Chapter 2

Sequences and Series

2.1 Sequences

Simply speaking, a sequence is a list of number, written in an order

$$\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots\}$$

where the elements a_i represents numbers. In this section, we only concentrate an infinite sequences.

Definition 1. An infinite sequence of number is a function whose domain is the set of positive integers.

Some examples are

(1) $1, 2, 3, \cdots$ (2) $2, 4, 5, 8, \cdots$ (3) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

In each case, the three dots are used to suggest that the sequence continuous indefinitely following the obvious pattern.

The most common way to specify a sequence is to give a formula that relate the terms in the sequence to their term numbers. For example, in the sequence $2, 4, 6, 8, \cdots$ each term is twice the term number; that is, the n^{th} term in the sequence is given by the formula $2n$.

We can write this sequence as $2, 4, 6, 8, \cdots, 2n, \cdots$ or $\{2n\}_{n=1}^{\infty}$ or $\{2n\}$

Example 10. Express the following sequences in the notation $\{a_n\}_{n=1}^{\infty}$.
(Find a formula for the n^{th} term of the following sequences):

1. $2, \frac{3}{4}, \frac{4}{9}, \cdots$
2. $1, \sqrt{2}, \sqrt{3}, \cdots$
3. $0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \cdots$
4. $1, -1, 1, -1, \cdots$
5. $1, 0, 1, 0, \cdots$
6. $1, -4, 9, -16, \cdots$

7. $0, 3, 8, 15, \cdots$

8. $0, 1, 1, 2, 2, 3, 3, 4, 4, \cdots$

9. $2, 6, 10, 14, 18, \cdots$

10. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \cdots$

11. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$

Solution:

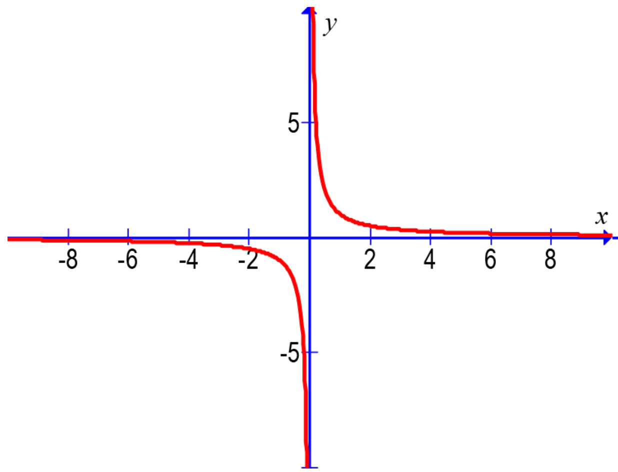
Homework

1. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$
2. $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \cdots$
3. $1, 5, 9, 13, 17, \cdots$
4. $-3, -2, -1, 0, 1, \cdots$
5. $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \cdots$
6. $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \cdots$

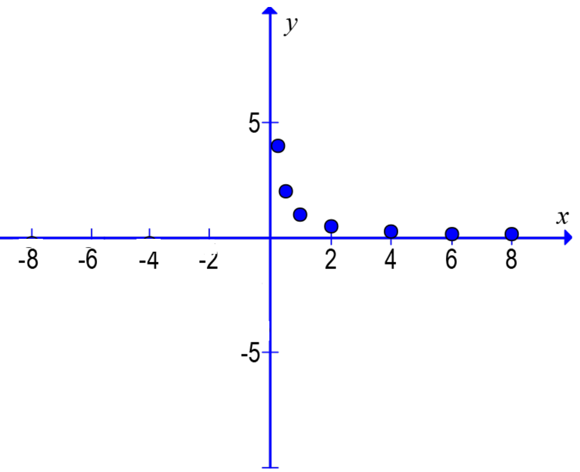
Graph of the Sequences

The sequences are functions, we may care about the graph of a sequence. For example, the graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is the graph of the equation

$$y = \frac{1}{n}, n = 1, 2, 3, \cdots$$



$f(x) = \frac{1}{x}$



$f(n) = \frac{1}{n}$

Definition 2. A sequence $\{a_n\}$ is said to be **converge** to the limit L if given any $\varepsilon > 0$, there is a positive integer N such that $|a_n - L| < \varepsilon$ for $n \geq N$. In this case we write

$$\lim_{n\to\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to be **diverge**

Example 11. Determine whether the following sequences converges or diverges. If it converges, find the limit.

1. $\left\{\frac{1-2n}{1+2n}\right\}$
2. $\{\sqrt{n}\}$
3. $\{(-1)^n\}$
4. $\left\{\left(\frac{1}{1+2n}\right\}\right.$
5. $\left\{(-1)^{n+1}\frac{1}{n}\right\}$
6. $\left\{\frac{n}{e^n}\right\}$

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Theorem 1. Suppose that the sequence $\{a_n\}$ and $\{b_n\}$ converge to L_1 and L_2 resepectively, and c is a constant. Then:

(a) $\lim\limits_{n\to\infty} c = c$

(b) $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n = cL_1$

(c) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L_1 \pm L_2$

(d) $\lim\limits_{n\to\infty} (a_n \quad b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = L_1 \cdot L_2$

(e) $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$

Theorem 2 (Sandwich Theorem)**.** If $a_n \leq b_n \leq c_n$ holds for all $n \geq N$, N is some index, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ also.

Example 12. Determine whether the following sequences converges or diverges. If it converges, find the limit.

1. $\left\{\frac{\ln n}{n}\right\}$
2. $\left\{\sqrt[n]{n}\right\}$
3. $\left\{\frac{\sin n}{n}\right\}$
4. $\left\{\frac{\cos n}{n}\right\}$
5. $\left\{\frac{1}{2^n}\right\}$
6. $\left\{\frac{(-1)^n}{n}\right\}$
7. $\left\{x^{\frac{1}{n}}\right\}, x > 0$
8. $\left\{\frac{n!}{n^n}\right\}$

Theorem 3. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Proof:

Since, $-|a_n| \le a_n \le |a_n|$ and $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty}|a_n| = 0$, by squeeze theorem, we must also have $\lim_{n\to\infty} a_n = 0$. □

Increasing, Decreasing and Bounded sequences:

A sequence $\{a_n\}$ is said to be

1. increasing if $a_n \le a_{n+1}$ for every n
2. decreasing if $a_n \ge a_{n+1}$ for every n
3. strictly increasing if $a_n < a_{n+1}$ for every n
4. strictly decreasing if $a_n > a_{n+1}$ for every n
5. monotonic if any of these four properties holds.

Note: To show that a sequence is increasing, we can try one of the following:

1. Show that $a_n < a_{n+1}$ for all n

2. Show that $a_{n+1} - a_n > 0$ for all n
3. If $a_n > 9$ for all n, then show that $\frac{a_{n+1}}{a_n} > 1$ for all n
4. If $f(n) = a_n$, show that $f'(x) < 0$.

Example Show that $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing sequence.

solution1:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+1)}$$
$$= \frac{1}{(n+1)(n+1)} > 0, \text{for any } n.$$

solution2:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2}\frac{n+1}{n}$$
$$= \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1, \text{for any } n.\text{(why)}.$$

Example 13. Determine whether the following sequences is increasing, strictly increasing, decreasing, strictly decreasing or not monotonic.

1. $\left\{1 - \frac{1}{n}\right\}$
2. $\left\{\frac{n}{4n-1}\right\}$
3. $\{n - 2^n\}$
4. $\left\{\frac{2^n}{1+2^n}\right\}$
5. $\{e^{-n} n\}$
6. $\left\{\frac{n^n}{n!}\right\}$
7. $\left\{\frac{1}{n+\ln n}\right\}$
8. $\left\{\tan^{-1} n\right\}$
9. $\{\sin n\pi\}$
10. $\{n + (-1)^n \sqrt{n}\}$
11. $\left\{\frac{n-1}{n+1}\right\}$
12. $\left\{\frac{\sqrt{n+1}}{n}\right\}$
13. $\{\ln n - \ln n + 2\}$

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1. Frequently one can guess whether a sequence is increasing or decreasing after writing out some of the initial terms. However, to be certain that the guess is correct, a precise mathematical proof is needed.
2. A sequence $\{a_n\}$ would called eventually monotone if there is some integer N such that the sequence is monotone for $n \geq N$

Example: Show that the sequence $\left\{\frac{10^n}{n!}\right\}$ is eventually.

Solution: we have $a_n = \frac{10^n}{n!}$, $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

So, $\frac{a_{n+1}}{a_n} = \frac{10}{n+1} \Rightarrow \frac{a_{n+1}}{a_n} < 1$ for all $n \geq 10$

Definition 3. A sequence $\{a_n\}$ is said to be bounded from above if there exists a number M such that $a_n \leq M$ for all n. M is called **an upper bound** of the sequence. A sequence $\{a_n\}$ is said to be bounded from below if there exists a number m such that $\{a_n \geq m\}$ for all n. m is called **lower bound** of the sequence. A sequence is **bounded** if it is bounded from above and below.

Example:

1. The sequence $\{(-1)^n\}$ has upper bound 1, and the lower bound -1.
2. The sequence $\{\sin n\}$ has upper bound 1, and the lower bound -1.
3. The sequence $\{2^n\}$ is bounded below by 2 but has no upper bound.
4. The sequence $\{(-1)^n\ 2^n\}$ is bounded neither below nor above.

Theorem 4. If the sequence $\{a_n\}$ is convergent, then it is bounded.
or (Every unbounded sequence is divergent) (without proof).

Note: The converse is not true, for example $(-1)^n, \sin n$ is bounded but not convergent.

Theorem 5. If the sequence $\{a_n\}$ is convergent, then the point of convergent is unique.

Theorem 6. A bounded monotonic sequence is convergent.

Example: Show that the sequence $\left\{(1+2^n)^{\frac{1}{n}}\right\}$ convergence use above theorem.

Example: Show that the sequence $\left\{(2^n+3^n)^{\frac{1}{n}}\right\}$ convergence use above theorem.

Definition 4. As sequence $\{a_n\}$ is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that,

$$|a_m - a_n| < \epsilon \quad \text{for all} \quad m, n \geq N$$

Theorem 7. A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence

Example: Prove the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$ is convergent and find the limit

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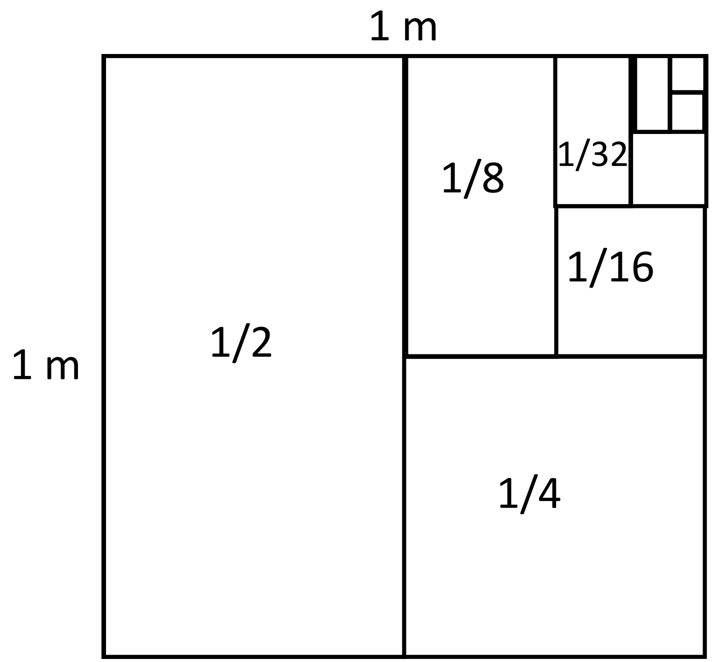
2.2 Infinite Series

An infinite series is the sum of an infinite sequence (say $\{a_n\}$) of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k,$$

where a_1 is called first term, a_2 is second term, and a_n is the nth.
Note: The sume of infinite number maybe has finite sum or not. For example: Suppose we have a square with area is $1m^2$. This area can be represented geometrically as infinity small rectangle by halving each rectangle repeatedly.



$$\text{So } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

and some infinite series don't have a finite sum such as $1 + 2 + 3 + \cdots$

Definition 5 (Partial Sums of a series)**.** Let $\sum_{n=1}^{\infty} a_n$ be a series then S_n is called the nth partial sum of a series where $S_1 = a_1, S_2 = a_1 + a_2$ and $S_n = a_1 + a_2 + \cdots + a_n$, for the series $\sum_{n=1}^{\infty} a_n$, the sequence $\{S_n\}$ is known by the sequence of partial sums of the series.

Definition 6 (Convergent Series)**.** If the sequence $\{S_n\}$ converges to a number

s, then the series $\sum_{n=1}^{\infty} a_n$ converges to s and written $s = \sum_{n=1}^{\infty} a_n$, where S_n is the nth partial sum of the series, i.e.

$$\lim_{n\to\infty} S_n = s \Rightarrow s = \sum_{n=1}^{\infty} a_n$$

Note:

- The letter we use to represent the index does not matter, for example:

$$\sum_{i=1}^{\infty} \frac{3}{i^2+1} = \sum_{k=1}^{\infty} \frac{3}{k^2+1} = \sum_{n=1}^{\infty} \frac{3}{n^2+1}$$

- The index may start at $n = 0$ or $n = 1$ or any other point we can change it to zero or one.

  example: Change the index of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+5}{2^n}$

Example 14. Show that the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent and find its sum.

Solution

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Example 15. Show that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$ is convergent and find its sum.

Solution: first we need to find the partial sums of the series,

$$S_n = \sum_{i=0}^{n} \frac{1}{i^2+3i+2}$$

Now,

$$\frac{1}{i^2+3i+2} = \frac{1}{(i+2)(i+1)} = \frac{1}{i+1} - \frac{1}{i+2}$$

So,

$$S_n = \sum_{i=0}^{n} \frac{1}{i^2+3i+2} = \sum_{i=0}^{n} \left[\frac{1}{i+1} - \frac{1}{i+2}\right]$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{n+2}$$

$$S_n \Rightarrow 1 \ \text{ as } \ n \to \infty$$

Homework 4. Show that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+4n+3}$, ans=5/12. This type of series

Note: The above type of series is called telescoping series. The telescoping is a series whose partial sums eventually only have a fixed number of terms after cancellation.

2.2.1 Some Special Series

Definition 7 (Geometric series)**.** A geometric series is any series that can be written in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$$

in which a and r are fixed real numbers and $a \neq 0$.

Theorem 8. The geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$ and the sume is

$$\frac{a}{1-r} = a + ar^2 + ar^3 + \cdots + ar^{k-1} + \cdots$$

Proof Let us treat the case $|r| = 1$. First if $r = 1$, then the series is

$$a + a + a + \cdots + a + \cdots \ .$$

So, the nth partial sum is $S_n = na$ and $\lim_{n\to\infty} na = \pm\infty$. This prove dievergence.

(The sign depending on whether a is positive or negative).
If $r=-1$, the series is

$$a-a+a-a+\cdots$$

, so the sequence of the partial sums is

$$a, 0, a, 0, \cdots$$

which diverges
Now, consider the case $|r| \neq 1$, the nth partial sum of the series is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots *$$

multiply both sides of (*) by r yields:

$$rS_n = ar + ar^2 + \cdots + ar^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots **$$

Subtracting (**) from (*) gives

$$S_n(1-r) = a - ar^n \text{since} r \neq 1 \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

if $|r| < 1$ i.e. $-1 < r < 1, \lim_{n\to\infty} r^n = 0$, so $\lim_{n\to\infty} S_n = \frac{a}{1-r}$
if $|r| > 1, \lim_{n\to\infty} r^n = \infty or -\infty$ i.e. S_n is divergent. □

Example 16. Determine whether the follwing series converges or diverges. If it converges, find the sum.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$

2. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{-3}{4}\right)^{k-1}$

3. $\displaystyle\sum_{k=1}^{\infty} 4^{k-1}$

4. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{4^k}$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

6. $\displaystyle\sum_{k=1}^{\infty} 9^{-n+2}4^{n+1}$

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A rubber ball dropped on a hard surface takes a sequence of vertical bounces. Each bounce is 3/4 as height as the preceding one. If the ball is dropped from the height of 10 m, **a.** find the total distance it has traveled when it hits the surface fifth time.**b.** find the total distance the ball will travel if it is allowed to bounce indefinitely.

Example 17 (Repeating decimal using geometric series)**.** Express the following repeating decimal as the ratio of two integers.

1. $5.232323\cdots$
2. $0.159159159\cdots$
3. $0.782178217821\cdots$
4. $0.4444\cdots$

$$
\begin{aligned}
5.232323\cdots =& 5 + \frac{23}{100} + \frac{23}{100^2} + \cdots \\
=& 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right) \\
=& 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99}
\end{aligned}
$$

Definition 8 (Harmonic series)**.** One of the most important among all diverging series is the harmonic series which is defined as:

$$\sum_{k=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

We will prove it is divergent in the next section.

Definition 9 (The P-series)**.** The P-series has the following form:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

where p is a real non-negative constant. converges if $p > 1$ and diverges if $0 < p \leq 1$.

2.2.2 Convergence divergence test

1. Divergence test

Theorem 9. (a) If $\lim\limits_{n\to\infty} U_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} U_n$ diverges.

(b) If $\lim_{n\to\infty} U_n = 0$, then the series $\sum\limits_{n=1}^{\infty} U_n$ may either converges or diverges

Example 18. (a) The series $\sum\limits_{n=1}^{\infty} n^2$ diverges, because $n^2 \to \infty$ as $n \to \infty$

(b) The series $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n}$ diverges, because $\frac{n+1}{n} \to 1$ as $n \to \infty$

(c) The series $\sum\limits_{n=1}^{\infty} \dfrac{-n}{2n+5}$ diverges, because $\frac{-n}{2n+5} \to \frac{-1}{2} \neq 0$ as $n \to \infty$

(d) The series $\sum\limits_{n=1}^{\infty} (-1)^n$ diverges, because the limit does not exist.

Theorem 10. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

(a) $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$

(b) $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n = kA$ (where k is any number)

Example 19 (Prove or disprove)**.** .

(a) $\sum_{n=1}^{\infty} a_n$ converges $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges. proof (How?)

(b) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges. disprove (How?)

2. Integral test

Theorem 11. Let $\sum_{k=1}^{\infty} a_k$ be a series with *positive* terms, and let $f(x)$ be the function that $f(n) = a_n$. If f is *decreasing* and *continuous* on the interval $[a, \infty)$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\int_a^{\infty} f(x)dx$, **both converge** or **both diverge**

Example 20. Determine whether the following series converge or diverge

$$1. \sum_{k=1}^{\infty} \frac{1}{k} \qquad 2. \sum_{k=1}^{\infty} \frac{1}{k^2} \qquad 3. \sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

Solution: If we replace k by x in the general term $1/k$, we obtain $f(x) = \frac{1}{x}$, which decreasing, continuous an positive function in the interval $[1, \infty)$. so to apply integral test, we nee to find

$$\int_1^{\infty} \frac{1}{x} = \lim_{l \to \infty} \int_1^{l} \frac{1}{x} dx = \lim_{l \to \infty} [\ln l - \ln 1] = \infty.$$

The integral diverges and consequently so does the series.
By the same manner we can apply the integral test for $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{l \to \infty} \int_1^{l} \frac{1}{x^2} dx = \lim_{l \to \infty} \left[1 - \frac{1}{l} \right] = 1.$$

So the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. (Note but the sum of the series is not 1)

Use integral test to show that the P-series converges if $p > 1$ and diverges if $0 < p < 1$. **Solution:** if $p = 1$, the series is the harmonic series, which was previously shown is divergent.
For $p \neq 1$, we shall use the integral test

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{l \to \infty} \int_1^l x^{-p} dx = \lim_{l \to \infty} \frac{x^{1-p}}{1-p}\bigg|_1^l = \lim_{l \to \infty} \left[\frac{l^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

if $p > 1$ then $1 - p > 0$, so $l^{1-p} \to 0$ as $l \to \infty$. Thus, the integral converges and it is value is $\frac{-1}{1-p}$ and consequently the series also converges. For $0 < p < 1$, if follow that $1 - p > 0$ and $l^{1-p} \to \infty$ as $l \to \infty$. So the integral and the series diverge. □

3. The comparison test

Let $\sum_{n=1}^{\infty} a_n$ be a series with no negative terms:

(a) $\sum_{n=1}^{\infty} a_n$ converges if there is a convergent series $\sum_{n=1}^{\infty} c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N

(b) $\sum_{n=1}^{\infty} a_n$ diverges if there is a divergent series $\sum_{n=1}^{\infty} d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N

Example 21. Determine the following series converges or diverge

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \qquad \sum_{n=1}^{\infty} \frac{1}{n!} \qquad \sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 n}$$

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4. A limit Comparison test

Theorem 12 (without proof)**.** Suppose $a_n > 0$ and $b_n > 0$ for all $n > N$

(a) if $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(c) if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Example 22. Determine which of the following series converge or diverge:

(a) $\sum \frac{2n+1}{n^2+2n+1}$

(b) $\sum \frac{1}{2^n-1}$

(c) $\sum \frac{1+n\ln n}{n^2+5}$

(d) $\sum \frac{1}{1+\sqrt{2}}$

(e) $\sum \frac{n+5}{n^2-2n+3}$

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Note: In order to apply the original version of the comparison test successfully, it is important to have an intuitive feeling for whether the given series converges or diverges. The form of the comparison will depend on whether your are trying to prove converges or diverges. For instance, if you did not know intuitively that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{100n+20000}$ would have diverges to infinity, you might try to argue that

$$\frac{1}{100n+20000} < \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

while true, this does not help at all, as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. We could of course, argue instead that

$$\frac{1}{100n+20000} \geq \frac{1}{20100n} \quad \text{if} \quad n \geq 1, \quad \text{(how?)}$$

and we conclude by comparison theorem $\sum_{n=1}^{\infty} \frac{1}{100n+20000}$ diverges to infinity by comparison with divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.

An easier way is to use limit comparison test and fact that

$$\lim_{n\to\infty} \frac{\frac{1}{100n+20000}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{100n+20000} = \frac{1}{100} > 0$$

However, the limit comparison test has disadvantage when compared to the ordinary comparison test. It can fail in certain cases because the limit L does not exist. In such cases it's possible that the ordinary comparison test may still work. The series $\sum_{n\to\infty} \frac{1+\sin n}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1+\sin n}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} (1+\sin n) \quad \text{does not exist}$$

The limit comparison test gives us no information. However, since $\sin n \leq 1$, we have

$$0 \leq \frac{1+\sin n}{n^2} \leq \frac{2}{n^2}.$$

The given series does, in fact, converge by comparison with $\sum_{n\to\infty} \frac{1}{n^2}$, using comparison test.

5. The Ratio Test

Theorem 13. Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then

(a) if $\rho < 1$, the series converges

(b) if $\rho > 1$ or $\rho = +\infty$, the series diverges

(c) if $\rho = 1$, the series may converges or diverges, so that another test must be tried

Example 23. Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{4^n}$ (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$

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6. The root test

Theorem 14. Let $\sum a_n$ be a series with positive terms and suppose that:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = \rho$$

(a) if $\rho < 1$, the series converges

(b) if $\rho > 1$ or $\rho = +\infty$, the series diverges

(c) if $\rho = 1$, the series may converges or diverges, so that another test must be tried

Example 24. Use the root test to determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$ (b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^5}$

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2.3 Alternating series

So far our emphasis has been on series with positive terms. In this section we shall discuss containing both positive and negative terms.

Definition 10 (Alternating Series)**.** A series in which terms are alternately positive and negative is called alternating series, for example: $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!}$

Definition 11 (The alternating series test)**.** Suppose $\{a_n\}$ is a sequence whose terms satisfy:

(i) $a_n a_{n+1} < 0$ for all integer number n.

(ii) $|a_{n+1}| < |a_n|$ for $n \geq N$ for some integer N.

(iii) $\lim_{n\to\infty} a_n = 0$

Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Example 25. Use the alternating series test to show that the following series converge:

$$(a) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad\qquad (b) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+3)}{k(k+1)}$$

Definition 12. A series $\sum_{n=1}^{\infty} u_n$ is said to be converges absolutely if the series of absolute value $\sum_{n=1}^{\infty} |u_n|$ converges.

For example: the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, as the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is p-series and converges.

Theorem 15 (without proof)**.** If a series converges absolutely, then it converges

The converse the above theorem is not true i.e. converges $\not\Rightarrow$ absolutely converges.

For example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent by the alternating series test, but it does not converge absolutely. If we replace the terms by their absolute values, we get the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Example: The series $1-\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}-\frac{1}{2^6}+\cdots$ converges absolutely (why), so it is convergent

Homework 5. Show that the series $\sum_{k=1}^{\infty}\frac{\cos k}{k^2}$ and $\sum_{k=1}^{\infty}\frac{(-1)2^k}{k!}$ converges.

Definition 13 (Conditional Convergence)**.** if $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say that it is *conditionally convergent* or *convergence conditionally*

2.4 Power Series

Definition 14. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is called a power series or a power series about c. The constants $a_0, a_1, a_2, \cdots, a_n, \cdots$ are called the coefficient of the power series.

Some examples are: $\sum_{k=0}^{\infty} x^k$, $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

Theorem 16. For a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ exactly one of the following is true:

1. The series converges only for $x = c$.
2. The series converges absolutely (and hence converges) for all real calues x.
3. The series converges for all x in some open interval $(c - R, c + R)$ and diverges $x < c - R$ or $x > c + R$. At either of the points $x = c - R$ or $x = c + R$. the series may converges absolutely, converges conditionally, or diverges, depending on the particular case.

Theorem 17. suppose power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exist and is equal to L

1. If $L = \infty$, then $R = 0$ and the series fall into the case (1) in theorem 16.
2. If $L = 0$, then $R = \infty$ and the series fall into the case (2) in theorem 16.
3. If $0 < L < \infty$, then $R = 1/L$ and the series fall iinto the case (2) in theorem 16.

with these two theorems in mind, we calculate the interval of convergence of a power series in one or two steps:

(i) Calculate R. If $R = 0$ the series converges only at 0, and if $R == \infty$ the interval of convergence is $(-\infty, \infty)$.

(ii) If $0 < R < \infty$ check the values $x = -R$ and $x = R$. Then the interval of convergence is $(-R, R), [-R, R), (-R, R] or [-R, R]$ depending on the convergence or divergence of the series at $x = R$ and $x = -R$.

Example 26. For what values of x do the following power series converge?

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$

$\sum_{n=1}^{\infty}\frac{(x-5)^n}{n^2}$

$\sum_{n=1}^{\infty} n!x^n$

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2.4.1 Differentiation and Integration of power series

If a power series has a positive radius of convergence, it can be differentiated or integrated term by term. The resulting series will converge to the appropriate derivative or integral of the sum of the original series everywhere except possibly at the end points of interval of convergence of the original series.

Example 27. Determine the interval of convergence and the sum of each of the following series

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$
2. $\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$
3. $\sum_{n=0}^{\infty}(n+3)x^n = \frac{(3-2x)}{(1-x)^2}$
4. $\sum_{n=0}^{\infty}(-1)^n(4x)^n$

Example 28. Determine power series representations for the functions indicated below. On what interval is each representation valid:

1. $\frac{1}{(1-x)^3}$ in power of x
2. $\ln(1+x)$ in power of x
3. $\frac{1}{2-x}$ in power of x
4. $\frac{1}{1+2x}$ in power of x
5. $\ln(2-x)$ in power of x
6. $\frac{1}{x}$ in power of $x-1$
7. $\frac{1}{x^2}$ in power of $x+2$

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2.4.2 Taylor and Maclaurin Series

if $f(x)$ has derivatives of all orders at $x = c$ (i.e. $f^{(x)}(c)$ exist) for $k = 0, 1, \cdots$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

is called *Taylor series of f about c*. If $c = 0$, the term Maclaurin series is usually used in place of Taylor series.

Example 29. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$, where does each series converge.

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Example 30. Find the Taylor series generated by $f(x) = 1/x$ at $c = 2$, where, if anywhere, does the series converge to $1/x$

Solution: We need to find $f(2), f'(2), f''(2), \cdots$. Taking derivatives we get:

$$f(x) = 1/x \Rightarrow f(2) = 1/2, f'(x) = -1/x^2 \Rightarrow f'(2) = -1/2^2$$

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