Chapter 2

Quantifiers and Written Proofs

In the previous chapter we showed that every propositional expression is logically equivalent to one that involves only the symbols $\wedge, \vee$ and $\neg$. Put differently, *for every* propositional expression *there exists* a logically equivalent expression in which the only logical symbols used are $\wedge, \vee$ and $\neg$. This statement contains two specification of a quantity: one arises when it says it applies to *all* propositional expressions, and the other arises when it says that, for of them, there is *at least one* logically equivalent expression that satisfies the given criteria of being logically equivalent and only using the given symbols (in fact there are many such expressions).

In the previous chapter we also constructed several types of proofs. One type was when we used the Laws of Logic to demonstrate (prove) the logical equivalence of two given statements. A second type was when we used logical equivalences and Inference Rules to demonstrate the validity of arguments. A third type was when we showed that certain assertions are not true – for example that a given statement is not a tautology, two statements are not logically equivalent, a given argument is not valid, when we reasoned in words that a given assertion was true, and so on. Arguments of this type are more common in mathematics than those of the first two types.

In the first part of this chapter we will study *quantifiers* which, as the name suggests, specify the quantity of objects to which a given assertion applies. Almost every mathematical statement contains one or more quantifiers. In the second part of the chapter we will begin the transition from writing symbolic proofs to writing convincing arguments (i.e., proofs) in English. A number of tools for proving mathematical statements will be developed, and then other items will be added to our toolkit as time goes by. (Note: no one gets to know for sure which tool will work in a given situation, but experience can help make it possible to know which ones are likely to work, and in which order various methods should be tried.) Each proof strategy will make use of the logic and structure developed in the previous chapter.

2.1 Open Statements

Here are two assertions in mathematics: "*Every non-zero number x has a multiplicative inverse y*" and "*There is a number x such that $x^2 = -1$*".

It is not possible to know the truth value of the either assertion unless you know what sort of numbers can be used to replace x and y. If they must be integers, then both assertions are false. If they can be real numbers, then the first assertion is true and the second one is false. If they can be complex numbers, then both assertions are true.

An *open statement* is an assertion that contains one or more variables. Usually the truth value of such a statement can not be determined until the values of the variables are known. An example is "*x is a root of $x^2 + 5x + 6$*". This statement is true if $x = 3$, and false if $x = 1$. It is never true if x is required to be a negative real number, and (as we've seen) can be true if x is required to be a positive real number.

When the context is clear, we will drop the qualifier "open", and refer to assertions that contain one or more variables as *statements*.

It is also the case that we can't tell if a statement containing variables is ever true, or always true, unless we know what sort of values can replace the variables. An example is $x^2 = 2$. If x can be any real number, then this statement is true when $x = \sqrt{2}$ and when $x = -\sqrt{2}$. If x must be an integer, then it is never true.

The point to remember about statements involving variables is that once the variables are assigned values (that they are allowed to have), then the resulting statement has a truth value. Before the variables have values, the only things we have a chance to know (and may not get to know) is whether such statement is always true or always false no matter which of the allowed values are assigned to the variables, or if it is sometimes true (and sometimes false) depending on which of the allowed values are assigned to the variables.

The Laws of Logic apply to statements involving variables because they apply once values are given to the variables (in exactly the same way each time). Thus, for example, if $p(x)$ and $q(x)$ are statements involving the variable x, the contrapositive of $p(x) \to q(x)$ is $\neg q(x) \to \neg p(x)$, and these statements have the same truth value for any x, so either one can replace the other whenever it occurs. Similarly, for every x we have that $\neg(p(x) \vee q(x))$ has the same truth value as $\neg p(x) \wedge \neg q(x)$, so either one can replace the other whenever it occurs, and so on.

2.2 Quantifiers

When we make an assertions like "*if* $x^2+3x+2=0$ *then* $x=-1$ *or* $x=-2$", the intention is to convey that the assertion holds for every real number x. Similarly, an assertion like "*some rectangles are squares*" is intended to convey that at least one rectangle is a square.

If an assertion contains one or more variables, it isn't possible to know its truth value until something about the variables is known. There are two options:

1. Give values to the variables. The assertion will then be either true or false. Giving different values to the variables might result in a different truth value for the assertion.

2. Specify the quantity (that is, number) of allowed replacements for each variable that result in the assertion being true. This specification is an assertion that is either true or false, that is, it is a statement.

The second option leads to the study of quantifiers.

The *universe* of a variable is the collection of values it is allowed to take.

The *universal quantifier* $\forall$ asserts that the given statement is true *for all* allowed replacements for a variable. Think of the upside-down "A" as representing "All". Synonyms for "*for all*", include "*all*", "*every*" and "*for each*".

An example of using a universal quantifier is: "*for all integers* n*, the integer* $n(n+1)$ *is even*". We could take a first step towards a symbolic representation of this statement by writing "$\forall n, n(n+1)$ *is even*", and specifying that the universe of n is the integers. (This statement is true.)

A universal quantifier is like the logical connective "and". Suppose the universe consists of the integers 1, 2, 3. Then the statement $\forall x, x > 1$ is precisely the statement $(1 > 1) \wedge (2 > 1) \wedge (3 > 1)$, which is false.

The comma following a universal quantifier is best read as a pause, as in the textual representation of the statement in the previous paragraph.

The *existential quantifier* $\exists$ asserts that *there exists* at least one allowed replacement for a variable for which the given statement is true. Think of the backwards "E" as representing "exists". Synonyms for "*there exists*" include "*there is*", "*there are*", "*some*", and "*at least one*".

An example of using an existential quantifier is "*there exists an integer* n *such that* $n^2 - n + 1 = 0$". A symbolic representation of this statement is obtained by writing $\exists n, n^2 - n + 1 = 0$, and specifying that the universe of n is the integers. (This statement is false.)

An existential quantifier is like the logical connective "or". Suppose the universe consists of the integers 1, 2, 3. Then the statement $\exists x, x < 3$ is precisely the statement $(1 < 3) \vee (2 < 3) \vee (3 < 3)$, which is true.

The comma following an existential quantifier is best read as "such that", as in the textual representation of the statement in the previous paragraph.

We can completely write the statement "$\forall n, n(n+1)$ *is even*" in symbols by remembering the definition of an even integer. An integer k is *even* when there is an integer t such that $k = 2t$. Symbolically, k is even when $\exists t, k = 2t$, where the universe of t is the integers. With this in mind "$\forall n, n(n+1)$ *is even*" becomes "$\forall n, \exists t, n(n+1) = 2t$".

When quantifiers are nested, they are read in order from left to right. For example, if x and y are understood to be numbers, "$\forall x, \exists y, x + y = 0$" is read as follows: *for all* x*, the statement "*$\exists y, x + y = 0$*" is true.* No matter the value of x, the number y can be chosen to be its negative. Hence, $\exists y, x + y = 0$ is true for any x. Consequently, $\forall x, \exists y, x + y = 0$ is true.

The order of quantifiers is important. The statement "$\exists x, \forall y, x + y = 0$" says that there is a real number x such that, for every real number y, the quantity $x + y = 0$, which is false. There is no real number x with the property that no matter what real number y is added to it, the sum is zero.

Suppose the universe consists of the integers 1, 2, and consider the statement $\forall x, \exists y, xy = 2$. When $x = 1$, the statement $\exists y, xy = 2$ is precisely $(1 \cdot 1 = 2) \vee (1 \cdot 2 = 2)$, which is true. When $x = 2$, the statement $\exists y, xy = 2$ is precisely $(2 \cdot 1 = 2) \vee (2 \cdot 2 = 2)$, which is true. Thus, $\forall x, \exists y, xy = 2$ is precisely the statement

$$[(1 \cdot 1 = 2) \vee (1 \cdot 2 = 2)] \wedge [(2 \cdot 1 = 2) \vee (2 \cdot 2 = 2)].$$

The first expression in square brackets corresponds to the statement $\exists y, xy = 2$ when $x = 1$, the second one corresponds to this statement when $x = 2$, and we are taking the conjunction of these statements because of the universal quantifier. The given statement is true. On the other hand, $\exists y, \forall x, xy = 2$ is precisely

$$[(1 \cdot 1 = 2) \wedge (2 \cdot 1 = 2)] \vee [(1 \cdot 2 = 2) \wedge (2 \cdot 2 = 2)],$$

which is false.

Here are more examples about determining the truth value of statements involving quantifiers.

First, suppose the universe is the integers and consider the statement $\forall x, \exists y, x + y < 10$. The first quantifier is "for all", and it applies to x. Thus, the quantifed statement is going to be true only if the statement that follows, $\exists y, x + y < 10$, is true no matter what x in the universe is used. The next quantifier is "there exists', and it applies to y This quantified statement is going to be true only if there is at least one y in the universe so that $x+y < 10$ is true. The statement $\forall x, \exists y, x + y < 10$ is true. Given any integer x, if we choose y to be $-x$ then $x + y = x + (-x) = 0$. Therefore, for any x, there exists y such that $x + y < 10$.

Second, suppose the universe is the integers and consider the statement $\exists x, \exists y, xy = 4$. The first quantifier is "there exists", and it applies to x. Thus, the quantifed statement is going to be true only if the statement that follows, $\exists y, xy = 4$, is true for at least one x in the universe. The next quantifier is also "there exists", and it applies to y. This quantified statement is going to be true only if, for the x determined by the first quantifier, the statement $xy = 4$ for at least one y in the universe. The statement $\exists x, \exists y, xy = 4$ is true. Since $1 \cdot 4 = 4$, we could take $x = 1$, and then it would be true that there exists y, namely $y = 4$, such that $xy = 4$. Alternatively, we could have chosen $x = 2$ and then $y = 2$, or $x = 4$ and then $y = 1$.

Finally, suppose again that the universe is the integers and consider the statement $\exists x, \forall y, x + y < 10$. The first quantifier is "there exists", and it applies to x. Thus, the quantifed statement is going to be true only if the statement that follows, $\forall y, x + y < 10$, is true for at least one x in the universe. The second quantifier is "for all", and it applies to y. This quantified statement is going to be true only if, for the x determined by the first quantifier, the statement $x + y < 10$ is true no matter which y in the universe is used. The statement $\exists x, \forall y, x + y < 10$ is false. To demonstrate that, we need to argue that no matter which x in the universe is used, the statement $x + y < 10$ is not true for all integers y in the universe. Suppose that an integer x is given. If we take $y = -x + 11$, then y is an integer and $x + y = x + (-x + 11) = 11$, so it is not the case that $x + y < 10$ for all integers y. Put differently, no matter which integer x is being used, we can always find y such that $x + y \geq 10$.

Let $s(x)$ denote a statement involving the variable x. Observe that if $\forall x, s(x)$ is true, then so is $\exists x, s(x)$, provided the universe contains a non-zero number of elements: if an assertion is true for all x is the universe, then it is true for at least one x (provided there is one). If the universe contains no elements, then $\forall x, s(x)$ is always true, and $\exists x, s(x)$ is never true (why?). Of course, the truth of $\exists x, s(x)$ tells us nothing about the truth of $\forall x, s(x)$.

Both universal and existential quantifiers can be (unintentionally) hidden, as in the example used to begin this section. Another example is the statement "if $(a \neq 0)$ and $(ax^2 + bx + c = 0)$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is meant to apply to all real numbers x. If the universal quantifier were made explicit, it would read "for all real numbers $x \ldots$". Similarly, "a real number can have more than one decimal expansion" is intended to assert the existence of one or more such numbers. If the existential quantifier were made explicit, it would read "there is a real number x such that x has more than one decimal expansion".

2.3 Negating Statements Involving Quantifiers

The negation of a universally quantified statement is an an existentially quantified statement. If it is not the case that a statement is true for all allowed replacements in the universe, then it is false for at least one allowed replacement.

For example "$\neg[\forall n \geq 0, n^2 - n + 41]$ *is prime*" says "it is not the case that for every positive integer n the number $n^2 - n + 41$ is prime", or in other words "there exists a positive integer n such that $n^2 - n + 41$ is not prime".

In symbols, if $s(x)$ is an assertion involving the variable x (and maybe some other variables and quantifiers) $\neg[\forall x, s(x)]$ is the same is $\exists x, \neg s(x)$.

The negation of an existentially quantified statement is a universally quantified statement. If it is not the case that a statement is true for at least one allowed replacement in the universe, then it is false for all allowed replacements.

For example "$\neg[\exists a, b, \frac{a}{b} = \sqrt{2}]$" says "it is not the case that there exists (integers) a and b such that $\frac{a}{b} = \sqrt{2}$", or in other words "for all (integers) a and b, $\frac{a}{b} \neq \sqrt{2}$", that is, "$\sqrt{2}$ is irrational".

In symbols, if $s(x)$ is an assertion involving the variable x (and maybe some other variables and quantifiers) $\neg[\exists x, s(x)]$ is the same is $\forall x, \neg s(x)$.

Using what we've done above, the statement $\neg[\exists x, \forall y, x + y = 0]$ is the same as $\forall x, \neg[\forall y, x + y = 0]$ (take $s(x)$ to be $\forall y, x + y = 0$). In turn, this is the same as $\forall x, \exists y, \neg(x + y = 0)$, or equivalently $\forall x, \exists y, x + y \neq 0$. The latter statement is easily seen to be true. No matter number x is, we can choose y to be any number different than $-x$, and $x + y \neq 0$.

2.4 Some Examples of Written Proofs

Suppose you want to write a proof in words for a statement of the form "if p then q". That is, you wish to establish the theorem $p \Rightarrow q$. There are many techniques (methods) that can be tried. There is no guarantee of which method will work best in any given situation. Experience is a good guide, however. Once a person has written a few proofs, they get a sense of the best thing to try first in any given situation.

To use the method of *direct proof* to show p logically implies q, *assume p is true* and then *argue using definitons, known implications and equivalences that q must be true.* The reason for assuming p is true comes from the definition of logical implication. In this case the first line of the proof is "*Assume p.*" and the last says, essentially, "*q is true*". What comes in between depends on p and q.

In the following example of a direct proof, we use the definition of an even integer: An integer n is *even* if there exists an integer k so that $n = 2k$. Put differently, the integer n is even if it leaves remainder 0 on division by 2. An integer n is *odd* if it leaves remainder 1 on division by 2, that is, if $n = 2k + 1$ for some integer k. Every integer is either even or odd, and not both.

Proposition 2.4.1 *If the integer n is even, then n^2 is even.*

Proof. Suppose that the integer n is even. Hence, there exists an an integer k so that $n = 2k$. Then, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. $\square$

It is customary in mathematics to use a box to indicate the end (or absence) of an argument.

Another proof technique is to *prove the contrapositive.* That is, assume q is false, and argue using the same things as above that p must also be false. This works since $p \to q \Leftrightarrow \neg q \to \neg p$. In this case the first line of the proof is "*Assume $\neg q$.*" and the last is, essentially, "*$\neg p$ is true*". This method is sometimes called giving an *indirect proof.* The motivation for the name comes from the fact that the logical implication is proved indirectly, by its contrapositive.

Proposition 2.4.2 *If the integer n^2 is even, then n is even.*

Proof. We will prove the contrapositive that if n is not even, then n^2 is not even.

Suppose that the integer n is not even, that is, it is odd. We want to show that n^2 is odd. Since n is odd, there exists an an integer k so that $n = 2k + 1$. Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. □

Yet another technique is *proof by contradiction*. Such a proof begins by assuming q is false and, again proceeding as above, until deriving a statement which is a (logical) contradiction. This enables you to conclude that q is true. In such a situation, the first line of the proof is "Suppose $\neg q$." and the proof ends with " We have obtained a contradiction. Therefore q."

Here is a classic example of proof by contradiction. It uses the definition of a rational number: a number x is *rational* if there exist integers a and b so that $x = a/b$. A number is *irrational* if it is not rational.

Put slightly differently, x is rational if it is a ratio of two integers. There are many ratios of integers that equal a given number. In particular, there is always one where the fraction a/b is in *lowest terms*, meaning that a and b have no common factors other than one.

Proposition 2.4.3 *$\sqrt{2}$ is not rational.*

Proof. Suppose $\sqrt{2}$ is rational. Then there exist integers a and b so that $\sqrt{2} = a/b$. The integers a and b can be chosen so that the fraction a/b is in lowest terms, so that a and b have no common factor other than 1. In particular, a and b are not both even.

Since $\sqrt{2} = a/b$, we have that $2 = (a/b)^2 = a^2/b^2$. By algebra, $2b^2 = a^2$. Therefore a^2 is even. By Proposition 2.4.2, a is even. Thus there exists an integer k so that $a = 2k$. It now follows that $2b^2 = a^2 = (2k)^2 = 4k^2$, so that $b^2 = 2k^2$. Therefore b^2 is even. By Proposition 2.4.2, b is even.

We have now derived the contradiction (a and b are not both even) and (a and b are both even). Therefore, $\sqrt{2}$ is not rational. □

Sometimes the hypotheses lead to a number of possible situations, and it is easier to consider each possibility in turn. In the method of *proof by cases*, one lists the cases that could arise (being careful to argue that all possibilities are taken into account), and then shows that the desired result holds in each case. It could be that different cases are treated with different proof methods. For example, one could be handled directly, and another by contradiction.

In the following example we make use the fact that every integer n can be uniquely written in the form $3k + r$, where k is an integer and r equals 0, 1, or 2. When the remainder, r, equals 0 we have $n = 3k$, so that n is a multiple of 3.

Proposition 2.4.4 *If the integer n^2 is a multiple of 3, then n is a multiple of 3.*

Proof. We prove the contrapositive: if n is not a multiple of 3, then n^2 is not a multiple of 3. Suppose n is not a multiple of 3. Then the remainder when n is divided by 3 equals 1 or 2. This leads to two cases:

Case 1. The remainder on dividing n by 3 equals 1.

Then, there exists an integer k so that $n = 3k+1$. Hence $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $(3k^2 + 2k)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

Case 2. The remainder on dividing n by 3 equals 2.

Then, there exists an integer k so that $n = 3k+2$. Hence $n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Since $(3k^2 + 4k + 1)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

Both cases have now been considered. In each of them, we have shown that n^2 is not a multiple of 3. It now follows that if n is not a multiple of 3, then n^2 is not a multiple of 3. This completes the proof. □