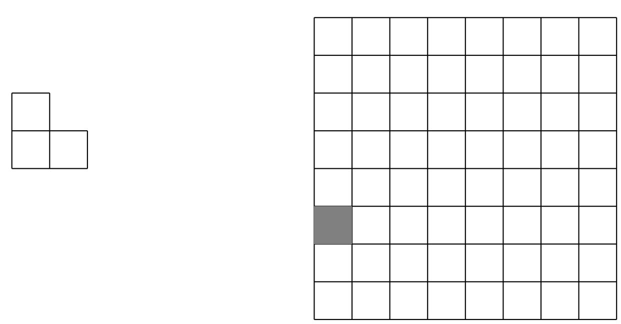
Chapter 4

Induction and Recursion

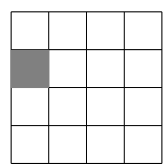
4.1 Induction: An informal introduction

This section is intended as a somewhat informal introduction to *The Principle of Mathematical Induction* (PMI): a theorem that establishes the validity of the proof method which goes by the same name. There is a particular format for writing the proofs which makes it clear that PMI is being used. We will not explicitly use this format when introducing the method, but will do so for the large number of different examples given later.

Suppose you are given a large supply of L-shaped tiles as shown on the left of the figure below. The question you are asked to answer is whether these tiles can be used to exactly cover the squares of an $2^n \times 2^n$ *punctured grid* – a $2^n \times 2^n$ grid that has had one square cut out – say the 8×8 example shown in the right of the figure.

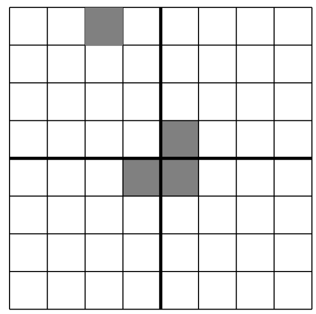


In order for this to be possible at all, the number of squares in the punctured grid has to be a multiple of three. By direct calculation we can see that it is true when $n = 1, 2$ or 3, and these are the cases we're interested in here. It turns out to be true in general; this is easy to show using congruences, which we will study later, and also can be shown using methods in this chapter. But that does not mean we can tile the punctured grid. In order to get some traction on what to do, let's try some small examples. The tiling is easy to find if $n = 1$ because 2×2 punctured grid is exactly covered by one tile. Let's try $n = 2$, so that our punctured grid is 4×4. By rotating, we can assume the missing square is in the upper left quadrant, say as illustrated below.



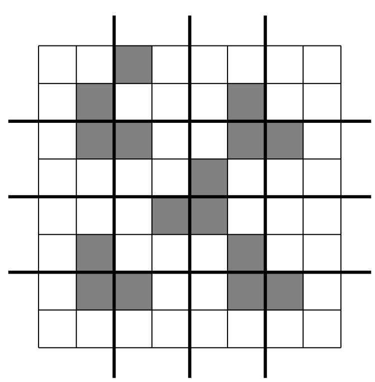
Imagine the punctured grid partitioned into four 2×2 grids, one of which has a square missing, as shown on the left of the figure below. As shown on the right of the figure, we can astutely place one tile to transform our problem into four 2×2 problems, each of which we know how to solve.

It is clear that this method works no matter which square in the upper left quadrant has been removed. Hence, if we can cover any 2×2 punctured grid, then we can cover and 4×4 punctured grid. Now we can see what to do to cover the 8×8 punctured grid: partition it into four 4×4 grids, one of which has a square removed, then astutely place a tile to transform the problem into four 4×4 problems we know how to solve because of our previous work.



There is nothing special about the numbers 4 and 8 in the previous examples. Once we know how to cover all possible punctured grids of size 2×2, 4×4, and 8×8, we can use the same method on any 16×16 punctured grid. And we can keep going. Once we know how to cover all punctured grids of size $2 \times 2,\ 4 \times 4,\ \ldots, 2^k \times 2^k$, we can use the same method to reduce the problem of covering a $2^{k+1} \times 2^{k+1}$ grid to four smaller problems we know how to solve because of previous work. Therefore, for any $n \geq 1$, the squares of a $2^n \times 2^n$ punctured grid can be exactly covered by L-shaped tiles.

The previous example illustrates the strong form of the *Principle of Mathematical Induction* (PMI). One meaning of the word *induction* is "the act of bringing forward". Above, we brought forward our knowledge of how to solve smaller instances of the problem to solve all instances of the next possible size. Notice also that the solution can be obtained recursively. For example, to cover an 8×8 punctured grid, we cover four 4×4 punctured grids, and each of these is covered via covering four 2×2 punctured grids. This is illustrated in the figure below. Completing the tiling of each 2×2 punctured grid gives the tiling of the 8×8 punctured grid.



4.2 More informal examples

4.2.1 The sum of the first n odd positive integers

Suppose that you are mathematically doodling and notice that:

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

and are led to wonder whether *the sum of the first* n *odd positive integers equals* n^2.

By the work above, this is true for $n = 1, 2, 3, 4$. Suppose you know that the sum of the first n odd positive integers equals n^2 for $n = 1, 2, \ldots, k$, where $k \geq 4$. That is, suppose we know that $1 + 3 + \cdots + 2n - 1 = n^2$, for $n = 1, 2, \ldots, k$ and $k \geq 4$.

With this supposition in hand, look at the sum of the first $k + 1$ odd positive integers, $1+3+\cdots+2(k+1)-1$. The goal is to "bring forward" our knowledge about the value of the sum of the first k (or fewer) odd positive integers to show that $1 + 3 + \cdots + 2(k + 1) - 1 = (k + 1)^2$.

By the meaning of the ellipsis, "$\cdots$", this equals $1 + 3 + \cdots + (2k - 1) + 2(k+1)-1$. The first k terms can be replaced by their value, which we know is k^2, so that we have

$$1+3+\cdots+(2k-1)+2(k+1)-1 = k^2+2(k+1)-1 = k^2+2k+1 = (k+1)^2.$$

Thus, if the sum of the first k odd integers equals k^2, then the sum of the first $(k + 1)$ off integers is $(k + 1)^2$.

Since we know that the sum of the first 4 odd integers is 4^2, it follows that the sum of the first 5 odd integers is 5^2. Now using this, it follows that the sum of the first 6 odd integers is 6^2. Repeating this argument over and over, we can eventually reach any integer $n \geq 4$. Combining this with the fact that we directly checked that the sum of the first n odd integers is n^2 for $n = 1, 2, 3, 4$, it follows that, for any $n \geq 1$, $1 + 3 + \cdots + 2n - 1 = n^2$.

4.2.2 A postage stamp problem

Problems of this type date back to the days when stamps came in denominations like 1 cent, 3 cents, 5 cents, and so on. People often kept a supply of stamps of various values, and then tried to combine them to make whatever postage was needed at the time. This is the same problem as writing a given positive integer as a sum of various other given positive integers, if possible.

Suppose you want to know which positive integers can be written as a sum of 3s and 5s. Clearly 1 and 2 can't, 3 can, 4 can't, 5 and 6 can, and 7 can't. After some further experimenting, it seems like *every positive integer* $n \geq 8$ *can be written as a sum of 3s and 5s.*

From directly checking, $8 = 5 + 3$, $9 = 3 + 3 + 3$, and $10 = 5 + 5$, so each of 8, 9, and 10 can be written as a sum of 3s and 5s. Suppose we know that each of the integers $8, 9, 10, \ldots, k$ can be written as a sum of 3s and 5s, where $k \geq 10$. Let's try to "bring forward" this knowledge to show that $k + 1$ can be written as a sum of 3s and 5s.

If we can write $k-2 = (k+1)-3$ as a sum of 3s and 3s, then we can add 3 and obtain $k + 1$ as a sum of 3s and 5s. Since $k \geq 10$ (because we checked from 8 to 10), $k - 2 \geq 8$. By our supposition, $k - 2$ can be written as a sum of 3s and 5s. Therefore, $k+1 = (k-2)+3$ can be written as a sum of threes and fives.

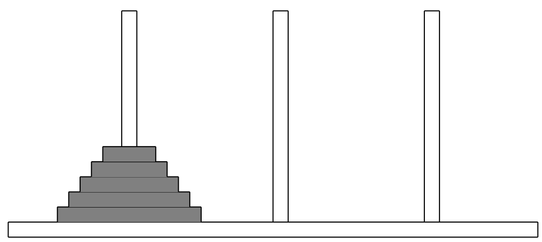
Since we know that 8, 9 and 10 can be written as a sum of 3s and 5s, it follows that 11 can be so written too. Using this, so can 12. Repeating as often as needed, we can eventually reach any integer $n \geq 8$. Therefore every positive integer $n \geq 8$ can be written as a sum of 3s and 5s.

We close this subsection with some notes. First, the reason is that this is possible is that 3 and 5 have no common positive divisors except for 1. It is not possible otherwise. For example, if 3 and 5 were replaced by two even numbers, then we could only achieve sums that are even. Second, the number 8 does not arise by coincidence. It is known that if the positive integers a and b have no common positive divisors except for 1, then every positive integer $n \geq (a-1)(b-1)$ can be written as a sum of a's and b's. Further, $(a-1)(b-1)-1$ can't be so written, so $(a-1)(b-1)$ is the smallest value for which the statement (that every integer at least as large can be written in the desired form) is true. *Finally, and most importantly, it was crucial that we directly checked all the way up to 10. If we had only checked up to*

9, say, then the argument would fail when $k+1=10$ *because* $k+1-3=7$*, which can not be expressed in the desired form.*

4.2.3 The Towers of Hanoi

The *Towers of Hanoi* is a puzzle that begins with $n \geq 1$ rings, each with a different diameter, stacked in decreasing order of size on one of three towers. An example with five rings is shown below. The objective is to move the rings one at a time so that they are eventually stacked in the same order on one of the other towers. At no point in time may a larger ring rest on top of a smaller one.



It is easy to directly check that a solution exists when there are 1 or 2 rings. To obtain a solution when there are 3 rings, first use the 2-ring solution to move the top 2 rings to one of the unused towers. Then move the bottom (largest) ring to the remaining unused tower. Finally, use the 2-ring solution again to move the 2 smaller rings to the tower containing the largest ring. It does not matter if, at any point in this process, any of the other rings is placed on top of the largest one (since it is largest).

Suppose we can solve the puzzle when n is any of the integers $1, 2, \ldots, k$, where $k \geq 3$. Let's try to "bring forward" this knowledge to obtain a solution when there are $k+1$ rings. We can proceed as in the 3-ring case. First, use the k-ring solution to legally move the k smallest rings to one of the other towers. Leaving the large ring in place will not cause the constraint that a larger ring may not rest atop a smaller one to be violated. Second, move the largest ring to the empty tower. Finally, use the k-ring solution to legally move the k smallest rings so that they are on top of the largest one.

Since we know how to solve the puzzle when there are 1, 2, and 3 rings, it follows that we can also solve it when there are 4 rings. Using this, we can also solve it when there are 5 rings. Repeating as often is needed, we can eventually obtain a solution for any integer $n \geq 1$. Therefore, for any $n \geq 1$ there is a solution to the Towers of Hanoi puzzle when there n rings.

It is possible go a bit farther and find the number of moves in the solution to the puzzle with n rings. Some experimenting with 1, 2, and 3 rings leads to the conjecture that if there are n rings, then the there is a solution that uses $2^n - 1$ moves. Let's outline the argument that this statement is true. When there is one ring, exactly one move is needed, so that statement is true when $n = 1$. Assume that we know solutions so that solving the puzzle with 1 ring takes $2^1 - 1$ moves, that solving it with 2 rings takes $2^2 - 1$ moves, and so on, until solving it with k rings takes $2^k - 1$ moves, where $k \geq 1$. Consider an instance of the puzzle with $k + 1$ rings. Before the largest ring can be moved, the k other rings must be stacked in decreasing order of size on one of the other towers. By assumption, this can be done using $2^k - 1$ moves. Moving the largest ring takes one move, and then properly stacking the remaining k rings on top of it can be done using $2^k - 1$ more moves. Thus the number of moves for tolve the puzzle with $k + 1$ rings is $2(2^k - 1) + 1 = 2^{k+1} - 1$, as desired. Therefore, for any $n \geq 1$, the puzzle with n rings can be solved using $2^n - 1$ moves.

Legend has it that the end of the world would come before a person could complete the solution to the puzzle with 64 rings. By the above, it would take $2^{64} - 1$ moves. There are $60 \times 60 \times 24 \times 365 = 31536000 \approx 2^{24.9}$ seconds in a year, ignoring leap years. Hence, if a person could move one ring per second, then solving the puzzle would take about 2^{39} years.

4.3 PMI: The Principle of Mathematical Induction

The Principle of Mathematical Induction (PMI) is a theorem that gives a method for establishing the truth of statements quantified over all integers greater than or equal to some given integer. An example of such a statement is "*For any $n \geq 1$, a $2^n \times 2^n$ punctured grid can be exactly covered by L-shaped tiles*". Another is "*Every integer greater than or equal to six can be written as a sum of twos and threes*". In computer science, statements like these regularly arise in the analysis of algorithms. But not only that, proofs by induction also tend to imply recursive algorithms for solving the problem at hand. Further, PMI is a main tool in proving the correctness of recursive algorithms. Witness the L-shaped tiles example in the previous paragraph. *Whenever you need to prove a statement that is quantified over all integers greater than of equal to some given integer, then one tool you should consider trying to use is PMI.* (As usual, it may or not be successful to complete the task at hand.)

It turns out that there are two forms of PMI – a so-called strong form and a so-called weak form – but they are of identical expressive power. In other words, any statement that can be proved by one of them can be proved by the other. However, it is often true that a proof using one form (usually the strong form) involves a lot less writing than a proof using the other form. The choice of which to use is really a matter of mathematical aesthetics, and sheer laziness (wanting to write less, or wanting the writing to be easier). We will begin our discussion of PMI with the strong form of induction, and come to the weak form later.

Theorem 4.3.1 (Strong Form of PMI) *Let $S(n)$ be a statement whose truth depends on the integer n. If the following two conditions hold:*

1. *the statement $S(n)$ is true when n is any of the integers $n_0, n_0+1, \ldots, t$, for some $t \geq n_0$;*
2. *the truth of the statement $S(n)$ for all of the integers $n_0, n_0+1, \ldots, k$, where $k \geq t$, logically implies the truth of $S(n)$ when $n = k+1$;*

then, the statement $S(n)$ is true for all integers $n \geq n_0$.

We discuss two important matters before giving many examples of how to use this theorem. The first is the proof method implied by the theorem. The second is why it implies such a method.

The strong form of PMI is commonly referred to as *strong induction* or sometimes just *induction*. We will discuss the qualifier "strong" after introducing "weak induction" The theorem says we can prove that a $S(n)$ is true for all $n \geq n_0$ by doing two things:

1. Directly check that $S(n)$ is true for the first few possible values of n, say $n = n_0, n = n_0 + 1, \ldots, n = t$, where $t \geq n_0$. (It turns out that the

size of t depends on what you're trying to prove.) This is called the *Basis* because it is the foundation that the rest of the argument rests on.

2. Prove that if $S(n)$ is true for all possible values of n from n_0 up to k, where $k \geq t$, then it is also true when $n = k + 1$. This is called the *Induction* because we use (bring forward) the truth of $S(n)$ for smaller values of n to prove that $S(n)$ is true for the next possible value of n. Usually the induction is separated into two parts. In the *Induction Hypothesis* one assumes the truth of $S(n)$ for all vales $n = n_0,\ n = n_0 + 1,\ \ldots,\ n = k$ for *some* k which is at least as large as the biggest value checked in the Basis. In the *Induction Step* one uses this information to show that $S(n)$ is also true when $n = k + 1$.

Now, why do these two points imply the conclusion we want? The first point says the statement $S(n)$ is true for all values of n from n_0 up to t. Using this, the second point says that the statement $S(n)$ is also true when $n = t+1$. So, now, we have that $S(n)$ is true or all values of n from n_0 up to $t + 1$. But using this and the second point (again), we get that $S(n)$ is true or all values of n from n_0 up to $t + 2$. This procedure can be repeated over and over. For any particular integer $x \geq n_0$, after enough applications of the second assertion we have that the statement $S(n)$ is true when $n = x$. But x is an arbitrary integer which is greater than or equal to n_0. Hence, after we know the first assertion is true, enough applications of the second assertion would allow us to conclude that $S(n)$ is true for any given integer $x \geq n_0$. Thus it follows that $S(n)$ is true for all integers $n \geq n_0$.

By the above discussion, a proof using PMI has four components:

1. A *Basis* in which it is checked that the statement $S(n)$ is true for all values of n from n_0 up to some $t \geq n_0$ (whose size depends on what is needed in the Induction Step, described below).

2. An *Induction Hypothesis*, in which it is assumed that the statement $S(n)$ is true for all values of n from n_0 up to k, where $k \geq t$.

3. An *Induction Step*, in which the Induction Hypothesis (an assumption) is used to argue that the statement $S(n)$ is true when $n = k + 1$.

4. A *Conclusion*, in which it is asserted that PMI implies that the statement $S(n)$ is true for all $n \geq n_0$.

It is customary to carry out these four steps in clearly labelled sections. Many examples follow in later subsections.

We conclude this subsection by discussing the proof of PMI. It depends on the Well-Ordering Principle (WOP), which is the following self-evident theorem:

Theorem 4.3.2 (Well Ordering Principle) *Let X be a non - empty set of integers that is bounded below (ie. every integer in the set is at least as big as some constant $n_0 \in \mathbb{Z}$). Then X has a smallest element.*

The WOP asserts that if this is the case, then X has a smallest element. If $n_0 \in X$, then it is the least element of X. Otherwise, since X is not empty, there is a first integer after n_0 which belongs to X (remember that infinity is not an integer!), and this integer is the smallest element of X.

We now explain how the WOP implies PMI. The proof is by contradiction. Suppose assertions (1) and (2) of PMI hold, but the conclusion that $S(n)$ is true for all $n \geq n_0$ is false. Then, the set X of integers greater than or equal to n_0 for which $S(n)$ is false is not empty. By assertion (1), none of the values $n_0, n_0 + 1, \ldots, t$ belong to X. Hence X is bounded below by n_0. By the WOP, the set X has a smallest element, call it $k + 1$. Note that $k + 1 \geq t + 1$, so that $k \geq t$. Since $k + 1$ is the smallest element in X, the statement $S(n)$ is true when n is any of the integers $n_0, n_0 + 1, \ldots, k$, where $k \geq t$. But then, by assertion (2) of PMI, the statement $S(n)$ is true when $n = k + 1$, a contradiction to $k + 1$ being the smallest integer n for which $S(n)$ is false. Hence $S(n)$ must be true for all $n \geq n_0$.

It transpires that if one assumes the truth of PMI, then one can use that assumption to prove the truth of the WOP. The WOP and PMI are regarded as equivalent in the sense that each logically implies the other.

4.4 Formalizing the informal examples

In this section we revisit the examples in Section 4.2 and write the proofs in the traditional format of a proof using PMI. It turns out that there is not a great difference between what we did previously and these more structured arguments.

4.4.1 The sum of the first n odd positive integers

The statement $S(n)$ we want to prove is $1 + 3 + \cdots + 2n - 1 = n^2$, and we want to prove it is true for all $n \geq 1$.

Basis. Since $1 = 1^2,\ 1 + 3 = 2^2, 1 + 3 + 5 = 3^2$ and $1 + 3 + 5 + 7 = 4^2$, the statement is true when $n = 1, 2, 3, 4$.

Induction Hypothesis. Suppose that $1 + 3 + \cdots + 2n - 1 = n^2$ for $n = 1, 2, \ldots, k$, where $k \geq 4$.

Induction Step. We want to show that $1 + 3 + \cdots + 2(k+1) - 1 = (k+1)^2$. Look at the LHS,

$$\begin{aligned} 1 + 3 + \cdots + 2(k+1) - 1 &= 1 + 3 + \cdots + (2k-1) + 2(k+1) - 1 \\ &= k^2 + 2(k+1) - 1 \quad \text{(by IH)} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2, \quad \text{as wanted.} \end{aligned}$$

Conclusion. Therefore. by PMI, for any $n \geq 1$, $1 + 3 + \cdots + 2n - 1 = n^2$. □

4.4.2 A postage stamp problem

The statement $S(n)$ that we want to prove is "*the integer n can be written as a sum of 3's and 5's*", and we want to prove it for all $n \geq 8$.

Basis. Since 8 = 5+3, 9 = 3+3+3, and 10 = 5+5, each of 8, 9, and 10 can be written as a sum of 3s and 5s.

Induction Hypothesis. Assume each of $8, 9, 10, \ldots, k$ can be written as a sum of 3's and 5's, where $k \geq 10$.

Induction Step We want to show that $k + 1$ can be written as a sum of 3's and 5's. Since $k \geq 10$, $k+1-3 \geq 8$, so by the Induction Hypothesis, $k+1-3$ can be written as a sum of 3's and 5's But then adding 3 to this sum gives $k + 1$ as a sum of 3's and 5's, which is what we wanted.

Conclusion. Therefore, by the strong form of PMI, any integer $n \geq 8$ can be written as a sum of 3's and 5's. □

4.4.3 The Towers of Hanoi

The statement $S(n)$ that we want to prove is "*There is a solution to the Towers of Hanio puzzle with n rings*", and we want to prove if for all $n \geq 1$.

Basis. It is clear how to solve the puzzle if $n = 1$. There is only one ring – just move it!

Induction Hypothesis. Suppose we know how to solve the puzzle with n rings for $n = 1, 2, \ldots, k$, where $k \geq 1$.

Induction Step. Consider the puzzle with $k + 1$ rings. It can be solved in three phases. First, legally move the k smallest rings to one of the other towers. We know how to do this by the Induction Hypothesis. Leaving the large ring in place will not cause the constraint that a larger ring may not rest atop a smaller one to be violated. Second, move the largest ring to the empty tower. Finally, legally move the k smallest rings so that they are on top of the largest one. Again, this is possible because we know how to solve the puzzle with k rings.

Conclusion. Therefore, by PMI, for any $n \geq 1$ there is a solution to the Towers of Hanoi puzzle when there n rings. $\Box$

A good exercise is to formally write out the proof by PMI that the puzzle with n rings can be solved in $2^n - 1$ moves.

4.5 Silly analogies for PMI

In carrying out a proof by PMI, it is important to carry out all four of the steps. The only two that require any real work are checking that the Basis holds (in enough cases so that the Induction Step works), and then proving that the logical implication needed for the Induction Step. The other two steps are important, however, especially for communication; *it is definitely worth making an effort to clearly state the Induction Hypothesis.*

Some people draw an analogy between PMI and climbing as high as you want on a really tall ladder, starting from rung n_0. Assertion (2) is saying that if you have climbed up the steps $n_0, n_0 + 1, \ldots, k$, where $k \geq t$, then you can climb up to step $k + 1$. By itself, this does not matter much. You have to be able to get on the ladder and complete the steps $n + 0, n_0 + 1, \ldots, t$, otherwise you can't use assertion (2) repeatedly to conclude that you can climb as high as you want on the ladder. Thus assertion (1) is of crucial importance in the argument. It might be that, depending on the situation, the rung t to which you have to climb before being assured of the ability to continue differs.

Other people draw an analogy between PMI and toppling dominoes. Suppose you have an infinite row of dominoes that are arranged close together, but that dominoes $n_0, n_0+1, \ldots, t$ are exceptionally heavy. Suppose you happen to be able to prove that if you can make dominoes $n_0, n_0+1, \ldots, k$ fall over, where $k \geq t$, then the next domino in the row is guaranteed to fall over. Pushing over domino n_0 alone won't help if domino n_0+1 is so heavy that it won't fall over when struck by domino n_0. And pushing over the first few of $n_0, n_0+1, \ldots, t$ won't help if the the next domino is also very heavy. The only thing to do is make sure you individually push over each of the dominoes $n_0, n_0+1, \ldots, t$. After you do that, you can conclude from your argument that all of the dominos will fall over.

4.6 An example of bad reasoning

A classical example of needing both of the assertions the fallacious argument that *in any group of $n \geq 1$ people, all people in the group have the same hair colour*. Certainly it is true that in any group of 1 people, all people in the group have the same hair colour. Suppose that it is true that in any group of 1 up to t people, all people in the group have the same hair colour, for some $t \geq 1$. Now consider a group of $t+1$ people. We want to use the Induction Hypothesis to argue that all people in this group have the same hair colour. Consider any member of the group. Call her Anna. By the Induction Hypothesis, all t members of the group who are not Anna have the same hair colour. Now consider any other member of the group. Call him Bill. By the Induction Hypothesis, all t members of the group who are not Bill have the same hair colour. But now Anna and Bill each have the same hair colour as all the remaining members of the group, and so all $t+1$ members of the group have the same hair colour. Therefore, by PMI, in any group of $n \geq 1$ people, all people in the group have the same hair colour.

Now, the statement "proved" in the previous paragraph is certainly not true, and so there must something wrong with the argument. In the Basis we checked only up to $t = 1$. (Had we checked up to $t = 2$, and been the least bit alert, there would have been trouble.) So this means the first application of the Induction Step is supposed to take us from the truth of the statement for all values of n from 1 up to 1, to the truth of the statement for all values of n from 1 up to 2. But the argument does not work as there are no group members besides Anna and Bill. In saying that they each have the same hair colour as all members the rest of the group we are assuming that there is at least one more person in the group. There isn't. Thus the argument given to establish the Induction Step is wrong, as it does not work when $k = 1$. A different way to view the problem is that we have an Induction Step that is valid so long as $k \geq 2$, but no Basis that supports the truth of the Induction Hypothesis in that case.

4.7 The weak form of induction

Look back at the induction examples we have done so far. In the sum of the first n odd positive integers, and the Towers of Hanoi, completing the Induction Step required only that we assume the statement $S(n)$ to be true when $n = k$ (and not all values from n_0 up to k). In the postage stamp problem, completing the Induction Step required the truth of $S(n)$ for a value between 8 and k.

Mathematicians care about aesthetics, and so we do not like to assume more than we need. If completing the Induction Step requires only that $S(n)$ when $n = k$, we don't want to assume any more than that. It is also true that some proofs become much easier to write using the weak form of induction because it is much easier to state the Induction Hypothesis.

Theorem 4.7.1 (Weak Form of PMI) *Let $S(n)$ be a statement whose truth depends on the integer n. If the following two conditions hold:*

1. *the statement $S(n)$ is true when $n = n_0$;*

2. *the truth of the statement $S(n)$ when $n = k$, where $k \geq n_0$, logically implies the truth of $S(n)$ when $n = k + 1$;*

then, the statement $S(n)$ is true for all integers $n \geq n_0$.

As before, a proof using the result given by this theorem has four parts: a Basis, an Induction Hypothesis, and Induction Step, and a Conclusion. There are two differences between the strong form of PMI and the weak form of PMI. One of them is that the weak form has only one case in the Basis, whereas the strong form may involve many cases in the Basis. The main difference, however, is that in the weak form the Induction Hypothesis only that the statement to be proved holds when $n = k$ (and not all values from n_0 up to k). The terms "strong form" and "weak form" arise from this comparison between the Induction Hypotheses. The strong form has a stronger Induction Hypothesis in the sense that more seems to be being assumed.

The reason the conclusion holds is the same as before. We know that the statement is true for n_0. The induction (assertion (2)) then allows us to conclude that the statement is true for $n_0 + 1$. Using this, the induction (assertion (2)) then allows us to conclude that the statement is true for n_0+2. And so on, until finally we can reach any integer $x \geq n_0$. Thus, as before, the only reasonable conclusion is that the statement is true for all integers $n \geq n_0$. Note, also, that by the time we have applied assertion (2) enough times to know the statement is true when $n = k$, we have actually proved that it is true for all integers between n_0 and k (identical to the assumption in the strong form of induction).

The proof of the weak form of PMI is virtually identical to the proof given for the strong form. It is a good exercise to write it out and see the underlying logic for yourself.

In what follows, we will give some more example of using PMI, and will freely use the weak form when it is possible to do so. How do you know which form to use? Sometimes you don't until after completing the Induction Step and seeing the smaller values for which you need the truth of the statement. *It is always safe to use the strong form of PMI.* but your proofs might look a lot prettier (and you might look more aware of what's being assumed) with the weak form.

4.8 Examples involving sums and inequalities

4.8.1 Summations

The key point in using PMI to prove summation identities occurs in the Induction Step: *remember the meaning of the ellipsis "...", and substitute the assumed value from the Induction Hypothesis for the first k terms in the sum (and don't forget to keep the $(k+1)$-st term, then do algebra to get what you want.*

Example. *Prove that, for any natural number $n \geq 1$,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Basis. When $n = 1$, we have LHS$= 1$ and RHS$= 1(1+1)/2 = 1$. Thus the statement is true when $n = 1$.

Induction Hypothesis. Assume that $1+2+3+\cdots+k = k(k+1)/2$ for some $k \geq 1$.

Induction Step. We want to prove that

$$1 + 2 + 3 + \cdots + (k+1) = (k+1)((k+1)+1)/2 = (k+1)(k+2)/2.$$

Consider the LHS:

$1 + 2 + \cdots + (k+1)$
$= 1 + 2 + \cdots + k + (k+1)$ (meaning of the ellipsis)
$= k(k+1)/2 + 2(k+1)/2$ (by IH, and getting a common denominator)
$= (k+1)(k+2)/2$ as desired.

Conclusion. Therefore, by induction, $1+2+3+\cdots+n = n(n+1)/2$ for all $n \geq 1$. $\square$

4.8.2 Summation identities to memorize

There are a number of summations that arise frequently. You should both memorize them, and know how to prove each one. Induction always works, though there can be other proofs as well.

- For any natural number $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

- For any natural number $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- For any natural number $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

  It is a fluke that the RHS is the square of the first identity above. The pattern does not continue.

- (Sum of a geometric series.) For any natural number $n \geq 1$ and any real number r,

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

4.8.3 Inequalities

The following sort of argument arises all the time in proving inequalities where one "side" is a polynomial. Suppose $n \geq 6$, and consider $n^3 + 4n^2 + 5n + 3$. Since $n \geq 6$, we can change the right hand 3 to n, that is $n^3 + 4n^2 + 5n + 3 \leq n^3 + 4n^2 + 5n + n = n^3 + 4n^2 + 6n$ In the same way, replacing $6n$ by n^2 only makes the expression larger (because $n \geq 6$). Thus, $n^3 + 4n^2 + 6n \leq n^3 + 4n^2 + n^2 = n^3 + 5n^2$. And, doing the same again to replace $5n^2$ by n^3 (because $n \geq 6$) gives $n^3 + 5n^2 \leq n^3 + n^3$. Putting this all together, we have just shown that if $n \geq 6$, then $n^3 + 4n^2 + 5n + 3 \leq 2n^3$. We use this method in the example below.

Example. *Prove that for all $n \geq 5$, $2^n > n^2$.*

Basis. When $n = 5$, $2^n = 2^5 = 32$ and $n^2 = 5^2 = 25$. As $32 > 25$ the statement is true for $n = 5$.

Induction Hypothesis. Suppose $2^k > k^2$, for some $k \geq 5$.

Induction Step. We want to show $2^{k+1} > (k+1)^2$. Consider $(k+1)^2 = k^2 + 2k + 1 < k^2 + 2k + k$ (as $k \geq 5 > 1$) $= k^2 + 3k < k^2 + k(k)$ (as $k \geq 5 > 3$) $= 2k^2 < 2(2^k)$ (by the induction hypothesis) $= 2^{k+1}$, which is what we wanted.

Conclusion. Therefore, by the Principle of Mathematical Induction, for all $n \geq 5$, $2^n > n^2$. □

There is a fairly established hierarchy of the growth rates of functions, and it is used all the time when comparing the performance of algorithms on inputs of given size (for example, algorithms that operate on n items usually use a number of steps proportional to n^2, or to $n \log_2(n)$; when n is large, this difference matters in terms of how long it takes for the task to be completed.). What that means is that for all large enough values (and maybe not for small ones), functions at a higher level in the hierarchy are greater that those at a lower level. Constants are at the bottom of the hierarchy, then logs. And then polynomials. The higher the degree, the faster the growth. Exponential functions always eventually become greater than any polynomial, and factorials always eventually become larger that exponentials. Finally, functions like n^n eventually become larger than factorials. Inequalities between functions at the various levels of this hierarchy can be proved with induction.

Example. *Prove that* $n! > 3^n$ *for all* $n \geq 7$.

Basis. When $n = 7$ we have $n! = 7! = 5040$ and $3^n = 3^7 = 2187$. Hence the statement to be proved is true when $n = 7$.

Induction Hypothesis. Assume $k! > 3^k$ for some $k \geq 7$.

Induction Step. We want to show that $(k+1)! > 3^{k+1}$. Consider the RHS. We have $3^{k+1} = 3 \cdot 3^k < 3 \cdot k!$ by the Induction Hypothesis. Now, since $k + 1 \geq 8 > 3$, we have $3 \cdot k! < (k+1)k! = (k+1)!$, as wanted.

Conclusion. Therefore, by PMI, $n! > 3^n$ for all $n \geq 7$. □

4.9 A subtraction game

Subtraction games are two-player games in which there is a pile of objects, say coins. There are two players, Alice and Bob, who alternate turns subtracting from the pile some number of coins belonging to a set S (the *subtraction set*). Alice goes first. The first player who is unable to make a legal move loses.

For example, suppose the initial pile contains 5 coins, and each player can, on his turn, remove any number of coins belonging to the set $S = \{1, 2, 3\}$. Who wins? Alice goes first. On her turn she removes 1, 2, or 3 coins from the pile. If she removes 3, then the game reduces to a 2-coin game with Bob going first. Bob wins on his next move. Similarly, if she removes 2, then the game reduces to a 3-coin game with Bob going first, and Bob wins on his next move. But, if she removes 1, then the game reduces to a 4-coin game with Bob going first, and no matter what move Bob makes, Alice wins on her next move.

In any subtraction game, the winner can be determined if we know how many coins are in the pile, and which player is next to play. Suppose there are n coins in the pile. If the player next to play take some coins and leave a position from which the opponent (who becomes the player next to play) has no winning strategy, then he can win. If he can not do this, then every legal move leaves a position from which the opponent has a winning strategy, and so the player whose turn it is ca not win.

In the table below, we enter N if the player next to play has a winning strategy, and O if the opponent has a winning strategy. The discussion above says that the n-th entry is N whenever there is a legal move so that the entry in the corresponding position is O, and otherwise (the entry corresponding to every legal move is N) it is O/

For the game at hand, we can summarize the winner for each value of n in a table.

n	1	2	3	4	5	6	7	8	9	10	11	12
Who	N	N	N	O	N	N	N	O	N	N	N	O

In making the table (do it!), a pattern of who wins for which values of n becomes apparent. It is summarized, and proved, below.

Example. *In the subtraction game with $n \geq 1$ coins and $S = \{1, 2, 3\}$, if n is not a multiple of 4 then the next player to play has a winning strategy, and if n is a multiple of 4 then the opponent has a winning strategy.*

<u>Basis:</u> If $n = 1, 2$ or 3, then the next player to play can win on their move by taking all of the coins. Thus the statement to be proved is true when

$n = 1, 2$, or 3.

Induction hypothesis: Suppose that the statement to be proved is true when n is any of $1, 2, \ldots, k$, where $k \geq 3$. That is, in each of these situations, if the number of coins in the pile is not a multiple of 4, then the next player to play has a winning strategy, and if the number of coins in the pile is a multiple of 4 , then the opponent has a winning strategy.

Induction step: Suppose the pile has $n = k + 1$ coins. There are 2 cases to consider

If $k + 1$ is a multiple of 4, then any legal move leaves a pile in which the number of coins is not a multiple of 4. By the induction hypothesis, from each of these positions the next player to play has a winning strategy. Hence, in this case, the position in which there are $k + 1$ coins is such that the opponent has a winning strategy.

If $k + 1$ is is not a multiple of 4, then the next player can remove $1, 2$ or 3 coins, as needed, so that the number of coins remaining in the pile will be a multiple of 4. By the induction hypothesis, the opponent has a winning strategy from resulting position. Hence, in this case, the position in which there are $k + 1$ coins is such that the next player to play wins.

Conclusion: By the Principle of Mathematical Induction, for any $n \geq 1$, if n is not a multiple of 4 then the next player to play has a winning strategy, and if N is a multiple of 4 then the opponent has a winning strategy.

4.10 Recursive definitions

The word "recursive" originates from the Latin word *recurs*, which means "returned", and which arises from a verb that means "go back". Informally, we will call a process "recursive" if it refers back to itself. In mathematics, a process is recursive if successive results depend on previous ones; a function is recursive if the value of the function at some elements of the domain depends on its value at other elements of the domain. In order to avoid an infinite regression of self-references, some basic outcomes (results, values) must be explicitly known without any self-reference.

A *recursive definition of a sequence* consists of two parts:

1. one or more *base cases* that explicitly state one or more terms of the

sequence, and

2. a *recursion* (that is, a function) that gives other terms of the sequence in terms of those already known.

Some examples follow.

- The sequence $1, 2, 4, 8, \ldots, 2^n, \ldots$ is recursively defined by $a_0 = 1$, and $a_{n+1} = 2a_n$, for all $n \geq 0$.
- The sequence $-5, -2, 1, 4, \ldots, 3n - 5, \ldots$ is recursively defined by $a_0 = -5$ and $a_{n+1} = a_n + 3$ for all $n \geq 0$.
- The sequence $a_1, a_2, \ldots$ where $a_n = 1 + 2 + \cdots + n$ is recursively defined by $a_1 = 1$, and $a_{n+1} = a_n + (n + 1)$.
- The *Fibonacci Sequence* $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ is recursively defined by $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$.

After describing the first few terms explicitly, the key to writing recursive definitions is to imagine that all terms up to the n-th are of the correct form, and then to describe how to get the $(n+1)$-st term from those already defined. Go back over the examples above with this in mind.

We now generalize the example in the first bullet point. A *geometric progression* (or geometric sequence) is a sequence $a, ar, ar^2, ar^3, \ldots$, where $a, r \in \mathbb{R}$. (Remember that $a = ar^0$, so the sequence can also be written as $ar^0, ar, ar^2, ar^3, \ldots$.) Geometric progressions (with common ratio r) have the property that the ratio of each term to the one immediately before it is (the same number) r. These sequences can be recursively defined by $g_0 = a$, and $g_{n+1} = rg_n$ for all $n \geq 0$.

We now generalize the example in the second bullet point. An *arithmetic progression* (or arithmetic sequence) is a sequence $a, a + d, a + 2d, a + 3d, \ldots$, where $a, d \in \mathbb{R}$. Arithmetic progressions are sequences such that the difference between any term and the one after it is (*the common difference*) d. These sequences can be recursively defined by $b_0 = a$, and $b_{n+1} = b_n + d$ for all $n \geq 0$.

The Fibonacci sequence has many wild and wonderful properties. Every third Fibonacci number is even, every fourth is a multiple of three, every fifth is a multiple of 5, every sixth is a multiple of 8. In general, every n-th Fibonacci number is a multiple of f_n. All of these facts can be proved using PMI. Another remarkable fact which can also be proved by these methods is that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

This is even more stunning when you stop to think that f_n is an integer! Just for the sake of interest, let's look at the right hand side a bit more closely. The quantity $\frac{1-\sqrt{5}}{2}$ is less than one, so $\left(\frac{1-\sqrt{5}}{2}\right)^n$ converges to zero (quickly) as n grows. Because of this, it turns out that f_n is the nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$, i.e., the integer that arises from rounding.

Other things can also be defined recursively in the same way. As with the example of sequences, a *recursive definition* consists of two parts:

1. one or more *bases cases* that explicitly describe some of the basic items, and

2. a *recursion* that gives other items in terms of those already known.

One example is the recursive definition of *n-factorial*, that is, the quantity $n! = 1 \times 2 \times \cdots \times n$, where n is a non-negative integer (remember that an empty product equals zero): $0! = 1$ *and* $n! = n \times (n-1)!,\ n \geq 1.$

Suppose you are given a machine that will add two numbers like, say, your calculator. How can you use it to compute the sum of n numbers? Surely you add the first two numbers, then add the third to the total, and then the fourth, and so on. It may not be apparent that doing so implicitly uses a recursive definition of summation. Given numbers $x_1, x_2, \ldots, x_n$, let $S_i = x_1 + x_2 + \cdots + x_i$. Then $S_2 = x_1 + x_2$, and $S_k = S_{k-1} + x_k$ for $k \geq 3$.

Other (associative) operations like multiplication, set union, set intersection, conjunction of logical propositions, and disjunction of logical propositions can be recursively defined in a similar way.

4.11 Examples involving recursively defined sequences

In this section we present a few examples of induction proofs involving recursively defined sequences. Typically these involve showing that some sort of formula holds. An important point to remember is that *the Basis of the induction often has the same number of cases as the Basis of the recursive definition*.

Proofs involving recursively defined sequences in which the recursion has more than one term are almost always by strong induction. The reason is that, in the Induction Step, you will want to apply the recursion and then make a substitution for each of the terms that arise from doing so. Typically this involves substituting for more than just the k-th term of the sequence.

4.11.1 Formulas for the terms of recursively defined sequences

This subsection illustrates a method to use when proving the correctness of a formula for the terms of a recursively defined sequence.

Example. *Let a_n be the sequence recursively defined by $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. Show that $a_n = 2^n$ for all $n \geq 0$.*

Basis. When $n = 0$ we have $a_0 = 1 = 2^0$ and when $n = 1$ we have $a_1 = 2 = 2^1$. Hence the statement is true when $n = 0$ and $n = 1$.

Induction Hypothesis. Assume that $a_n = 2^n$ when $n = 0, 1, \ldots, k$, where $k \geq 1$. That is, assume $a_0 = 2^0, a_1 = 2^1, \ldots, a_k = 2^k$.

Induction Step. We went to show that $a_{k+1} = 2^{k+1}$. Consider a_{k+1}. Since $k + 1 \geq 2$ we have

$$a_{k+1} = 3a_k - 2a_{k-1} = 3 \times 2^k - 2 \times 2^{k-1}$$

by the Induction Hypothesis. The RHS of this expression equals $3 \times 2^k - 2^k = 2^k(3 - 1) = 2^{k+1}$, as needed.

Conclusion. Therefore, by PMI, $a_n = 2^n$ for all $n \geq 0$. □

4.11.2 A property of the Fibonacci numbers

Example. *Show that every third Fibonacci number is even.*

Let's first translate the problem. We want to show that, for all $n \geq 1$, f_{3n} is even.

Basis. We have $f_{3 \cdot 1} = f_3 = 2$, which is clearly even. Thus, the statement is true when $n = 1$.

Induction Hypothesis. Suppose, for some $k \geq 1$, that f_{3k} is even.

Induction Step. We want to show that $f_{3(k+1)} = f_{3k+3}$ is even Consider f_{3k+3}. Since $k \geq 1,\ 3k + 3 \geq 6$ so we can use the recursion to write

$$f_{3k+3} = f_{3k+2} + f_{3k+1} = (f_{3k+1} + f_{3k}) + f_{3k+1} = f_{3k} + 2f_{3k+1}.$$

Now, the last term on the RHS is even, and the first term on the RHS is even by the Induction Hypothesis. Therefore, $f_{3k} + 2f_{3k+1} = f_{3k+3}$ is even, as desired.

Conclusion. Therefore, by PMI, for all $n \geq 1$, f_{3n} is even. □

Much more is true. Every fourth Fibonacci number is a multiple of 3, every fifth one is a multiple of 5, every sixth one is a multiple of 8, and in general every n-th Fibonacci number is a multiple of f_n.

4.11.3 Bounds for the n-th Fibonacci number

Recall the Fibonacci numbers $f_1, f_2, \ldots$ are defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$.

Example. *Use the strong form of mathematical induction to prove that $f_n \leq 2^{n-1}$ for any natural number $n \geq 1$.*

Basis. We have $f_1 = 1 \leq 2^0$ and $f_2 = 1 \leq 2^1$. Thus the statement is true when $n = 1$ and when $n = 2$.

Induction Hypothesis. Assume that $f_n \leq 2^{n-1}$ when $n = 1, 2, \ldots, k$, where $k \geq 2$. That is, assume $f_1 \leq 2^0,\ f_2 \leq 2^1, \ldots, f_k \leq 2^{k-1}$.

Induction Step. We want to prove that $f_{k+1} \leq 2^{(k+1)-1} = 2^k$.
Consider f_{k+1}. Since $k + 1 \geq 3$ we have

$$
\begin{array}{rcll}
f_{k+1} & = & f_k + f_{k-1} & \text{(by definition of } f_{k+1}) \\
& \leq & 2^{k-1} + 2^{(k-1)-1} & \text{(by IH)} \\
& = & 2^{k-2}(2+1) & \text{(algebra)} \\
& \leq & 2^{k-2}2^2 & \text{(because } 3 \leq 4 = 2^2) \\
& = & 2^k & \text{as wanted.}
\end{array}
$$

Conclusion. Therefore, by PMI, $f_n \leq 2^{n-1}$ for all natural numbers $n \geq 1$. □

It is worth emphasizing the importance of having two cases in the Basis. In the Induction Step we want to take f_{k+1} and replace it by $f_k + f_{k-1}$. The recursive part of the definition can only be applied when $k+1$ is at least 3.

By using a bit more algebra, better upper bounds are possible. For example, for all integers $n \geq 1$, $f_n \leq (7/4)^{n-1}$.

4.11.4 Iterating one term recurrences and proving the formula obtained to be correct

For our work in this subsection it is important to know the value of the sum of a geometric progression of finite length, that is, of $a + ar + ar^2 + \cdots + ar^n$, for any integer n. Since $a + ar + ar^2 + \cdots + ar^n = a(1 + r + \cdots + r^n)$, it is enough to know the value of the bracketed sum. Suppose $S = 1 + r + \cdots + r^n$. Then $rS = r + r^2 + \cdots + r^{n+1}$, so that $rS - S = r^{n+1} - 1$. All other terms cancel. Therefore, factoring the left hand side, $S(r-1) = r^{n+1} - 1$ so that $S = \frac{r^{n+1}-1}{r-1}$.

Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 2$ and $a_n = 7a_{n-1} + 2$ for $n \geq 2$.

By direct computation,

$$
\begin{array}{rcl}
a_1 & = & 2 \\
a_2 & = & 7a_1 + 2 = 16 \\
a_3 & = & 7a_2 + 2 = 114 \\
a_4 & = & 7a_3 + 2 = 800
\end{array}
$$

Computing the exact values in this way does no help find a formula for the n-therm of the sequence unless you happen to have amazing powers of observation. The best way to obtain formula is to write out the derivation for the first few cases, but don't perform any additions or multiplications (except for collecting exponents with the same base), and then try to recognize what you have as something you know. *If there is a pattern, it is typically fairly apparent after working out enough cases that the calculation is routine and boring – typically that means working out about 4 cases.*

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= 7a_1 + 2 = 7 \cdot 2 + 2 \\
a_3 &= 7a_2 + 2 = 7(7 \cdot 2 + 2) + 2 = 7^2 \cdot 2 + 7 \cdot 2 + 2 \\
a_4 &= 7a_3 + 2 = 7(7^2 \cdot 2 + 7 \cdot 2 + 2) + 2 = 7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 + 2 \\
a_5 &= 7a_4 + 2 = 7(7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 + 2) + 2 \\
&= 7^4 \cdot 2 + 7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 + 2
\end{aligned}$$

At this point it seems reasonable to conjecture that

$$a_n = 2(7^{n-1} + 7^{n-2} + \cdots + 1) = 2\frac{7^n - 1}{7 - 1} = \frac{7^n - 1}{3}$$

for all $n \geq 1$.

We can prove the conjectured formula is correct using PMI.

Basis. When $n = 1$ we have $a_1 = 2 = \frac{7^1 - 1}{3}$, as desired. Thus the statement is true when $n = 0$.

Induction Hypothesis Assume that $a_i = \frac{7^i - 1}{3}$ for $i = 0, 1, \ldots, k$, for some $k \geq 0$.

Induction Step We want to show that $a_{k+1} = \frac{7^{(k+1)} - 1}{3}$. Since $k + 1 \geq 2$ we can use the recursion to write $a_{k+1} = 7a_k + 2 = 7(\frac{7^k - 1}{3} + 2$, by the Induction Hypothesis. Hence $a_{k+1} = \frac{7^{k+1} - 7}{3} + \frac{6}{3} = \frac{7^{k+1} - 1}{3}$, as desired.

Conclusion. Therefore, by PMI, $a_n = \frac{7^n - 1}{3}$ for all $n \geq 1$. □

Example. *Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$ and $a_n = a_{n-1} + 3n^2$ for $n \geq 1$. Find, with proof, a formula for a_n for all $n \geq 0$.*

Finding the formula. Write out the first few values, but be very judicious about doing multiplications or collecting terms. Keep going until the calculation becomes boring.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= a_0 + 3 \cdot 1^2 = 0 + 3 \cdot 1^2 \\
a_2 &= a_1 + 3 \cdot 2^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 \\
a_3 &= a_2 + 3 \cdot 3^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 + 3 \cdot 3^2 \\
a_4 &= a_3 + 3 \cdot 4^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 + 3 \cdot 3^2 + 3 \cdot 4^2
\end{aligned}$$

At this point is seems reasonable to conjecture that $a_n = 3(1^2 + 2^2 + \cdots + n^2)$ for all $n \geq 0$. The bracketed expression is a known sum, so our conjecture really is that $a_n = 3n(n+1)(2n+1)/6 = n(n+1)(2n+1)/2$ for all $n \geq 0$. We now prove the conjecture by induction.

Basis When $n = 0$ we have $a_n = a_0 = 0$ and $n(n+1)(2n+1)/2 = 0(1)(1)/2 = 0$. Thus the statement is true when $n = 0$.

Induction Hypothesis. Assume that $a_k = k(k+1)(2k+1)/2$ for some $k \geq 0$.

Induction Step. We want to show that $a_{k+1} = (k+1)((k+1)+1)(2(k+1)+1)/2 = (k+1)(k+2)(2k+3)/2$. Look at a_{k+1}. Since $k+1 \geq 1$, we can use the recursion to write

$$\begin{aligned}
a_{k+1} &= a_k + 3(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{2} + 3(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{2} + \frac{6(k+1)^2}{2}
\end{aligned}$$

where, in the last two steps, we used the Induction Hypothesis, then got a common denominator. Now,

$$\begin{aligned}
\frac{k(k+1)(2k+1)}{2} + \frac{6(k+1)^2}{2} &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{2} \\
&= \frac{(k+1)[2k^2 + 7k + 6)]}{2} \\
&= \frac{(k+1)(k+2)(2k+3)}{2}
\end{aligned}$$

which is what we wanted.

Conclusion. Therefore, by PMI, $a_n = n(n+1)(2n+1)/2$ for all $n \geq 0$. $\square$