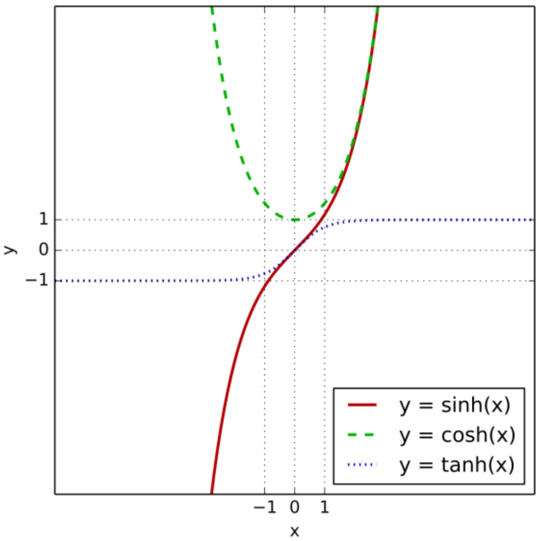
Salahaddin University
College of Engineering
Electrical Department

Chapter Five

Integration methods



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Integration methods

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5.1 Standard form of integration

Below is a list of some basic integrals. These are integrals that should be memorized. All of the integration techniques that we use to compute more complicated integrals are aimed at reducing the more complicated integrals to one of the forms in the basic list.

1. If n is any fixed real number (except -1), then

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

2. $$\int x^{-1} \, dx = \ln |x| + C$$

3. $$\int e^x \, dx = e^x + C$$

4. $$\int \cos(x) \, dx = \sin(x) + C$$

5. $$\int \sin(x) \, dx = -\cos(x) + C$$

5.1 Standard form of integration

Below is a list of some basic integrals. These are integrals that should be memorized. All of the integration techniques that we use to compute more complicated integrals are aimed at reducing the more complicated integrals to one of the forms in the basic list.

6. $$\int \sec^2(x)\,dx = \tan(x) + C$$

7. $$\int \csc^2(x)\,dx = -\cot(x) + C$$

8. $$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

9. $$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

10. $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

11. $$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C.$$

5.1 Standard form of integration

Example 1 *Evaluate the indefinite integral*

$$\int x^2 \left(4x^3+5\right)^8 \, dx.$$

Solution 2 *Make the substitution*

$$\begin{aligned} u &= 4x^3+5 \\ du &= 12x^2 dx \end{aligned}$$

and note that

$$x^2 dx = \frac{1}{12} du.$$

The above integral can then be written as

$$\int \frac{1}{12} u^8 \, du.$$

Since

$$\int \frac{1}{12} u^8 \, du = \frac{1}{12} \cdot \frac{1}{9} u^9 + C,$$

we see that

$$\int x^2 \left(4x^3+5\right)^8 \, dx = \frac{1}{108} \left(4x^3+5\right)^9 + C.$$

5.1 Standard form of integration

Example 2 *Evaluate the definite integral*

$$\int_0^{\pi/4} \tan^2(x)\sec^2(x)\ dx.$$

Solution 4 *Making the substitution*

$$\begin{aligned} u &= \tan(x) \\ du &= \sec^2(x)\,dx, \end{aligned}$$

we see that

$$\int \tan^2(x)\sec^2(x)\ dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3(x) + C.$$

By the FTC, we then have

$$\int_0^{\pi/4} \tan^2(x)\sec^2(x)\ dx = \frac{1}{3}\tan^3\left(\frac{\pi}{4}\right) - \frac{1}{3}\tan^3(0) = \frac{1}{3}.$$

(Recall that $\tan\left(\frac{\pi}{4}\right) = 1$ *and* $\tan(0) = 0$.*)*

5.2 Tabular Integration

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize

the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

Example Using Tabular Integration

Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

5.2 Tabular Integration

Example: Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

5.3 Integration by parts

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

5.3 Integration by parts

Example: **Integration by Parts**

Evaluate $\int x \cos x \, dx$.

SOLUTION

We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \qquad dv = \cos x \, dx.$$

To complete the formula, we take the differential of u and find the simplest antiderivative of $\cos x$.

$$du = dx \qquad v = \sin x$$

Then,

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

5.3 Integration by parts

Example:

Evaluate $\int x^2 e^x \, dx$.

SOLUTION

With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = \underbrace{x^2 e^x}_{u \quad v} - \underbrace{2\int x e^x \, dx}_{v \, du}.$$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$

5.3 Integration by parts

Example:

Find $\int \ln x \, dx$.

SOLUTION

If we want to use parts, we have little choice but to let $u = \ln x$ and $dv = dx$.

$$\int \ln x \, dx = (\ln x)(x) - \int (x)\left(\frac{1}{x}\right) dx \qquad \int u \, dv = uv - \int v \, du$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

5.4 Integration containing trigonometric function

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero).

use the identity $\sin^2 x = 1 - \cos^2 x$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

5.4 Integration containing trigonometric function

Example: $$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

5.4 Integration containing trigonometric function

Example:

$$\int \cos^5 x\, dx.$$

Solution

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x)$$

$$= \int (1 - u^2)^2\, du$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

5.5 Integration by partial fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated.

$$\int \frac{5x-3}{x^2-2x-3}\,dx =$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

To find A and B, we obtaining

$$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

5.5 Integration by partial fractions

can be rewritten as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3},$$

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

5.5 Integration by partial fractions

Example: Evaluate
$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$
using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C \end{aligned}$$

5.5 Integration by partial fractions

Example:

$$\int \frac{6x + 7}{(x + 2)^2}\, dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$6x + 7 = A(x + 2) + B \qquad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x + 7}{(x + 2)^2}\, dx = \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx$$

$$= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2}\, dx$$

$$= 6 \ln |x + 2| + 5(x + 2)^{-1} + C$$

5.5 Integration by partial fractions

Example $$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3\,\overline{\smash{)}\,2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx \\ &= x^2 + 2\ln|x + 1| + 3\ln|x - 3| + C. \end{aligned}$$

5.5 Integration by partial fractions

Example

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx$$

5.5 Integration by partial fractions

Solution

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$u = x^2 + 1,$
$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.$$