Quantifiers and Written Proofs Questions

1. Write each statement in plain English. Do not use any symbols except the letters that denote elements of the universe.

   (a) $\forall x, \forall y, (x \neq -y) \rightarrow (x+y) \neq 0$, where the universe is the real numbers.

   (b) $\exists s, \forall t, p(s) \wedge [(t \neq s) \rightarrow \neg p(t)]$, where the universe of s and t is the collection of all students who completed Math 122 last fall, and $p(s)$ is the assertion "s got 100% on the final exam".

2. Suppose the collection of allowed replacements for the variables is the integers. Let $p(n)$ be "*n is even*" and $q(n)$ be "*n is odd*". Determine the truth value of each statement and provide a brief explanation of your reasoning.

   (a) $\forall n, p(n) \vee q(n)$

   (b) $[\exists n, p(n)] \wedge [\exists n, q(n)]$

   (c) $\exists n, p(n) \rightarrow q(n)$

   (d) $[\forall n, p(n)] \wedge [\forall n, q(n)]$

   (e) $\forall n, \exists m, n+m=0$

   (f) $\exists n, \forall m, n+m=0$

3. Write each statement in plain English.

   (a) $\forall x, [(x \neq \text{"Quebec"}) \rightarrow v(x)] \wedge \neg v(\text{"Quebec"})$, where the universe of x is the collection of all major Canadian cities, and $v(x)$ is the assertion "Gary has visited x".

   (b) $\exists s, \forall t, p(s) \wedge [(t \neq s) \rightarrow \neg p(t)]$, where the universe of s and t is the collection of all students who completed Math 122 last fall, and $p(s)$ is the assertion "s got 100% on the final exam".

4. Suppose that the collection of allowed replacements for the variable p is {*Gary*, *Christi*} and the collection of allowed replacements for the variable c is {*Whitehorse*, *Ottawa*, *Halifax*}. Let $v(p,c)$ be the statement "p has visited c". Write each statement in symbolic form without quantifiers.

   (a) Christi has visited every city.

   (b) There is a city Gary has not visited.

   (c) For every person there is a city which they have visited.

5. According to Robert Plant, the original first line of the Led Zepplen song *Stairway to Heaven* was "*There's a lady who knows all is glitters, is gold, and she is buying a stairway to heaven.*" Explain why, when this statement is written in symbols, either 3 or 4 quantifier appear, and the two formulations are logically equivalent.

6. Determine if each statement below is true or false, and explain your reasoning.

   (a) The negation of "*Every golf shot is a hook or a slice*" is "*Some golf shots are hooks and slices*".

   (b) The negation of "*All enforcers skate slowly and pass badly*" is "*Some enforcers skate fast and pass well*".

   (c) For integers m and n, arguing that if mn is odd then m and n are odd proves that if m or n is even then mn is even.

7. Suppose the universe of m and n is $\{-1, 0, 1\}$. Then, for example,

$$\exists n, n^2 + n > 0 \Leftrightarrow ((-1)^2 + (-1) > 0) \vee (0^2 + 0 > 0) \vee (1^2 + 1 > 0).$$

   For each of the following statements,

   (i) write a compound statement involving neither quantifiers nor variables that is logically equivalent to the given quantified statement,

   (ii) determine whether the statement is TRUE or FALSE, and

   (iii) write the negation of the quantified statement in symbols, with quantifiers, and without using negation ($\neg$) or any negated mathematical symbols like $\neq$ or $\not<$.

   (a) $\forall n, n^3 - n = 0$

   (b) $\exists n, \forall m, n + m < 1$.

8. Determine if each statement below is true or false, and explain your reasoning.

   (a) When the statement "*There is no largest integer.*" is written is symbols, both of the quantifiers $\forall$ and $\exists$ appear.

   (b) For the universe of real numbers, $\forall x, \exists y, xy = 1$ is false.

   (c) For the universe of integers, $\exists x, (x^2 < 0) \rightarrow (x > 10)$ is true.

9. Let $(0, 1)$, the open interval consisting of the real numbers x such that $0 < x < 1$, be the universe. Consider the statement $\mathcal{A} : \exists x, \forall y,\ y \leq x$.

   (a) Write $\mathcal{A}$ in English without using symbols except $x,\ y,\ (0, 1)$.

   (b) Write down the negation of statement $\mathcal{A}$ in symbols without using either of $\neg$ and $\not\leq$.

   (c) Explain why $\mathcal{A}$ is false.

10. Consider the following (correct) argument in which all variables represent integers.

    Suppose n and k are odd.
    Then $n = 2t + 1$ for some integer t, and $k = 2\ell + 1$ for some integer ℓ.
    Hence, $nk = (2t + 1)(2\ell + 1) = 4t\ell + 2t + 2\ell + 1$.
    Therefore, nk is odd.

(a) Write the implication proved by the argument in plain English.

(b) Write the contrapositive of the implication in plain English. Is it also proved by the argument?

(c) Write the converse of your statement in (a). Is it also proved by the argument?

11. Consider the following. All variables represent integers.

Proposition: If n^2 is a multiple of 8, then n is a multiple of 8.
Proof: Let $n = 8m$. Then $n^2 = 64m^2 = 8(8m^2)$, which is a multiple of 8, as desired. $\square$.

Why does the given argument not prove the proposition? Either give a correct proof, or give an example to show that the proposition is false.

12. (a) Let n be in integer. Explain what is wrong with the following argument which "shows" that *if n is a multiple of 2 and a multiple of 3, then n is a multiple of 6.*

Suppose n is a multiple of 6. Then $n = 6k$ for some integer k.
Since $6 = 2 \times 3$, we have that $n = 2 \times (3k)$, so it is a multiple of 2,
and $n = 3 \times (2k)$, so it is a multiple of 3. $\square$

(b) Give a correct proof of the assertion.

13. (a) Suppose that m and n are integers. It is claimed that the argument below proves that *if mn is odd, then m and n are both odd.* Does it? Explain your reasoning.

Suppose that the integers m and n are both even. Then there exists an integer k such that $m = 2k$, and there exists an integer ℓ such that $n = 2\ell$. Thus,

$$mn = (2k)(2\ell) = 2(2k\ell).$$

Since $2k\ell$ is an integer, mn is even.

14. Suppose that the integer a is a multiple of 3, and the integer b is a multiple of 4. Give a direct proof that ab is a multiple of 12.

15. Prove that if the integer n^2 is a multiple of 5, then the integer n is a multiple of 5. (Hint: prove the contrapositive using a proof by cases; there are 4 cases.)

16. Prove that $\sqrt{5}$ is irrational. (Hint: In the proof that $\sqrt{2}$ is irrational, read the phrase "*is even*" as "*is a multiple of 2*", and then try using the same argument with 2 replaced by 5. Also, use the result in Question 15.)

17. Prove that:

(a) The sum of two even inters is even.

(b) The sum of an even integer and an odd integer is odd.

(c) The sum of two odd integers is even.

(d) The product of two even integers is even. Further, it is a multiple of 4.

(e) The product of an even integer and an odd integer is even.

(f) The product of two odd integers is odd.

(g) If a and b are integers such that $a+b$ is even, then a and b are both even or both odd.

(h) If a and b are integers such that $a+b$ is odd, then a is even and b is odd, or a is odd and b is even.

(i) If a and b are integers such that ab is even, then a is even or b is even.

(j) If a and b are integers such that ab is odd, then a and b are both odd.