# Stage: Second
# Subject: Linear Algebra Application

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# Chapter Three

## 1- Orthogonal Matrix المصفوفة المتعامدة:

**Definition: A square matrix A is called orthogonal matrix if**

$$A.A' = A'.A = I$$

***Defn:*** **An orthogonal matrix is a matrix whose row vectors are orthonormal (** يعني العمود نضربها بنفسها كصف **)=** $A_1 A_1' = 1$ **. then** $AA' = A'A = I$

**Ex: Show that** $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$ **is a orthogonal matrix.**

**Solution: (A is orthogonal matrix if** $A.A' = A'.A = I$ )

Then we must to show $A.A' = A'.A = I$

$$A' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$A'A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}+0 & 0+0+0 & \frac{1}{2}-\frac{1}{2}+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ \frac{1}{2}-\frac{1}{2}+0 & 0+0+0 & \frac{1}{2}+\frac{1}{2}+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA' = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+0+\frac{1}{2} & \frac{1}{2}+0-\frac{1}{2} & 0+0+0 \\ \frac{1}{2}+0-\frac{1}{2} & \frac{1}{2}+0+\frac{1}{2} & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$then\ AA' = A'A = I$

$there\ fore\ A\ is\ an\ orthogonal\ matrix.$

## Properties of orthogonal matrix:

Let A be an_orthogonal matrix

**1-** $AA' = I$

**2-** $AA^{-1} = I \quad \Rightarrow \quad A' = A^{-1}$

**3-** Product of two_orthogonal matrices is a orthogonal matrix.

**4-** Transpose of orthogonal matrices is a orthogonal matrix.

**5-** Invers of orthogonal matrices is a orthogonal matrix.

**6-** If A is orthogonal matrix then $|A| = \pm 1$.

**Ex:** Prove that the following matrix is orthogonal matrix and hens find $B^{-1}$

$$B = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

***Solution:*** **(A is orthogonal matrix if** $B.B' = B'.B = I$ **)**

Then we must to show $B.B' = B'.B = I \quad \Rightarrow B' = B^{-1}$

$$B' = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$B.B' = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4+1+4 & -4+2+2 & -2-2+4 \\ -4+2+2 & 4+4+1 & 2-4+2 \\ -2-2+4 & 2-4+2 & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$then\, B.B' = I$

$there\ fore\, \mathbf{B}\, is\, an\, orthogonal\, matrix.$

$\because \mathbf{B}$ $is\, an\, orthogonal\, matrix\ then\ B.B' = I\ \ and\ \ B B^{-1} = I \ \ \Rightarrow\ B' = B^{-1}$

$$\therefore\ B^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

**Ex:** Show that $C = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is orthogonal matrix .

***Solution:*** **(A is orthogonal matrix if** $B.B' = B'.B = I$ **)**

Then we must to show $B.B' = B'.B = I \ \Longrightarrow\ B' = B^{-1}$

$$B' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$B.B' =$

**Ex:** Show that $A=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal matrix.

***Solution:*** **(A is orthogonal matrix if** $A.A' = A'.A = I$ )

Then we must to show $A.A' = A'.A = I$

$$A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$A.A' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$then\, A.A' = I$

$there\ fore\, A\, is\, an\, orthogonal\, matrix$

**Ex:** Show that $A=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ is orthogonal matrix.

## 2- Orthonormal Matrix:

**Definition: A square matrix A is called orthonormal matrix if**

$$A^{-1} = A' \quad \det(A) = \pm 1$$

We have $AA^{-1} = A^{-1}A = I \ \ and \ \ AA' = A'A = I$

**Ex:** Show that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthonormal matrix.

**Solution:**

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$|A| = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = (\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}) - (\frac{-1}{\sqrt{2}}.\frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} = 1$$

$$adj(A) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore A' = A^{-1} \Rightarrow A \, is \ orthonormal \, matrix$$