Functions and Relations Questions

1. Answer each question true or false, and briefly explain your reasoning.
   (a) Cartesian product is commutative on sets: $A \times B = B \times A$ for all A, B.
   (b) If $A \times B = B \times A$ then either $A = \emptyset$ or $B = \emptyset$.

2. Let A, B and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

3. Let A, B and C be sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

4. Prove that for all sets A, B, C and D, if $A \cap C = \emptyset$, then $(A \times B) \cap (C \times D) = \emptyset$.

5. For each of the following, if the statement is true then prove it, and if it is false then give an example or explanation demonstrating it is false.
   (a) The function $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = x$ is invertible.
   (b) The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x - 2$ is onto.
   (c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x + 9$ is 1-1.

6. List all of the functions from $\{a, b, c\}$ to $\{a, b\}$ and identify the ones that are (i) one-to-one, (ii) onto, (iii) both one-to-one and onto, (iv) neither one-to-one nor onto.

7. (a) Give an example of a function from $\mathbb{N}$ to $\mathbb{Z}$ that is onto. Is your function also 1-1?
   (b) Give an example of a 1-1 function from $\mathbb{Z}$ to $\mathbb{N}$. Is your function also onto?

8. Let a and b be integers, with $a \neq 0$.
   (a) Is the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = ax + b$, 1-1 and onto?
   (b) When is the function $f : \mathbb{Q} \to \mathbb{Q}$, where $f(x) = ax + b$, 1-1 and onto?
   (c) Repeat part (b) with $\mathbb{Q}$ replaced by $\mathbb{Z}$.

9. Suppose that f is a function from A to B. Let $g = \{(y, x) : (x, y) \in f\}$. Explain why g being a function from B to A implies that f is 1-1 and onto (hint: the definition of function).

10. Let f and g be the functions from $\{a, b, c, d, e, f\}$ to $\{a, b, c, d, e, f\}$ given in the following table:

$x =$	a	b	c	d	e	f
$f(x) =$	c	d	a	e	f	b
$g(x) =$	b	c	a	e	f	d

   (a) Find $f \circ g$ and $g \circ f$.
   (b) Show that $g^{-1} = g^2$. The notation g^2 means $g \circ g$. In general, g^n means $g \circ g \circ g \cdots \circ g$, where g appears n times ($n - 1$ compositions).

(c) Find f^2 and $f^4 = (f^2)^2$. What does this tell you about f^{-1}?

11. Let $f : A \to B$ and $g : B \to C$ be functions with $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $C = \{w, x, y, z\}$ such that $g \circ f = \{(a, y), (b, x), (c, w), (d, w)\}$ and $g = \{(1, y), (2, w), (3, x)\}$. Find f.

12. Let $f : A \to B$ and $g : B \to C$ be functions. Prove:

    (a) If $g \circ f$ is one-to-one and f is onto, then g is one-to-one.

    (b) If $g \circ f$ is onto and g is one-to-one, then f is onto.

    (c) Give an example to show that, in (a) and (b) above, f need not be onto and g need not be one-to-one.

    (d) Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Let the functions f and g be

$$f = \{(1, a), (2, b)\} \quad \text{and} \quad g = \{(a, 1), (b, 2), (c, 1)\}.$$

    Verify that $g \circ f = \iota_A$, and then explain why g is not the inverse of f.

13. Indicate whether each statement is true or false, and briefly justify your answer.

    (a) The relation $\{(x, y) : y^2 = (x - 2)^2 + 4\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

    (b) Suppose $|A| \geq 6$. Every function $f : A \to \{1, 2, 3, 4, 5, 6\}$ that is onto contains exactly six ordered pairs.

    (c) If $f : \{a, b, c, d\} \to \{1, 2, 3\}$ and $g : \{1, 2, 3\} \to \{a, b, c, d\}$ are such that $f \circ g(x) = x$ for every $x \in \{1, 2, 3\}$, then g is the inverse of f.

    (d) If $x \in \mathbb{R} \setminus \mathbb{Z}$, then $\lfloor x \rfloor = \lceil x \rceil - 1$.

    (e) Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is a 1-1 correspondence, then $g \circ f$ has an inverse and $|A| = |C|$.

14. Let a and b be integers, with $a \neq 0$.

    (a) Is the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = ax + b$, 1-1 and onto?

    (b) When is the function $f : \mathbb{Q} \to \mathbb{Q}$, where $f(x) = ax + b$, 1-1 and onto?

    (c) Repeat part (b) with $\mathbb{Q}$ replaced by $\mathbb{Z}$.

15. Let $A = \{1, 2, 3\}$. Give an example of a relation on A (that is, list the ordered pairs in your example) that is:

    (a) reflexive, but neither symmetric nor transitive;

    (b) symmetric, but neither reflexive nor transitive;

    (c) reflexive and transitive, but not symmetric;

    (d) antisymmetric and transitive;

    (e) neither symmetric nor antisymmetric.

16. Answer each question true or false, and briefly explain your reasoning.

   (a) $\emptyset$ is a binary relation on any set A.

   (b) If $|A| = 4$, then there are exactly 2^{16} relations on A.

   (c) If $\mathcal{R}$ is an anti-symmetric relation on $\mathbb{Z}$ and $(1,2) \notin \mathcal{R}$, then $(2,1) \in \mathcal{R}$.

   (d) For any set A, there is exactly one relation on A which is reflexive, symmetric, transitive and anti-symmetric.

   (e) The relation $\sim$ on $\{2,3\}$, defined by $x \sim y$ if and only if xy is odd, is reflexive.

   (f) The set of all relations from A to B is $\mathcal{P}(A \times B)$.

   (g) For the set $A = \{1,2,3\}$, if the relation $\mathcal{R}$ on A is anti-symmetric and $(1,3) \in \mathcal{R}$, then $\mathcal{R}$ is not symmetric.

   (h) For any set A, there is a relation $\mathcal{R}$ on A that is both symmetric and anti-symmetric.

17. Let $\sim$ be a reflexive, symmetric and transitive relation on $A = \{1,2,3\}$ such that $2 \sim 3$ and $1 \not\sim 2$. Write $\sim$ as a set of ordered pairs.

18. Suppose $\mathcal{R}$ is a symmetric and transitive relation on $A = \{1,2,3,4\}$ such that $(3,1), (3,2), (2,4) \in \mathcal{R}$. Must $\mathcal{R} = A \times A$?

19. (a) Suppose A is a non-empty set and $\mathcal{R}$ is a symmetric and transitive relation on A. Suppose further that each element $x \in A$ appears in some ordered pair in $\mathcal{R}$ (as either the first coordinate or the second coordinate). Prove that $\mathcal{R}$ is reflexive.

   (b) Why is the statement in part (a) true if $A = \emptyset$?

20. Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by $(a,b) \in \mathcal{R}$ if and only if $a - b \leq 1$. Determine, with a proof or counterexample as appropriate, whether $\mathcal{R}$ is (i) reflexive, (ii) symmetric, (iii) anti-symmetric, (iv) transitive.

21. Let $A = \{1,2,3,4\}$. Determine, with proof, whether each statement below is True or False.

   (a) If a relation $\mathcal{R}$ on A is anti-symmetric, then $\mathcal{R}$ can not be symmetric.

   (b) If a relation $\mathcal{R}$ on A is symmetric and transitive, and $(1,2), (1,3), (1,4) \in \mathcal{R}$, then $\mathcal{R}$ is reflexive.

22. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are relations on a non-empty set A. Determine if each of the following statements is true or false. Prove each true statement. For each false statement, give a counterexample using $A = \{1,2,3\}$.

   (a) If $\mathcal{R}$ and $\mathcal{S}$ are both anti-symmetric, then $\mathcal{R} \setminus \mathcal{S}$ is anti-symmetric.

   (b) If neither $\mathcal{R}$ nor $\mathcal{S}$ is symmetric, then $\mathcal{R} \cup \mathcal{S}$ is not symmetric.

(c) If $\mathcal{R}$ and $\mathcal{S}$ are both equivalence relations, then so is $\mathcal{R} \cap \mathcal{S}$.

23. Let C be the set of all circles drawn in the plane with centre at $(0, 0)$. Let $\mathcal{R}$ be the relation on C defined by $c_1 \mathcal{R} c_2$ if and only if the radius of c_1 is at least as large as the radius of c_2, Prove that $\mathcal{R}$ is anti-symmetric.

24. Let $\sim$ be the relation on $\mathbb{N} = \{1, 2, \ldots\}$ defined by $x \sim y$ if and only if x/y is an integer. Prove that $\sim$ is anti-symmetric.

25. Let $\mathcal{R}$ be the relation on $\mathbb{N}$ defined by $(a, b) \in \mathcal{R}$ if and only if b is a multiple of a, that is, $b = ak$ for some integer k. Prove that $\mathcal{R}$ is reflexive, anti-symmetric and transitive. Which of these three properties would no longer hold if the relation $\mathcal{R}$ were on $\mathbb{Z}$ instead?

26. Let $\mathcal{E}$ be the relation on $\mathbb{Q}$ defined by $(a/b, c/d) \in \mathcal{E}$ if and only if $ad = bc$.

    (a) Show that $\mathcal{E}$ is reflexive, symmetric and transitive, but not anti-symmetric.

    (b) What can you say about the fractions a/b and c/d if $(a/b, c/d) \in \mathcal{E}$? And why?

27. Let S be a set that contains at least two different elements. Let $\mathcal{R}$ be the relation on $\mathcal{P}(S)$, the set of all subsets of S, defined by $(X, Y) \in \mathcal{R}$ if and only if $X \cap Y = \emptyset$. Determine whether $\mathcal{R}$ is reflexive, symmetric, antisymmetric, or transitive. Why is it important that S has at least 2 different elements? Would any of the answers change if S was empty or had only one element?

28. Repeat the previous question using the relation $\mathcal{R}$ defined by $(X, Y) \in \mathcal{R}$ if and only if $X \subsetneqq Y$.

29. Let $\sim$ be an equivalence relation on $A = \{v, w, x, y, z\}$ with three equivalence classes. Suppose $v \sim y$ and $z \in [x]$, where $[x]$ denotes the equivalence class of x. Write $\sim$ as a subset of $A \times A$ and find the partition of A determined by the equivalence classes.

30. Let $\sim$ be an equivalence relation on the set $A = \{1, 2, \ldots, 8\}$, and denote the equivalence class of $x \in A$ by $[x]$.

    (a) Suppose that $1 \in [3], 4 \in [2]$, and $2 \in [1]$. Prove that $[4] = [3]$.

    (b) Ignore part (a) and suppose now that $\sim$ has 3 equivalence classes. If $[1]$ has 2 elements, $[2]$ has 3 elements, $1 \sim 6$, $2 \sim 5$, and $7 \sim 5$, then

        i. Write $\sim$ as a set of ordered pairs.

        ii. Find the partition of A induced by $\sim$.

31. Let T be a equilateral triangle with each side having length 1. Imagine T in a fixed position in the plane, say with the bottom side on the x-axis and the opposite angle above it. Let S be the set of coloured triangles obtainable from T by painting each side with one of the colours red and blue. Any combination

of colours is allowed, for example all sides could have the same colour. Note that S has 8 elements: for example the bottom side being red and all other sides being blue is a different painting than the leftmost side being red and all other sides being blue.

Define a relation $\mathcal{R}$ on S by $s_1 \mathcal{R} s_2$ if and only if s_1 can be rotated so that the rotated coloured triangle is identical to s_2. Prove that $\mathcal{R}$ is an equivalence relation and find the equivalence classes. (The elements of your sets can be pictures of the coloured triangles.)

32. Let $\sim$ be the relation on $T = \{10, 11, \ldots, 99\}$ defined by $a \sim b \Leftrightarrow a$ has the same first digit as b (that is, the same leftmost digit as b). Prove that $\sim$ is an equivalence relation.

33. Take it as given that the relation $\mathcal{R}$ on $A = \{1, 2, \ldots, 46\}$ defined by $x\mathcal{R}y$ if and only if $x - y$ is a multiple of 10 is an equivalence relation.

    (a) How many of the equivalence classes $[6], [13], [16], [28], [38], [46]$ are different? Why? Explain in at most two sentences.

    (b) How many subsets belong to the partition of A induced by $\mathcal{R}$? Why?