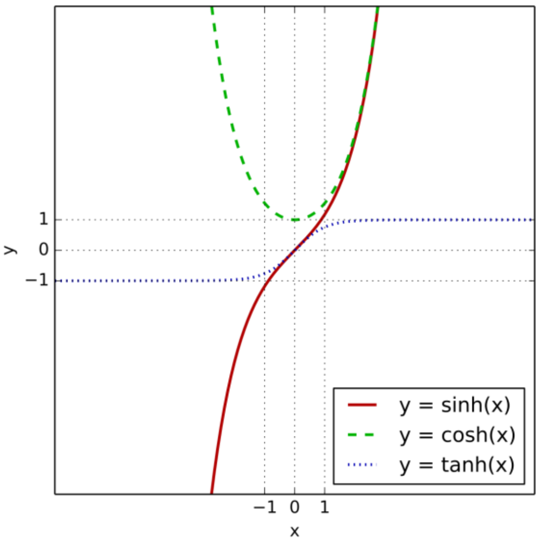
Salahaddin University
College of Engineering
Electrical Department

Chapter Four

Hyperbolic Functions



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Subject: Math II
Class: 1st Year 2nd Sem.

Outline
Hyperbolic Functions

4.1 **Definition of hyperbolic functions**

4.2 **Graph of hyperbolic functions**

4.3 **Applications of hyperbolic functions**

4.4 **Identities of hyperbolic functions**

4.5 **Derivatives of hyperbolic functions**

4.1 Definition of hyperbolic functions

The hyperbolic functions are defined as combinations of the exponential functions e^x and e^{-x}.

The basic hyperbolic functions are the hyperbolic sine function and the hyperbolic cosine function. They are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{csch} x$ are obtained from $\sinh x$ and $\cosh x$ in exactly the same way as the trigonometric functions $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are defined in terms of $\sin x$ and $\cos x$:

$$\tanh x = \frac{\sinh x}{\cosh x}; \quad \coth x = \frac{\cosh x}{\sinh x} \ (x \neq 0);$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \quad \operatorname{csch} x = \frac{1}{\sinh x} \ (x \neq 0).$$

4.1 Definition of hyperbolic functions

In mathematics, **hyperbolic functions** are analogues of the ordinary trigonometric **functions**, but defined using the **hyperbola** rather than the circle. Just as the points (cos t, sin t) form a circle with a unit radius, the points (cosh t, sinh t) form the right half of the unit **hyperbola**.

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

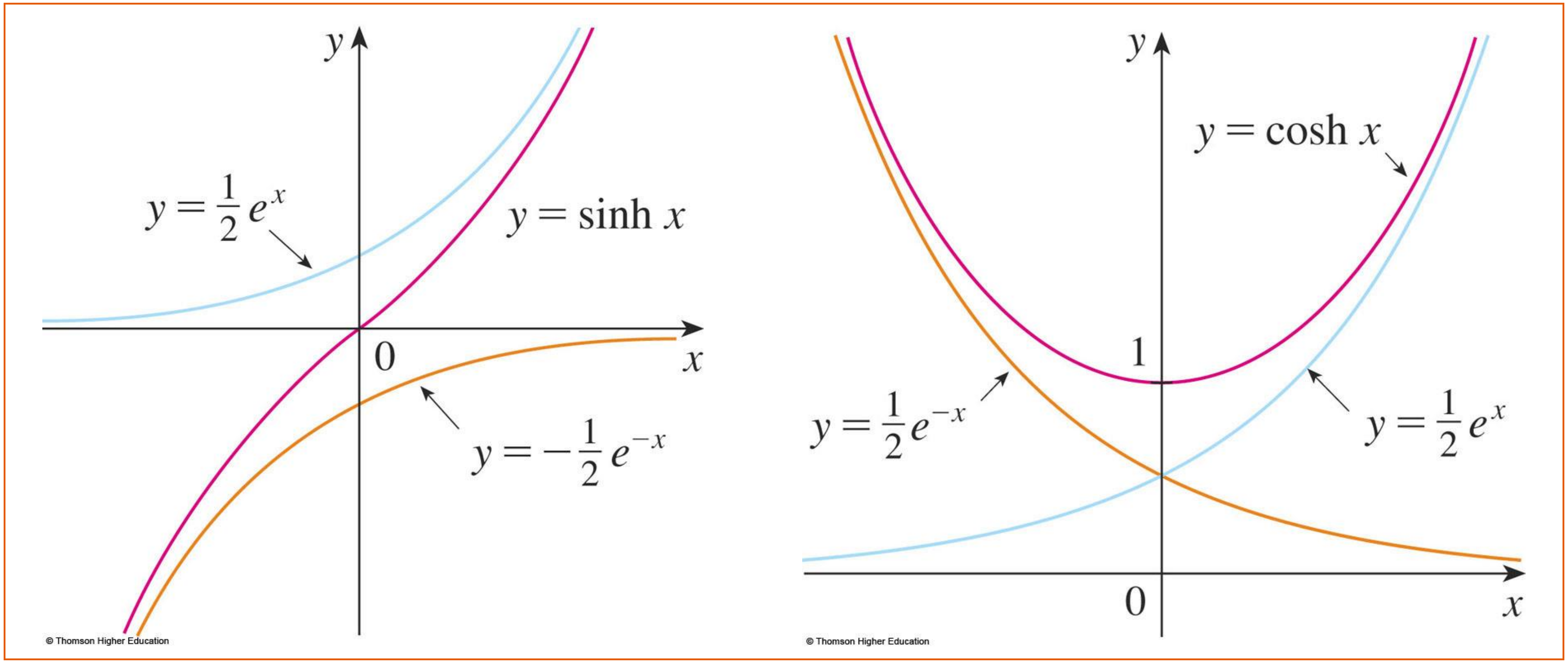
$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

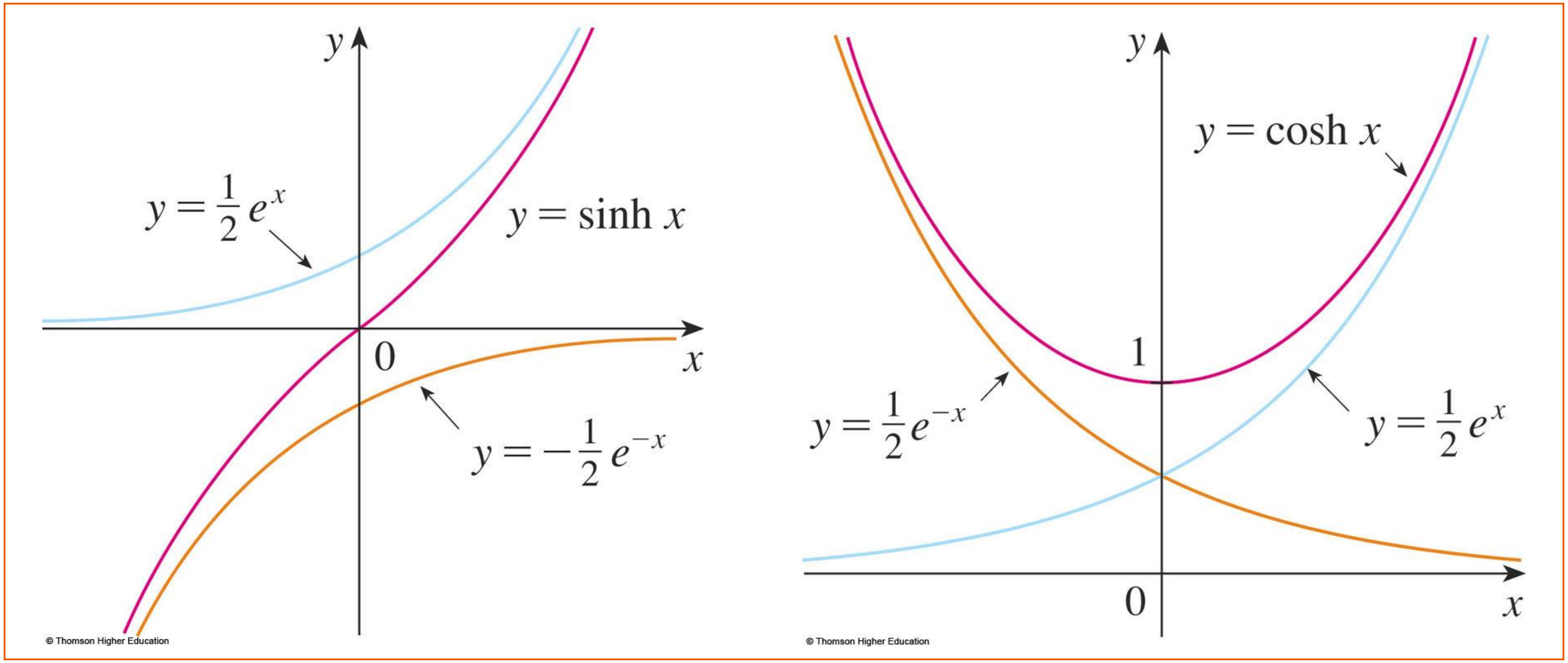
4.2 Graph of hyperbolic functions

The graphs of hyperbolic sine and cosine can be sketched using graphical addition, as in these figures.



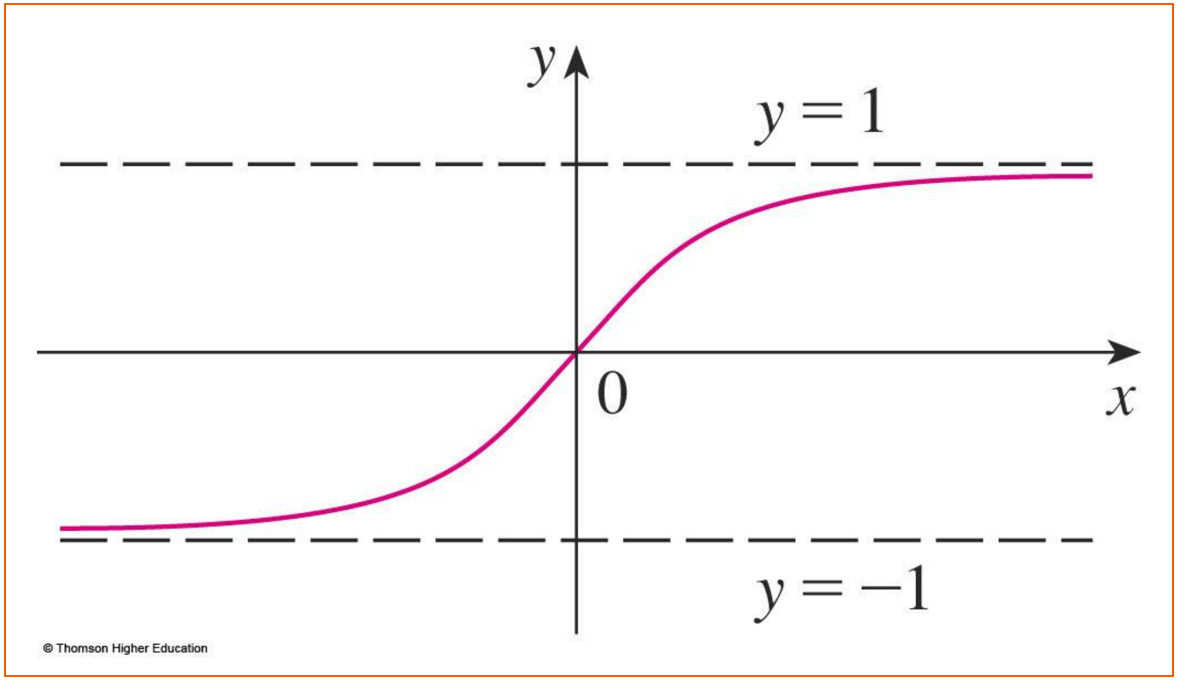
4.2 Graph of hyperbolic functions

Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, whereas cosh has domain $\mathbb{R}$ and range $[1, \infty)$.



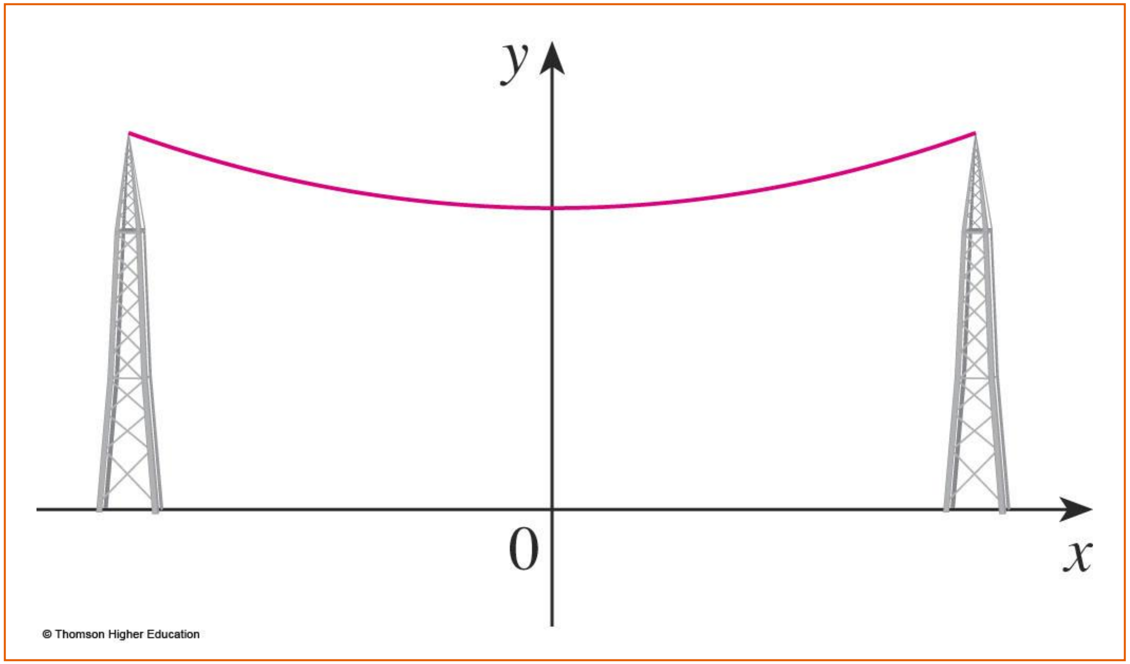
4.2 Graph of hyperbolic functions

- The graph of tanh is shown.
  - It has the horizontal asymptotes $y = \pm 1$.



4.3 Applications of hyperbolic functions

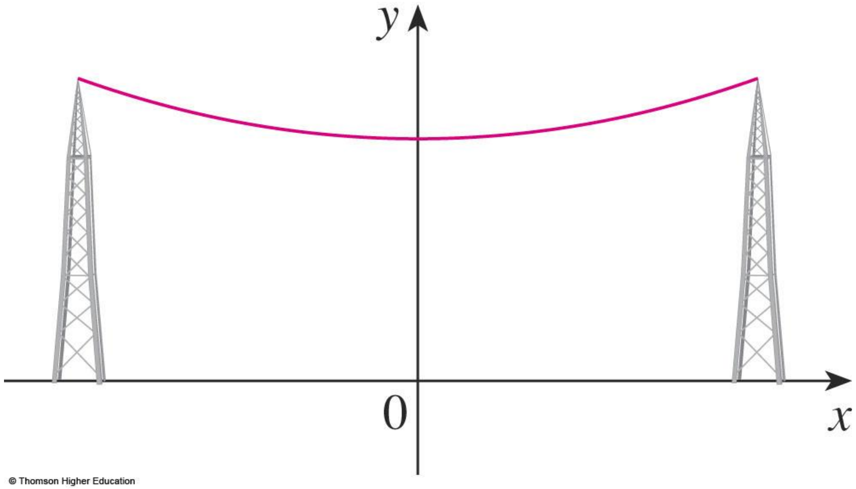
The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire.



4.3 Applications of hyperbolic functions

It can be proved that, if a heavy flexible cable is suspended between two points at the same height, it takes the shape of a curve with equation $y = c + a\cosh(x/a)$ called a catenary.

The Latin word catena means 'chain.'

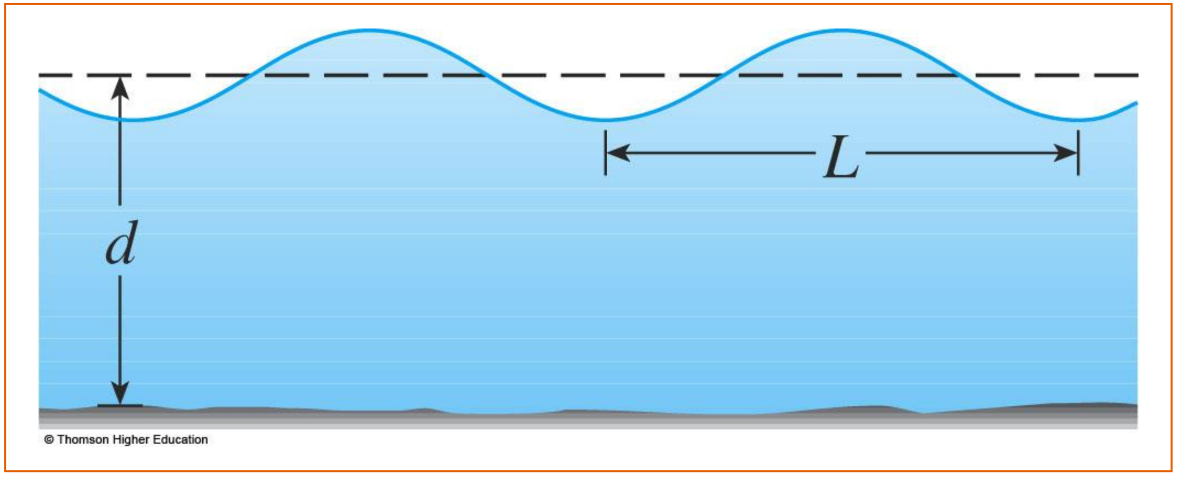


4.3 Applications of hyperbolic functions

- Another application occurs in the
- description of ocean waves.
  - The velocity of a water wave with length *L* moving across a body of water with depth *d* is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where *g* is the acceleration due to gravity.



4.4 Identities of hyperbolic functions

- We list some identities here.

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

4.4 Identities of hyperbolic functions

Example: Prove

a. $\cosh^2 x - \sinh^2 x = 1$

b. $1 - \tanh^2 x = \text{sech}^2 x$

4.4 Identities of hyperbolic functions

Example a

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

4.4 Identities of hyperbolic functions

Example b

- We start with the identity proved in (a):
- $\cosh^2 x - \sinh^2 x = 1$

  – If we divide both sides by $\cosh^2 x$, we get:

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

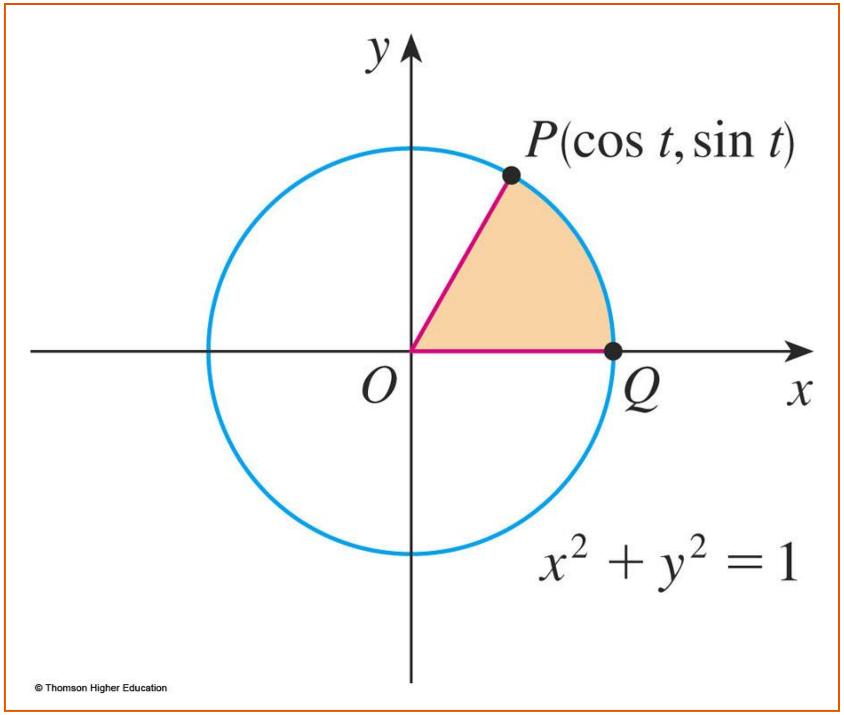
$$\text{or}\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

4.4 Identities of hyperbolic functions

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$.

- In fact, t can be interpreted as the radian measure of $\angle POQ$ in the figure.

For this reason, the trigonometric functions are sometimes called circular functions.



4.5 Derivatives of hyperbolic functions

The derivatives of hyperbolic functions can be easily found as these functions are defined in terms of exponential functions. So, the derivatives of the hyperbolic sine and hyperbolic cosine functions are given by

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x,$$

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x.$$

4.5 Derivatives of hyperbolic functions

We can easily obtain the derivative formula for the hyperbolic tangent:

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)' \cosh x - \sinh x (\cosh x)'}{\cosh^2 x}$$

$$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}.$$

It is known that the hyperbolic sine and cosine are connected by the relationship

$$\cosh^2 x - \sinh^2 x = 1.$$

Therefore, the derivative of the hyperbolic tangent is written as

$$(\tanh x)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

4.5 Derivatives of hyperbolic functions

Similarly, we can find the differentiation formulas for the other hyperbolic functions:

$$(\coth x)' = \left(\frac{\cosh x}{\sinh x}\right)' = \frac{(\cosh x)'\sinh x - \cosh x(\sinh x)'}{\sinh^2 x} = -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x}$$

$$= -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x,$$

$$(\operatorname{sech} x)' = \left(\frac{1}{\cosh x}\right)' = -\frac{1}{\cosh^2 x}\cdot(\cosh x)' = -\frac{1}{\cosh^2 x}\cdot\sinh x$$

$$= -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x} = -\operatorname{sech} x\tanh x,$$

$$(\operatorname{csch} x)' = \left(\frac{1}{\sinh x}\right)' = -\frac{1}{\sinh^2 x}\cdot(\sinh x)' = -\frac{1}{\sinh^2 x}\cdot\cosh x$$

$$= -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x} = -\operatorname{csch} x\coth x \quad (x \neq 0)\,.$$

4.5 Derivatives of hyperbolic functions

List of the differentiation formulas for the hyperbolic functions here.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) - \operatorname{csch}^2 x$$

4.5 Derivatives of hyperbolic functions

Example 1.

$$y = \coth \frac{1}{x}$$

Solution.

Differentiating as a composite function, we find:

$$y'\left(x\right) = \left(\coth \frac{1}{x}\right)' = -\operatorname{csch}^2\left(\frac{1}{x}\right) \cdot \left(\frac{1}{x}\right)' = -\operatorname{csch}^2\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = \frac{\operatorname{csch}^2\left(\frac{1}{x}\right)}{x^2}.$$

Example 2.

$$y = \ln(\sinh x), \;\; x > 0.$$

Solution.

$$y'\left(x\right) = \left[\ln(\sinh x)\right]' = \frac{1}{\sinh x} \cdot (\sinh x)' = \frac{\cosh x}{\sinh x} = \coth x.$$

4.5 Derivatives of hyperbolic functions

Example 3.

$$y = \sinh(\tan x)$$

Solution.

Using the chain rule, we obtain:

$$y'(x) = [\sinh(\tan x)]' = \cosh(\tan x) \cdot (\tan x)' = \cosh(\tan x) \cdot \frac{1}{\cos^2 x} = \frac{\cosh(\tan x)}{\cos^2 x},$$

where $x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$.

4.5 Derivatives of hyperbolic functions

Example 4.

$$y = \sinh(\ln x).$$

Solution.

By the chain rule,

$$y' = [\sinh(\ln x)]' = \cosh(\ln x) \cdot (\ln x)' = \cosh(\ln x) \cdot \frac{1}{x} = \frac{e^{\ln x} + e^{-\ln x}}{2x}.$$

We simplify the numerator using the logarithmic identity:

$$e^{\ln x} = x;$$

$$e^{-\ln x} = e^{\ln x^{-1}} = e^{\ln \frac{1}{x}} = \frac{1}{x}.$$

Hence

$$y' = \frac{e^{\ln x} + e^{-\ln x}}{2x} = \frac{x + \frac{1}{x}}{2x} = \frac{x^2 + 1}{2x^2}.$$

4.5 Derivatives of hyperbolic functions

Example 5.

$$y = \tanh(x^2)$$

Solution.

By the chain rule, we have:

$$y'(x) = [\tanh(x^2)]' = \frac{1}{\cosh^2(x^2)} \cdot (x^2)' = \frac{2x}{\cosh^2(x^2)}.$$

Example 6.

$$y = x \sinh x - \cosh x$$

Solution.

Using the difference and product rule, we have:

$$\begin{aligned} y'(x) &= (x \sinh x - \cosh x)' = (x \sinh x)' - (\cosh x)' = x' \sinh x + x(\sinh x)' \\ &- (\cosh x)' = 1 \cdot \sinh x + x \cdot \cosh x - \sinh x = \cancel{\sinh x} + x \cosh x - \cancel{\sinh x} \\ &= x \cosh x. \end{aligned}$$

4.5 Derivatives of hyperbolic functions

Example 7.

$$y = \sinh x \cosh x - x.$$

Solution.

Using the hyperbolic identity

$$\sinh 2x = 2\sinh x \cosh x,$$

we can write the equation in the form

$$y = \sinh x \cosh x - x = \frac{1}{2}\sinh 2x - x.$$

Applying the chain rule, we have

$$y' = \left(\frac{1}{2}\sinh 2x - x\right)' = \frac{1}{2}\cosh 2x \cdot (2x)' - 1 = \frac{1}{2}\cosh 2x \cdot 2 - 1 = \cosh 2x - 1.$$

Now we use another hyperbolic identity

$$\cosh 2x - 1 = 2\sinh^2 x.$$

So

$$y' = 2\sinh^2 x.$$

4.5 Derivatives of hyperbolic functions

Example 8.

$$y = \sinh^2 x$$

Solution.

$$y'(x) = \left(\sinh^2 x\right)' = 2\sinh x \cdot (\sinh x)' = 2\sinh x \cosh x.$$

We can simplify the answer using the double angle identity $\sinh 2x = 2\sinh x \cosh x$. Hence,

$$y'(x) = 2\sinh x \cosh x = \sinh 2x.$$

Example 9.

$$y = \sinh x \tanh x$$

Solution.

Using the product rule for differentiation, we obtain:

$$y'(x) = (\sinh x \tanh x)' = (\sinh x)' \tanh x + \sinh x (\tanh x)' = \cosh x \cdot \tanh x$$

$$+ \sinh x \cdot \frac{1}{\cosh^2 x} = \frac{\cancel{\cosh x} \sinh x}{\cancel{\cosh x}} + \sinh x \operatorname{sech}^2 x = \sinh x \left(1 + \operatorname{sech}^2 x\right).$$