = \frac{-1}{5} + \frac{2}{5} * (0.332) - \frac{3}{5} * -0.421 = 0.185$$

$$x_2^{(5)} = \frac{2}{9} + \frac{3}{9} * (0.181) - \frac{1}{9} * -0.421 = 0.329$$

$$x_3^{(5)} = \frac{-3}{7} + \frac{2}{7} * (0.181) - \frac{1}{7} * 0.332 = -0.424$$

The Jacobi Iterative Methods

- The 6th approximation is:

$$x_1^{(6)} = \frac{-1}{5} + \frac{2}{5} * (0.329) - \frac{3}{5} * -0.424 = 0.186$$

$$x_2^{(6)} = \frac{2}{9} + \frac{3}{9} * (0.185) - \frac{1}{9} * -0.424 = 0.331$$

$$x_3^{(6)} = \frac{-3}{7} + \frac{2}{7} * (0.185) - \frac{1}{7} * 0.329 = -0.423$$

The Jacobi Iterative Methods

- The 7th approximation is:

$$x_1^{(7)} = \frac{-1}{5} + \frac{2}{5} * (0.331) - \frac{3}{5} * - 0.423 = 0.186$$

$$x_2^{(7)} = \frac{2}{9} + \frac{3}{9} * (0.186) - \frac{1}{9} * - 0.423 = 0.331$$

$$x_3^{(7)} = \frac{-3}{7} + \frac{2}{7} * (0.186) - \frac{1}{7} * 0.331 = -0.423$$

The Jacobi Iterative Methods

By Gauss-Jacobi:

$x_1 = 0.186$

$x_2 = 0.331$

$x_3 = -0.423$

The Jacobi Iterative Methods

n	k=0	k=1	k=2	k=3	k=4	k=5	k=6
$x_1^{(k)}$	0.000	-0.200	0.146	0.192	0.181	0.185	0.186
$x_2^{(k)}$	0.000	0.222	0.203	0.328	0.332	0.329	0.331
$x_3^{(k)}$	0.000	0.429	-0.517	-0.416	-0.421	-0.424	-0.423

The Jacobi Iterative Methods

- **H.W:** Solve this equations using Gauss-Jacobi method

  $3x+y=11$

  $2x+5y=16$

  Choose the initial guess: x=0; y=0, considered three decimal digits

  $x^1 = \frac{11}{3} - \frac{1}{3}y$

  $y^1 = \frac{16}{5} - \frac{2}{5}x$

Gauss–Seidel Method

- In numerical linear algebra, the Gauss–Seidel method, also known as the method of successive displacement, is an iterative method used to solve a system of linear equations.
- It can be applied to any matrix with non-zero elements on the diagonals.
- With the Gauss-Jacobi method, the values of x_i^k obtained in the k^{th} iteration remain unchanged until the entire $(k+1)^{th}$ iteration has been calculated.

Gauss–Seidel Method

- With the Gauss-Seidel method, we use the new values x_i^{k+1} as soon as they are known.
- For example, once we have computed x_1^{k+1} from the first equation, its value is then used in the second equation to obtain the new x_2^{k+1} and so on.

Gauss–Seidel Method

- This method requires fewer iteration to produce the same degree of accuracy.
- This method is almost identical (modification) with Gauss –Jacobi method except in considering the iteration equations.
- The sufficient condition for convergence in the Gauss – Seidal method is that the system of equation must be strictly diagonally dominant.

Gauss–Seidel Method

- **E.X:** Solve Equations, using Gauss Seidel method, considered three decimal digits:

  $3x+y=11$

  $2x+5y=16$

- From the above equations:

  $$x^{1}=\frac{11}{3}-\frac{1}{3}\,y$$

  $$y^{1}=\frac{16}{5}-\frac{2}{5}\,x$$

Gauss–Seidel Method

- 1^{st} Approximation

$$x^1 = \frac{11}{3} - \frac{1}{3} * 0 = 3.667$$

$$y^1 = \frac{16}{5} - \frac{2}{5} *(3.667) = 1.733$$

Gauss–Seidel Method

2nd Approximation

$$x^2 = \frac{11}{3} - \frac{1}{3} * 1.733 = 3.089$$

$$y^2 = \frac{16}{5} - \frac{2}{5} *(3.089) = 1.964$$

Gauss–Seidel Method

3^{rd} Approximation

$$x^3 = \frac{11}{3} - \frac{1}{3} * 1.964 = 3.012$$

$$y^3 = \frac{16}{5} - \frac{2}{5} *(3.012) = 1.995$$

Gauss–Seidel Method

- 4^{th} Approximation

$$x^4 = \frac{11}{3} - \frac{1}{3} * 1.995 = 3.002$$

$$y^4 = \frac{16}{5} - \frac{2}{5} * (3.002) = 1.999$$

Gauss–Seidel Method

- 5^{th} Approximation

$$x^5 = \frac{11}{3} - \frac{1}{3} * 1.999 = 3.000$$

$$y^5 = \frac{16}{5} - \frac{2}{5} *(3.000) = 2$$

Gauss–Seidel Method

- 6^{th} Approximation

$$x^6 = \frac{11}{3} - \frac{1}{3} * 2 = 3$$

$$y^6 = \frac{16}{5} - \frac{2}{5} * (3) = 2$$

Gauss–Seidel Method

n	k=0	k=1	k=2	k=3	k=4	k=5	k=6
$x^{(k)}$	0.000	3.667	3.089	3.012	3.002	3.000	3
$y^{(k)}$	0.000	1.733	1.964	1.995	1.999	2	2

The Seidel Iterative Methods

- **H.W:** Apply the Seidel method to solve, considered three decimal digits

$$5x_1 - 2x_2 + 3x_n = -1$$
$$-3x_1 + 9x_2 + x_n = 2$$
$$2x_1 - x_2 - 7x_n = 3$$

- Continue iterations until two successive approximations are identical when rounded to three decimal digits.