

Chapter 1

Propositional Logic

1.1 Statements and Compound Statements

A *statement* or *proposition* is an assertion which is either true or false, though you may not know which. That is, a statement is something that has a *truth value*.

Here are some examples of statements.

- There are no integers a and b so that $\sqrt{2} = \frac{a}{b}$. (True.)
- For all integers $n \geq 0$, the number $n^2 - n + 41$ is prime. (False.)
- Every even positive integer except 2 is the sum of two prime numbers. (Goldbach's Conjecture: unknown.)

We usually use letters to denote statements. A good way to think of these letters is as variables that can take the values “true” and “false”. Variables that can take two possible values are sometimes called *Boolean variables*, after the British logician George Boole.

A *compound statement* is one formed by joining other statements together with *logical connectives*. Several such connectives are defined below. The statements that are joined together can themselves be compound statements.

Let p and q be statements.

The *conjunction of p and q* (read: p and q) is the statement $p \wedge q$ which asserts that p and q are both true. Notice that the wedge symbol looks vaguely like the letter “n” in and.

The *disjunction of p and q* (read: p or q) is the statement $p \vee q$ which asserts that either p is true, or q is true, or both are true. Notice that this is the inclusive sense of the word “or”. Also, the vee symbol looks vaguely like the letter “r” in or.

The *implication $p \rightarrow q$* (read: p implies q , or if p then q) is the statement which asserts that if p is true, then q is also true. We agree that $p \rightarrow q$ is true when p is false. The statement p is called the *hypothesis* of the implication, and the statement q is called the *conclusion* of the implication.

The *biconditional or double implication $p \leftrightarrow q$* (read: p if and only if q) is the statement which asserts that p and q have the same truth value. That is, if p is true then q is true, and if q is true then p is true.

A *truth table* gives the truth values of a statement for all possible combinations of truth values of the other statements from which it is made. Here and elsewhere, 0 and 1 will represent the truth values “false” and “true”, respectively.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Examples.

- “ $(e < \pi) \wedge (57 \text{ is prime})$ ” is false because “ (57 is prime) ” is false.
- “ $(\sqrt{2} \text{ is rational}) \vee (\frac{1+\sqrt{5}}{2} < 2)$ ” is true because “ $(\frac{1+\sqrt{5}}{2} < 2)$ ” is true (the statement “ $(\sqrt{2} \text{ is rational})$ ” is false).
- “ $(5^2 < 0) \rightarrow (1 < 2)$ ” is true because the hypothesis “ $(5^2 < 0)$ ” is false.
- “ $(1 < 2) \rightarrow (5^2 < 0)$ ” is false because the hypothesis is true and the conclusion is false. This example demonstrates that $p \rightarrow q$ and $q \rightarrow p$ are not the same.

- “ $(1 = 2) \leftrightarrow (\text{the number of primes is finite})$ ” is true because both “ $(1 = 2)$ ” and “ $(\text{the number of primes is finite})$ ” are false.

1.2 Negation of Statements

The **negation** of a statement p (read: **not** p) is the statement $\neg p$ which asserts that p is not true. Sometimes it is helpful to think of $\neg p$ as asserting “*it is not the case that p is true*”. Thus, $\neg p$ is false when p is true, and true when p is false.

p	$\neg p$
0	1
1	0

Notice that “ \neg ” is not a logical connective. It does not join two statements together. Instead, it applies to a single (possibly compound) statement.

Negation has precedence over logical connectives. Thus $\neg p \vee q$ means $(\neg p) \vee q$.

The negation of $\neg p$ is the statement with the opposite truth value as $\neg p$, thus $\neg(\neg p)$ is just another name for p .

The negation of $p \wedge q$ asserts “it is not the case that p and q are both true”. Thus, $\neg(p \wedge q)$ is true exactly when one or both of p and q is false, that is, when $\neg p \vee \neg q$ is true.

Similarly, $\neg(p \vee q)$ can be seen to be the same as $\neg p \wedge \neg q$.

Our reasoning can be checked on the truth tables below. Observe that the pairs of statements in question have the same truth value given any combination of possible truth values of p and q .

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	0	1	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	0	0	1	0	0	1	0	0

To find the negation of $p \rightarrow q$, we return to its description. The statement is false only when p is true and q is false. Therefore $\neg(p \rightarrow q)$ is the same as

$p \wedge \neg q$. Using the same reasoning, or by negating the negation, we can see that $p \rightarrow q$ is the same as $\neg p \vee q$.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$
0	0	1	1	1	0	0
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Finally, the statement $p \leftrightarrow q$ asserts that p and q have the same truth value. Hence $\neg(p \leftrightarrow q)$ asserts that p and q have different truth values. This happens when p is true and q is false, or when p is false and q is true (that is, when p *exclusive or* q is true). Thus, $\neg(p \leftrightarrow q)$ is the same as $(p \wedge \neg q) \vee (\neg p \wedge q)$.

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
0	0	1	1	1	0	0	0	0
0	1	1	0	0	1	0	1	1
1	0	0	1	0	1	1	0	1
1	1	0	0	1	0	0	0	0

1.3 Making and Using Truth Tables

Recall that the truth values of a statement can be summarized in a truth table. Also recall that “ \neg ” has precedence over logical connectives, and otherwise there is no implied order. This means that $p \vee \neg q$ is $p \vee (\neg q)$, and that $p \vee q \rightarrow r$ is actually $(p \vee q) \rightarrow r$, though it is far better to simply regard the statement as ambiguous and insist on proper bracketing.

To make a truth table, start with k columns corresponding to the most basic statements (usually represented by letters). How many rows are needed? We will argue below that if there are k of these you will need 2^k rows to list all possible combinations of truth values for these statements. After listing all 2^k possible combinations of truth values of the k most basic statements then, working with what’s inside the brackets first (just like algebra!), add a new column for each connective in the expression, and fill in the truth values using the definitions from before.

Here is how to see that a truth table that involves k basic statements needs 2^k rows. It is clear that two rows are needed when $k = 1$: one for when the statement is true and one for when it is false. Now consider the case when $k = 2$. When the first statement is true, the second can be true or false, and when the first statement is false, the second can be true or false. Thus, when $k = 2$ there are four rows needed in the truth table. The same sort of argument applies when $k = 3$: eight rows are needed. Four rows are needed to cover the situations where the first statement is true, as (from before) there are four combinations of truth values for the other two statements, and four more rows are needed to cover the situations where the first statement is false. Continuing in this way, the numbers of rows needed doubles for each additional statement, so if there are k statements you will need 2^k rows to list all possible combinations of truth values.

The truth table for $(\neg p \rightarrow r) \rightarrow (q \vee \neg r)$ will have 8 rows. Starting with the collection of truth possible values for p, q and r , we add columns to obtain the truth values of $\neg p$, $(\neg p \rightarrow r)$, $\neg r$, $(q \vee \neg r)$, and then, finally, the entire statement we want.

p	q	r	$\neg p$	$\neg p \rightarrow r$	$\neg r$	$q \vee \neg r$	$(\neg p \rightarrow r) \rightarrow (q \vee \neg r)$
0	0	0	1	0	1	1	1
0	0	1	1	1	0	0	0
0	1	0	1	0	1	1	1
0	1	1	1	1	0	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	0	0	0
1	1	0	0	1	1	1	1
1	1	1	0	1	0	1	1

Sometimes only part of the truth table needs to be made. For example, suppose it is given a and b are false, and c is true. Then the truth value of $\neg a \vee (b \vee \neg c)$ can be found by completing the single row of the truth table where a, b and c have the given truth values.

If we are given that p is false and q is true, then we can find all possible truth values of $\neg(p \leftrightarrow r) \rightarrow (q \rightarrow s)$ by completing the four rows of the truth table where p and q have the truth values given, and all possible truth values for r and s occur.

Sometimes information about truth values can be given a more indirectly.

Suppose we're given that $\neg a \rightarrow (b \leftrightarrow \neg c)$ is false, and asked to determine all possible truth values of $(a \vee b) \wedge (\neg b \vee \neg c)$. The information that given implication is false, lets us conclude that its hypothesis, $\neg a$, is true (so a is false), and its conclusion, $(b \leftrightarrow \neg c)$, is false (so b and $\neg c$ have different truth values, that is, b and c have the same truth value. Hence we need a truth table with only two rows:

a	b	c	$\neg b$	$\neg c$	$a \vee b$	$\neg b \vee \neg c$	$(a \vee b) \wedge (\neg b \vee \neg c)$
0	0	0	1	1	0	1	0
0	1	1	0	0	1	0	0

Therefore, if $\neg a \rightarrow (b \leftrightarrow \neg c)$ is false, so is $(a \vee b) \wedge (\neg b \vee \neg c)$.

1.4 Converse and Contrapositive

The *converse* of the implication $p \rightarrow q$ is $q \rightarrow p$. The example above shows that an implication and its converse can have different truth values, and therefore can not be regarded as the same.

The *contrapositive* of the implication $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

For example, the contrapositive of “if a and b are odd integers then the integer ab is odd” is “if the integer ab is even then it is not the case that the integers a and b are odd”, or equivalently “if the integer ab is even then the integer a is even or the integer b is even”.

p	q	$\neg p$	$\neg q$	implication		contrapositive		converse	
				$p \rightarrow q$	$\neg p \vee q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$		
0	0	1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1	0	0
1	0	0	1	0	0	0	0	1	1
1	1	0	0	1	1	1	1	1	1

Notice that the implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ have exactly the same truth table, that is, they are each true for exactly the same truth values of p and q . It is reasonable to regard these statements as being “the same” (and we will) in a similar way as we regard 0.25 and 2/8 as being the same.

For completeness, we note that the *inverse* of $p \rightarrow q$ is the statement $\neg p \rightarrow \neg q$. It is the contrapositive of the converse (or the other way around) of $p \rightarrow q$.

1.5 Logical Equivalence

In the process of making the truth table for $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$, we see that the two bracketed statements have the same truth values for given truth value assignments to p and q . Hence the double implication is always true.

A statement which is always true is called a *tautology*. A statement which is always false is called a *contradiction*.

For example, $p \wedge (\neg p)$ is a contradiction, while $p \vee (\neg p)$ is a tautology. Most statements are neither tautologies nor contradictions.

One way to determine if a statement is a tautology is to make its truth table and see if it (the statement) is always true. Similarly, you can determine if a statement is a contradiction by making its truth table and seeing if it is always false.

Informally, two statements s_1 and s_2 are logically equivalent if they have the same truth table (up to the order of the rows). This happens exactly when the statement $s_1 \leftrightarrow s_2$ is a tautology.

Formally, two statements s_1 and s_2 are *logically equivalent* if $s_1 \leftrightarrow s_2$ is a tautology.

We use the notation $s_1 \Leftrightarrow s_2$ to denote the *fact* (theorem) that $s_1 \leftrightarrow s_2$ is a tautology, that is, that s_1 and s_2 are logically equivalent. Notice that $s_1 \leftrightarrow s_2$ is a statement and can in general be true or false, and $s_1 \Leftrightarrow s_2$ indicates the (higher level) fact that it is always true.

Logical equivalence plays the same role in logic that equals does in algebra: it specifies when two expressions are “the same”. In the same way that equal expressions can be freely substituted for each other without changing the meaning of an expression, logically equivalent statements can be freely substituted for each other without changing the meaning of a compound statement. And, if two statements are each equivalent to the same statement, they are equivalent to each other.

Since logical equivalence is defined in terms of a statement being a tau-

tology, a truth table can be used to check if (prove that) two statements are logically equivalent. Soon we will have other methods to do this as well.

1.6 Necessity and Sufficiency

The words in the title of this subsection are used in reference to implications. Suppose the implication $p \rightarrow q$ is true. Take a good look at the three rows for the truth table that describe the situations where that occurs.

Is there a condition on q that is required (i.e. *necessary*) in order for p to be true? It may not guarantee the truth of p ; it just is a condition that is needed in the sense that p is not true unless the condition holds. From the truth table, assuming $p \rightarrow q$ is true, we see that p is not true unless q is also true. If q is false, then we know from the truth table that p must be false, so in order for p to be true, we need that q be true. Note, however, that the truth of q does not guarantee the truth of p as $p \rightarrow q$ is true when p is false and q is true. The words “ q is necessary for p ” are therefore considered to be equivalent to “if p then q ”, or $p \rightarrow q$.

An example is the statement “you must be at least four feet tall in order to ride the roller coaster”. In order to guarantee that it is possible for you to ride the roller coaster, it is necessary that you be at least four feet tall. So, if f is the statement “you are at least four feet tall”, and r is the statement “you can ride the roller coaster”, then the assertion is that f is necessary for r , i.e., the assertion being made is $r \rightarrow f$. If you are at least four feet tall, then there may be many other reasons why riding the roller coaster is impossible for you.

Is there a condition on p that’s enough (i.e. *sufficient*) to guarantee the truth of q ? It may not describe all situations where q is true; it just is a condition such that, if it holds, q is guaranteed to be true. From the truth table, if p is true, then q must also be true. Thus the truth of p suffices to guarantee the truth of q . Note, however, that we don’t need for p to be true in order for q to be true, as $p \rightarrow q$ is true when p is false and q is true. The words “ p is sufficient for q ” are therefore considered to be equivalent to “if p then q ”, or $p \rightarrow q$.

An example is the statement “a square is a rectangle”. If we know a shape is a square, then we know it is a rectangle. Thus, in order to guarantee a

shape is a rectangle, it suffices to guarantee that it is a square. Hence, if s is the statement “this shape is a square” and r is the statement “this shape is a rectangle”, then the assertion is that $s \rightarrow r$. There are plenty of objects which are rectangles but not squares, however.

Combining the above, what does it mean to say that p is a necessary and sufficient condition for q ? The statement “ p is sufficient for q ” is rendered symbolically as $p \rightarrow q$. The statement “ p is necessary for q ” is rendered symbolically as $q \rightarrow p$. Thus, “ p is a necessary and sufficient condition for q ” is the same as $(q \rightarrow p) \wedge (p \rightarrow q)$, which is logically equivalent to $p \leftrightarrow q$, or “ p if and only if q ”.

An example is the statement that for a triangle to be equilateral, it is necessary and sufficient that the three interior angles are equal. Every equilateral triangle has three equal interior angles, so that the condition is sufficient. And every triangle with three equal interior angles is equilateral, so that the condition is necessary. Thus, the statements “this triangle has 3 equal interior angles” and “this triangle is equilateral” are logically equivalent and can be used interchangeably.

1.7 The Laws of Logic

We now set out to develop an algebra of propositions. To do so, we need some basic operations (logical equivalences) that can be used. Each of the following can be verified (proved) with a truth table. It is a good idea to memorize them, so that they are at your fingertips when needed.

In what follows, **1** denotes a statement that is always true (i.e. a tautology), and **0** denotes a statement that is always false (i.e. a contradiction).

- Idempotence: $p \vee p \Leftrightarrow p$, $p \wedge p \Leftrightarrow p$
- Commutative: $p \wedge q \Leftrightarrow q \wedge p$, $p \vee q \Leftrightarrow q \vee p$
- Associative: $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$, $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- Distributive: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- Double Negation: $\neg(\neg p) \Leftrightarrow p$

- DeMorgan's Laws: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$, $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- Identity: $p \wedge \mathbf{1} \Leftrightarrow p$, $p \vee \mathbf{0} \Leftrightarrow p$
- Dominance: $p \wedge \mathbf{0} \Leftrightarrow \mathbf{0}$, $p \vee \mathbf{1} \Leftrightarrow \mathbf{1}$

The following are some other useful logical equivalences.

- $p \rightarrow q \Leftrightarrow \neg p \vee q$
- $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

It is apparent that the Laws of Logic come in pairs. The *dual* of a statement is obtained by replacing \vee by \wedge ; \wedge by \vee ; $\mathbf{0}$ by $\mathbf{1}$; and $\mathbf{1}$ by $\mathbf{0}$, wherever they occur. It is a theorem of logic that if s_1 is logically equivalent to s_2 , then the dual of s_1 is logically equivalent to the dual of s_2 .

For an example of using the Laws of Logic, we show that $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$.

$$\begin{aligned}
 p \leftrightarrow q & \\
 \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q) & \text{Known L.E.} \\
 \Leftrightarrow (\neg p \wedge (p \vee \neg q)) \vee (q \wedge (p \vee \neg q)) & \text{Distributive} \\
 \Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee (q \wedge \neg q)] & \text{Distributive (twice)} \\
 \Leftrightarrow [\mathbf{0} \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee \mathbf{0}] & \text{Known contradictions} \\
 \Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge p) & \text{Dominance} \\
 \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q) & \text{Commutative (3 } \times \text{)}
 \end{aligned}$$

There are two other forms of the Distributive Laws. These can be derived from the versions given above:

$$\begin{aligned}
 (q \wedge r) \vee p & \\
 \Leftrightarrow p \vee (q \wedge r) & \text{Commutative} \\
 \Leftrightarrow (p \vee q) \wedge (p \vee r) & \text{Distributive} \\
 \Leftrightarrow (q \vee p) \wedge (r \vee p) & \text{Commutative (twice)}
 \end{aligned}$$

Similarly $(q \vee r) \wedge p \Leftrightarrow (q \wedge p) \vee (r \wedge p)$.

The Laws of Logic can be used in several other ways. One of them is to prove that a statement is a tautology without resorting to a truth table. This

amounts to showing it is logically equivalent to $\mathbf{1}$. For example, $\neg q \vee (p \rightarrow q)$ is a tautology because:

$$\begin{aligned}
 \neg q \vee (p \rightarrow q) & \\
 \Leftrightarrow \neg q \vee (\neg p \vee q) & \quad \text{Known L.E.} \\
 \Leftrightarrow \neg q \vee (q \vee \neg p) & \quad \text{Commutative} \\
 \Leftrightarrow (\neg q \vee q) \vee \neg p & \quad \text{Associative} \\
 \Leftrightarrow \mathbf{1} \vee \neg p & \quad \text{Known tautology} \\
 \Leftrightarrow \mathbf{1} & \quad \text{Dominance}
 \end{aligned}$$

Similarly, a statement is proved to be a contradiction when it is shown to be logically equivalent to $\mathbf{0}$.

Another use of the Laws of Logic is to “simplify” statements. While the term “simplify” needs to be explained (quantified somehow) to be meaningful, or so we know when we are done, sometimes it is clear that an equivalent expression found is simpler than the one that was started with. For example:

$$\begin{aligned}
 \neg(\neg p \rightarrow q) \vee (p \wedge \neg q) & \\
 \Leftrightarrow \neg(\neg\neg p \vee q) \vee (p \wedge \neg q) & \quad \text{Known L.E.} \\
 \Leftrightarrow \neg(p \vee q) \vee (p \wedge \neg q) & \quad \text{Double Negation} \\
 \Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge \neg q) & \quad \text{DeMorgan} \\
 \Leftrightarrow (p \vee \neg p) \wedge \neg q & \quad \text{Dist've (from right to left)} \\
 \Leftrightarrow \mathbf{1} \wedge \neg q & \quad \text{Known tautology} \\
 \Leftrightarrow \neg q & \quad \text{Identity}
 \end{aligned}$$

This section concludes with one last example. Suppose we are asked to show that $(p \wedge q) \wedge [(q \wedge \neg r) \vee (p \wedge r)] \Leftrightarrow \neg(p \rightarrow \neg q)$. Use *LHS* to denote the expression on the left hand side. Then

$$\begin{aligned}
 & \text{LHS} \\
 \Leftrightarrow [(p \wedge q) \wedge (q \wedge \neg r)] \vee (p \wedge q) \wedge (p \wedge r) & \quad \text{Distributive} \\
 \Leftrightarrow [((p \wedge q) \wedge q) \wedge \neg r] \vee [(p \wedge q) \wedge p] \wedge r & \quad \text{Associative} \\
 \Leftrightarrow [(p \wedge (q \wedge q)) \wedge \neg r] \vee [(p \wedge p) \wedge q] \wedge r & \quad \text{Commutative, Associative} \\
 \Leftrightarrow [(p \wedge q) \wedge \neg r] \vee [(p \wedge q) \wedge r] & \quad \text{Idempotence} \\
 \Leftrightarrow (p \wedge q) \wedge (\neg r \vee r) & \quad \text{Distributive} \\
 \Leftrightarrow (p \wedge q) \wedge \mathbf{1} & \quad \text{Known tautology} \\
 \Leftrightarrow (p \wedge q) & \quad \text{Identity} \\
 \Leftrightarrow \neg\neg(p \wedge q) & \quad \text{Double Negation} \\
 \Leftrightarrow \neg(\neg p \vee \neg q) & \quad \text{DeMorgan} \\
 \Leftrightarrow \neg(p \rightarrow \neg q) & \quad \text{Known L.E.}
 \end{aligned}$$

1.8 Using Only And, Or, and Not

It turns out that any statement is logically equivalent to one that uses only the connectives \wedge , \vee , and \neg . The logical equivalences above allow statements involving the logical connectives \rightarrow and \leftrightarrow to be replaced by equivalent statements that use only \wedge , \vee , and \neg .

It is also possible to do this directly from the truth table, as will now be demonstrated. Let s be the statement involving p and q for which the truth table is given below.

p	q	s
0	0	1
0	1	1
1	0	0
1	1	1

First, for each row of the truth table where the statement s is true, write a statement that's true only when p and q have the truth values in that row. This statement will involve the logical connective “and”. For the truth table above:

- Row 1: $\neg p \wedge \neg q$
- Row 2: $\neg p \wedge q$
- Row 4: $p \wedge q$

Now, to get an expression that's logically equivalent to s , take the disjunction of these statements: it will be true exactly when the truth values of p and q correspond to a row of the truth table where s is true (row 1 or row 2 or row 4). Thus $s \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$. The process is exactly the same if there are more than two statements involved.

There is some terminology and an important fact (important in computer science) associated with what we have done. The expression associated with each row of the truth table – a conjunction of variables or their negations – is called a *minterm*. The compound statement derived using the process consists of the disjunction of a collection of minterms (that is, they are all joined together using “or”). It is called the *disjunctive normal form* of the statement s . Since every statement has a truth table, and every truth table

leads to a statement constructed as above, a consequence of the procedure just described is the theorem that *every statement is logically equivalent to one that is in disjunctive normal form*.

It can be observed directly from the truth table that

$$\begin{aligned} s &\Leftrightarrow p \rightarrow q \\ &\Leftrightarrow \neg p \vee q \quad \text{Known L.E.} \end{aligned}$$

The principle that things that are logically equivalent to the same statement are logically equivalent to each other now implies it should be true that that $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q) \Leftrightarrow \neg p \vee q$. This can be shown with the Laws of Logic.

It is possible to go beyond writing statements so they involve only \wedge , \vee , and \neg . With careful use of DeMorgan's Laws, really only need \vee and \neg , or \wedge and \neg , are needed. For example, from before, $p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. By DeMorgan's Law, $(\neg p \vee q) \wedge (\neg q \vee p) \Leftrightarrow \neg(\neg(\neg p \vee q) \vee \neg(\neg q \vee p))$, so $p \leftrightarrow q \Leftrightarrow \neg(\neg(\neg p \vee q) \vee \neg(\neg q \vee p))$. The latter statement uses only \vee and \neg . If you use DeMorgan's Law in a different way, then you can get an expression for $p \leftrightarrow q$ than involves only \wedge and \neg .

One can go a bit farther and introduce the logical connective “nand” (not and), so that “ p nand q ” is the statement $\neg(p \wedge q)$. It transpires that any proposition can be expressed (in a possibly complicated way) using only “nand”. The same thing applies to “nor”, where “ p nor q ” is the statement $\neg(p \vee q)$.

1.9 Logical Implication

It is apparent from examining the truth table for $a \wedge b$ that, if this statement is true, then so is a . Since an implication is true by definition when the hypothesis is false, this means that the statement $(a \wedge b) \rightarrow a$ is a tautology. If, in the midst of an argument, we were to discover that $a \wedge b$ is true, we would be entitled to conclude (infer, or deduce) that a is true (and the same for b). In what follows we develop a collection of basic rules for making inferences.

Informally, a statement p logically implies a statement q if the truth of p guarantees the truth of q . This happens exactly when $p \rightarrow q$ is a tautology.

Note that we are not concerned about what happens if p is false. This is because of the truth table for implies: $p \rightarrow q$ is true (by definition) when p is false.

Formally, we say p *logically implies* q when $p \rightarrow q$ is a tautology.

We use the notation $p \Rightarrow q$ to denote the fact (theorem), that $p \rightarrow q$ is a tautology, that is, that p logically implies q . Notice that $p \rightarrow q$ is a statement and can in general be true or false, and $p \Rightarrow q$ indicates the (higher level) fact that it is always true.

Suppose $p \Leftrightarrow q$. Then $p \leftrightarrow q$ is a tautology. Since $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, the latter statement is also a tautology. Using the reasoning in the first paragraph of this section, this means that each of $(p \rightarrow q)$ and $(q \rightarrow p)$ is a tautology. Therefore $p \Rightarrow q$ and $p \Leftarrow q$ (which has the obvious intended meaning: $q \Rightarrow p$). In the same way, if both $p \Rightarrow q$ and $p \Leftarrow q$, then $p \Leftrightarrow q$.

1.10 Valid Arguments and Inference Rules

An *argument* is an implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$. The statements p_1, p_2, \dots, p_n are called *premises*, and the statement q is called the *conclusion*. Put differently, an argument is an assertion. Since the truth table for implies says that an implication is true when its hypothesis is false, and since the hypothesis is the conjunction of all of the premises, the assertion being made is that *if the premises are all true, then so is the conclusion*.

An argument is called *valid* if the implication is a tautology (i.e., if the premises logically imply the conclusion, so that the conclusion is guaranteed to be true when all of the hypotheses are true), otherwise it is *invalid*.

To show that an argument is invalid, it needs to be demonstrated that the implication is not a tautology. From the truth table for implies, this amounts to describing a single row of a truth table where each premise is true and the conclusion is false. Such a collection of truth values is called a *counterexample* to the argument.

Arguments are usually presented in the tabular format shown below for the example $[(p \rightarrow \neg q) \wedge (\neg r \rightarrow p) \wedge q] \rightarrow \neg r$. The premises are listed first,

and then the conclusion is listed below a separating line.

$$\begin{array}{c}
 p \rightarrow \neg q \\
 \neg r \rightarrow p \\
 \qquad q \\
 \hline
 \therefore \neg r
 \end{array}$$

The argument given above is invalid. To demonstrate that, we need to give a counterexample: a truth value assignment for p, q and r such that the premises are all true and the conclusion is false. The best way to get started is to find truth values that make the conclusion false. Here there is only one: the statement $\neg r$ is false only when r is true. The next step is to try to assign truth values so that the premises are all true. The truth of third premise gives us that q is true. Since r is false, the second premise is true no matter truth value of p (that is, it places no restriction on the truth value of p). Finally, we want to choose a truth value for p , if necessary, so that the first premise is true. When p is false, the implication $p \rightarrow \neg q$ is true. (And since $\neg q$ is false, this is the only possible truth value assignment to p that makes the first premise true.) Thus, if p, q, r have the truth values 0, 1, 1, respectively, the premises are all true and the conclusion is false. Therefore, the argument is not valid.

A truth table can, in principle, be used to show an argument is valid. But, if the number of premises involved is large, so is the table. A better way is to give a *proof*: a chain of logical equivalences and implications involving the premises (which are assumed to be true because an implication is true when its hypothesis is false). The idea is that every statement you write down is true, and is either a premise, or an allowed additional hypothesis, or is derived from statements known to be true via logical equivalences and implications.

Our ultimate goal is to write mathematical proofs in words. Proving logical implications using inference rules and logical equivalences is a step towards that goal.

The two following *inference rules* are each a logical implication. They are just common sense, but can be formally proved with a truth table. These will get used frequently in arguments and hence need to be at your fingertips, so they should be memorized.

- Modus Ponens: $(p \rightarrow q) \wedge p \Rightarrow q$
- Chain Rule (Law of Syllogism): $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

We use these inference rules to prove some other rules. The rules above are worth memorizing. The rules below are easy consequences of them and need not be remembered.

Modus Tollens: $[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$.

Proof.

1. $p \rightarrow q$ Premise
2. $\neg q \rightarrow \neg p$ L.E. to 1
3. $\neg q$ Premise
4. $\therefore \neg p$ 2, 3, M.P.

Disjunctive Syllogism: $[(p \vee q) \wedge \neg p] \Rightarrow q$.

Proof.

1. $p \vee q$ Premise
2. $\neg p \rightarrow q$ L.E. to 1
3. $\neg p$ Premise
4. $\therefore q$ 2, 3, M.P.

Resolution: $[(p \vee r) \wedge (q \vee \neg r)] \Rightarrow p \vee q$.

Proof.

1. $p \vee r$ Premise
2. $\neg p \rightarrow r$ L.E. to 1
3. $q \vee \neg r$ Premise
4. $\neg r \vee q$ 3, Commutative
5. $r \rightarrow q$ L.E. to 4
6. $\neg p \rightarrow q$ 2, 5, Chain Rule
7. $p \vee q$ L.E. to 6

Here are two more inference rules which are clearly true, and which can be formally proved with a truth table.

- Disjunctive Amplification: $p \Rightarrow p \vee q$

- Conjunctive Simplification: $p \wedge q \Rightarrow p$

Here is another example of using inference rules to prove an argument is valid.

$$\begin{array}{l}
 \neg p \leftrightarrow q \\
 \neg q \rightarrow r \\
 p \\
 \hline
 \therefore r
 \end{array}$$

Proof.

- | | | |
|----|--|-------------------------------|
| 1. | $\neg p \leftrightarrow q$ | Premise |
| 2. | $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$ | L.E. to 1 |
| 3. | $q \rightarrow \neg p$ | 2, Conjunctive Simplification |
| 4. | $p \rightarrow \neg q$ | 3, Contrapositive |
| 5. | $\neg q \rightarrow r$ | Premise |
| 6. | $p \rightarrow r$ | 4, 5, Chain Rule |
| 7. | p | Premise |
| 8. | r | 6, 8, M.P. |

In the next example, the argument is given in words. We can still check its validity using inference rules. First, we need to translate the argument into symbols.

If I run, then my ankle does not hurt
 If I am not injured, then I run
 My ankle hurts

\therefore I am injured

Let p , q , and r denote the statements “I run”, “My ankle hurts”, and “I am injured”, respectively. Then the argument is:

$$\begin{array}{l}
 p \rightarrow \neg q \\
 \neg r \rightarrow p \\
 q \\
 \hline
 \therefore r
 \end{array}$$

This argument is valid (unlike most stories about sports injuries). We can prove it using inference rules.

- | | | |
|----|-----------------------------|-------------------|
| 1. | $p \rightarrow \neg q$ | Premise |
| 2. | $\neg r \rightarrow p$ | Premise |
| 3. | $\neg r \rightarrow \neg q$ | 2, 1, Chain Rule |
| 4. | $q \rightarrow r$ | 3, Contrapositive |
| 5. | q | Premise |
| 6. | $\therefore r$ | 4, 5, M.P. |

We conclude this section with two more inference rules that can be proved with a truth table, and then some discussion about them.

- Proof by Contradiction: $(\neg p \rightarrow \mathbf{0}) \Rightarrow p$
- Proof by Cases: $(p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow (p \vee q) \rightarrow r$

The rule “Proof by Contradiction” is best illustrated by a proof in words. An example will be given in the next section. The idea behind the rule is that one should only be able to obtain true statements when starting with true statements, and using logical equivalences and logical implications. Hence, if falsity of the desired conclusion leads to a statement that is never true (that is, a contradiction), then the conclusion can not be false. Here, we illustrate the use of this rule in a proof of the type above by giving a second proof of the rule “Resolution”.

Alternate proof of the rule “Resolution”.

- | | | |
|-----|-----------------------------|---|
| 1. | $\neg(p \vee q)$ | Negation of conclusion, for proof by contradiction |
| 2. | $\neg p \wedge \neg q$ | 1, DeMorgan |
| 3. | $\neg p$ | 2, Conjunctive Simplification |
| 4. | $\neg q$ | 2, Conjunctive Simplification |
| 5. | $p \vee r$ | Premise |
| 6. | $\neg p \rightarrow r$ | L.E. to 5 |
| 7. | r | 3, 6, M.P. |
| 8. | $q \vee \neg r$ | Premise |
| 9. | $\neg q \rightarrow \neg r$ | L.E. to 8 |
| 10. | $\neg r$ | 4, 9, M.P. |
| 11. | $r \wedge \neg r$ | $(\Leftrightarrow \mathbf{0})$ Known contradiction from 7, 10 |
| 12. | $p \vee q$ | 1, 11, Proof by Contradiction |

The rule “Proof by Cases” is also best illustrated by a proof in words. It was mentioned in this section (as with proof by contradiction) to illustrate that the proof methods we will use have a basis in logic. The intuition for proof by cases is simple enough. If the truth of p guarantees the truth of a conclusion, r , and the truth of q guarantees the truth of r , and one of p and q must be true, then r must be true. The way this rule is applied is that *if one of several cases must arise, and the desired conclusion holds in each case, then the premises logically imply the conclusion*. An example is given in the next section.