

Stage: Second
Subject: Linear Algebra
Application

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Chapter Four

Eigenvalues, Eigenvectors and Characteristic Equation of a

matrix القيم والمتجهات الذاتية لمصفوفة والمعادلة المميزة لها

Definition: If A is $(n \times n)$ matrix, Then a non-zero vector \underline{X} is called **eigenvector** of A if $(A \underline{X})$ is a scalar multiple (مضاعفاً عددياً) of \underline{X} ; that is $(A \underline{X} = \lambda \underline{X})$ for some scalar λ .

The scalar λ is called an **eigenvalue** (القيم الذاتية) of A , and \underline{X} is said to be an eigenvector (متجه ذاتي) corresponding (مناظر) to λ المميزة المعادلة.

To find the eigenvalue:

$$A \underline{X} = \lambda \underline{X}$$

$$A \underline{X} - \lambda \underline{X} = 0$$

$$(A - \lambda I) \underline{X} = 0$$

$$\det(A - \lambda I) = 0 \text{ Characteristic Equation}$$

$$\text{or } |A - \lambda I| = 0 \leftarrow \text{المعادلة المميزة}$$

λ : هي الجذور المميزة eigenvalue (الكامنة) للمصفوفة A , يعني λ هي حل حقيقي للمعادلة المميزة.

Ex: Find the eigenvalue & eigenvector of A if $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution:

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(3-\lambda) + 2 = 0$$

$$-3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda - 1 = 0 \Rightarrow \lambda_1 = 1$$

$$\lambda - 2 = 0 \Rightarrow \lambda_2 = 2$$

for $\lambda = 1$:

$$(A - \lambda I) \underline{X} = 0$$

$$\begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x+2y \\ -x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x+2y=0 \Rightarrow y=-x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ifor $\lambda = 2$:

$$(A - \lambda I) \underline{X} = 0$$

$$\begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-2 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+2y \\ -x-2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x+2y=0 \Rightarrow y = -\frac{1}{2}x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -\frac{1}{2}x \end{bmatrix} = x \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

Ex: If $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Find λ .

Solution:

$$BX = \lambda X$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6-2 \\ 2+0 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \lambda = 2$$

Ex: Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ Find the eigenvalue λ_1 and λ_2 .

Solution:

$$AX_1 = \lambda_1 X_1$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+4 \\ 2+3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \lambda_1 = 5$$

$$2) AX_2 = \lambda_2 X_2$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2-4 \\ 4-3 \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$-1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \lambda_2 = -1$$

Ex: Find the eigenvalue & eigenvector of A if $A = \begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix}$

Solution:

$$A - \lambda I = \begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -2-\lambda & 1 \\ 12 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \quad \text{eigenvalue}$$

$$\begin{vmatrix} -2-\lambda & 1 \\ 12 & -3-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(-3-\lambda) - 12 = 0$$

$$6 + 2\lambda + 3\lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 + 5\lambda - 6 = 0$$

$$(\lambda + 6)(\lambda - 1) = 0$$

$$\left. \begin{array}{l} \lambda - 1 = 0 \Rightarrow \lambda_1 = 1 \\ \lambda + 6 = 0 \Rightarrow \lambda_2 = -6 \end{array} \right\} \text{eigenvalue}$$

for $\lambda = 1$:

$$(A - \lambda I)X = 0 \quad \text{eigenvector}$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 12 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-1 & 1 \\ 12 & -3-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3x + y \\ 12x - 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -3x + y = 0 \Rightarrow x = \frac{1}{3}y$$

$$\text{Let } y = t \Rightarrow x = \frac{1}{3}t$$

$$\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \text{eigenvector}$$

For $\lambda = -6$:

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 12 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+6 & 1 \\ 12 & -3+6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x + y \\ 12x + 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x + y = 0 \Rightarrow x = -\frac{1}{4}y$$

$$\text{Let } y = t \Rightarrow x = -\frac{1}{4}t$$

$$\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \quad \text{eigenvector}$$

Ex: Find the eigenvalue & eigenvector of A if $A = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$

Solution:

$$A - \lambda I = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -2 & -5 \\ 4 & -1-\lambda & -5 \\ -2 & -1 & -3-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0$ eigenvalue

$$\begin{vmatrix} 3-\lambda & -2 & -5 \\ 4 & -1-\lambda & -5 \\ -2 & -1 & -3-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 3-\lambda & -2 & -5 \\ 4 & -1-\lambda & -5 \\ -2 & -1 & -3-\lambda \end{vmatrix} \begin{matrix} 3-\lambda & -1 \\ 4 & -1-\lambda \\ -2 & -1 \end{matrix} = 0$$

$$(3-\lambda)(-1-\lambda)(-3-\lambda) - 20 + 20 - [-8(-3-\lambda) + 5(3-\lambda) + 10(-1-\lambda)] = 0$$

$$(3-\lambda)(3+\lambda)(1+\lambda) - [-10 - 10\lambda + 15 - 5\lambda + 24 + 8\lambda] = 0$$

$$(9 - \lambda^2)(1 + \lambda) + 29 + 7\lambda = 0$$

$$9 + 9\lambda - \lambda^2 - \lambda^3 + 29 + 7\lambda = 0$$

$$[-\lambda^3 - \lambda^2 + 16\lambda - 20 = 0] \times (-1)$$

$$\lambda^3 + \lambda^2 - 16\lambda + 20 = 0$$

هاوكله كهي ژماره (معامل رقم) 20 برتبه له: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

مجموعه حل المعادلة هي مجموعة قيم λ التي تحقق المعادلة والتي يقبل القسمة على (20) القسمة على كل منهما بدون باقير

$$\text{if } \lambda = 1 \Rightarrow 1 + 1 - 16 + 20 \neq 0$$

$$\lambda = -1 \Rightarrow -1 + 1 + 16 + 20 \neq 0$$

$$\lambda = 2 \Rightarrow 8 + 4 - 32 + 20 = 0 \Rightarrow 32 - 32 = 0$$

$$\lambda^3 + \lambda^2 - 16\lambda + 20 = 0$$

$$(\lambda - 2)(\lambda^2 + 3\lambda - 10) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda + 5) = 0 \Rightarrow \lambda = 2, \lambda = 2, \lambda = -5 \text{ eigenvalue}$$

$$\text{When } \lambda = 2 \Rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & -2 & -5 \\ 4 & -1-\lambda & -5 \\ -2 & -1 & -3-\lambda \end{bmatrix} = \begin{bmatrix} 3-2 & -2 & -5 \\ 4 & -1-2 & -5 \\ -2 & -1 & -3-2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 4 & -3 & -5 \\ -2 & -1 & -5 \end{bmatrix}$$

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 1 & -2 & -5 \\ 4 & -3 & -5 \\ -2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x - 2y - 5z \\ 4x - 3y - 5z \\ -2x - y - 5z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y - 5z = 0 \quad \dots(1)$$

$$4x - 3y - 5z = 0 \quad \dots(2)$$

$$-2x - y - 5z = 0 \quad \dots(3)$$

equation(1) - equation(2)

$$\begin{array}{r} x - 2y - 5z = 0 \quad \dots(1) \\ - \mp 4x \pm 3y \pm 5z = 0 \quad \dots(2) \\ \hline -3x + y = 0 \Rightarrow y = 3x \end{array}$$

$$-2x - y - 5z = 0 \quad \dots(3)$$

$$-2x - 3x - 5z = 0 \Rightarrow -5x - 5z = 0 \Rightarrow z = -x$$

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 3x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad \text{eigenvector}$$

$$\text{When } \lambda = -5 \Rightarrow A - \lambda I = \begin{bmatrix} 3 - \lambda & -2 & -5 \\ 4 & -1 - \lambda & -5 \\ -2 & -1 & -3 - \lambda \end{bmatrix} = \begin{bmatrix} 3 + 5 & -2 & -5 \\ 4 & -1 + 5 & -5 \\ -2 & -1 & -3 + 5 \end{bmatrix} = \begin{bmatrix} 8 & -2 & -5 \\ 4 & 4 & -5 \\ -2 & -1 & 2 \end{bmatrix}$$

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 8 & -2 & -5 \\ 4 & 4 & -5 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 8x - 2y - 5z \\ 4x + 4y - 5z \\ -2x - y + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 2y - 5z = 0 \quad \dots(1)$$

$$4x + 4y - 5z = 0 \quad \dots(2)$$

$$-2x - y + 2z = 0 \quad \dots(3)$$

equation(1) - equation(2)

$$\begin{array}{r} 8x - 2y - 5z = 0 \quad \dots(1) \\ - \mp 4x \mp 4y \pm 5z = 0 \quad \dots(2) \\ \hline 4x + 6y = 0 \Rightarrow \end{array}$$

$$2x - 3y = 0 \Rightarrow y = \frac{2}{3}x$$

$$-2x - y + 2z = 0 \quad \dots(3)$$

$$\left[-2x - \frac{2}{3}x + 2z = 0 \right] \times \left(\frac{3}{2} \right) \Rightarrow -3x + x + 3z = 0$$

$$-4x + 3z = 0 \Rightarrow z = \frac{4}{3}x$$

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ \frac{2}{3}x \\ \frac{4}{3}x \end{bmatrix} = x \begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{bmatrix} \quad \text{eigenvector}$$