Salahaddin University
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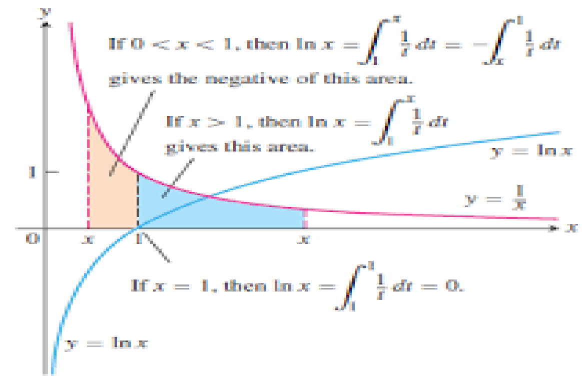

Chapter One

Transcendental functions

Natural Logarithmic function

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1.1 Graph of Logarithmic (y=ln)

Definition of the Natural Logarithm Function:

The natural logarithm of a positive number *x*, written as ln *x*, is the value of an integral.

DEFINITION The Natural Logarithm Function

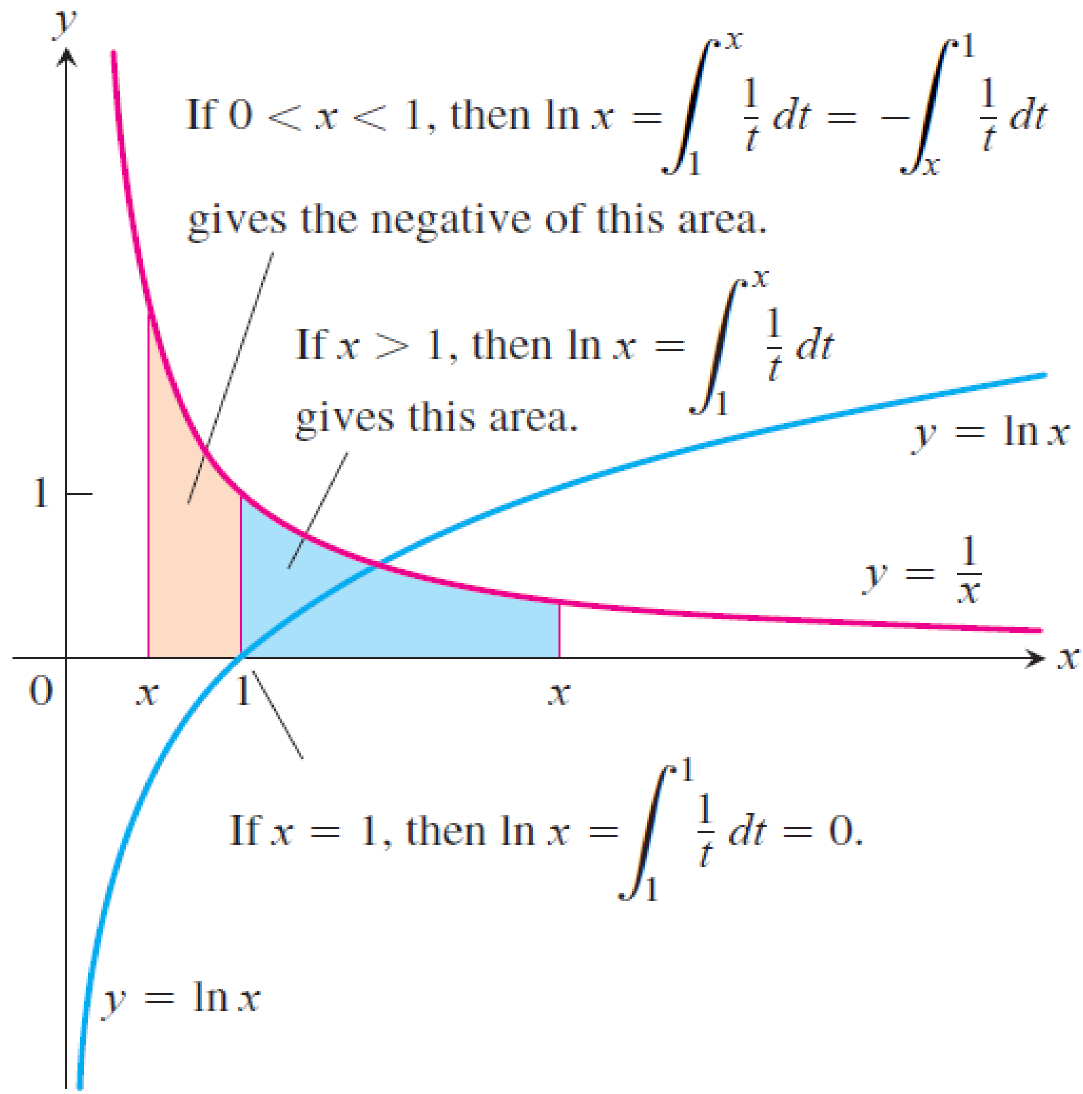
$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$
For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x tc 1.
The function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

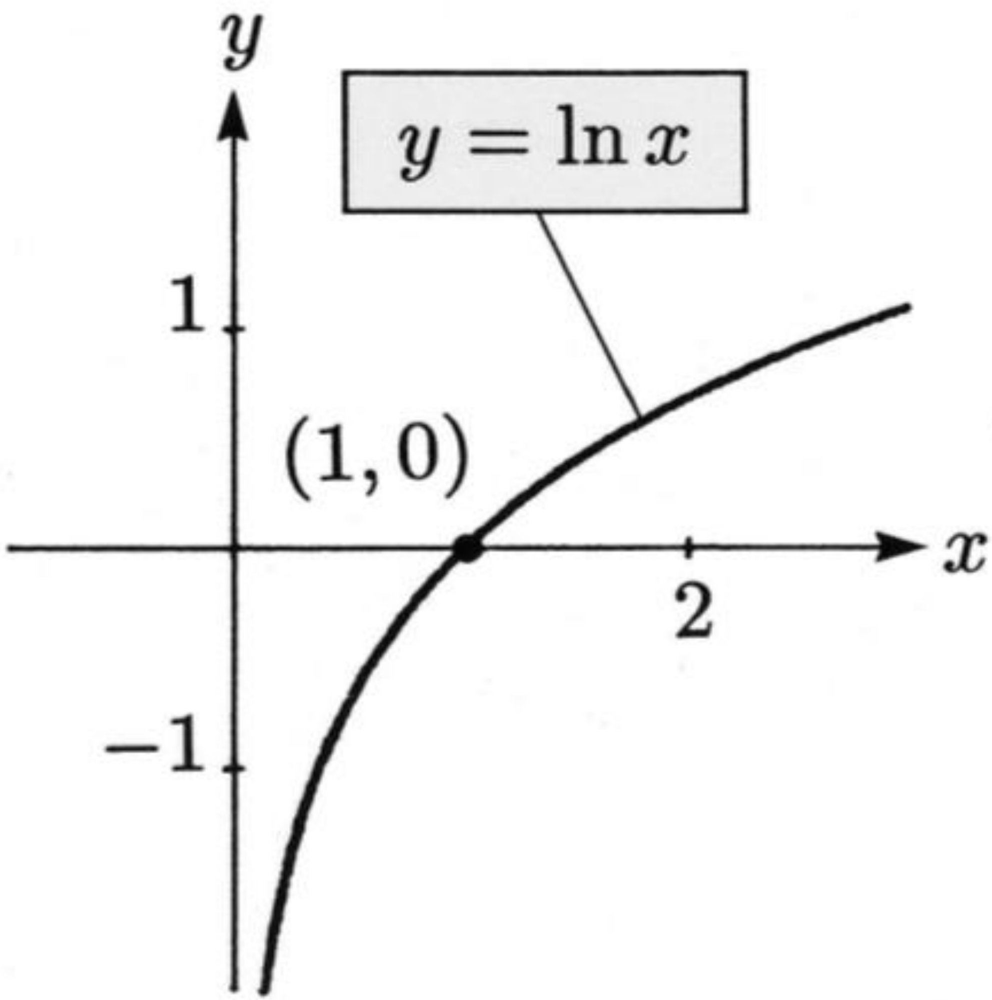
1.1 Graph of Logarithmic (y=ln)

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30



The graph of the logarithm rises above the *x*-axis as *x* moves from 1 to the right, and it falls below the axis as *x* moves from 1 to the left.

1.1 Graph of Logarithmic (y=ln)



Graph of $y = \ln x$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing

1.2 Prosperities of Logarithmic

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Writing Equivalent Expressions

Write an equivalent logarithmic equation.

$e^x = 23$ $\quad \log_e 23 = x$

$\ln 23 = x$

Write an equivalent logarithmic equation.

$e^x = 6$ $\quad \ln 6 = x$

1.2 Prosperities of Logarithmic

THEOREM **Properties of Logarithms**

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ r rational

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln (e^x) = x \qquad (\text{all } x)$$

DEFINITION **The Number e**

The number e is that number in the domain of the natural logarithm satisfying

$$\ln (e) = 1$$

1.2 Prosperities of Logarithmic

Example Expand $\ln(\frac{e^2\sqrt{a^2+1}}{b^3})$

using the rules of logarithms.

Solution

$$\ln(\frac{e^2\sqrt{a^2+1}}{b^3}) = \ln(e^2\sqrt{a^2+1}) - \ln(b^3) = \ln(e^2) + \ln(\sqrt{a^2+1}) - 3\ln b$$

$$= 2\ln e + \frac{1}{2}\ln(a^2+1) - 3\ln b = 2 + \frac{1}{2}\ln(a^2+1) - 3\ln b.$$

1.2 Prosperities of Logarithmic

EXAMPLE Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$ Product

(b) $\ln 4 - \ln 5 = \ln \frac{4}{5} = \ln 0.8$ Quotient

(c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3 \ln 2$ Power

1.2 Prosperities of Logarithmic

EXAMPLE Applying the Properties to Function Formulas

(a) $\ln 4 + \ln \sin x = \ln (4 \sin x)$ Product

(b) $\ln \frac{x+1}{2x-3} = \ln (x+1) - \ln (2x-3)$ Quotient

(c) $\ln \sec x = \ln \frac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln (x+1)^{1/3} = \frac{1}{3} \ln (x+1)$ Power

1.3 Derivative of Logarithmic (ln)

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

If u is a differentiable function of x, then we have

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

THEOREM **Laws of Exponents for e^x**

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

1.3 Derivative of Logarithmic (ln)

- **EXAMPLE** **Using the Formula**

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ then $y = e^u$. Then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \qquad \text{and} \qquad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u \dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x)$.

1.3 Derivative of Logarithmic (ln)

Derivative of ln *x*

Now that we know the derivative of e^x, it is relatively easy to find the derivative of its inverse function, $\ln x$.

$$y = \ln x$$

$$e^y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Differentiate implicitly.}$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

1.3 Derivative of Logarithmic (ln)

EXAMPLE Derivatives of Natural Logarithms

(a) $\dfrac{d}{dx}\ln 2x$

$$= \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$$

(b) $\dfrac{d}{dx}\ln(x^2 + 3)$

$$= \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$$

1.3 Derivative of Logarithmic (ln)

EXAMPLE Derivatives of Natural Logarithms

Find $\frac{d}{dx}\ln(\sin x)$.

Solution: $\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x)$

$$= \frac{1}{\sin x}\cos x \quad = \cot x$$

EXAMPLE

$f(x) = \sqrt{\ln x}$ find $f'(x)$

$$\frac{d}{dx}\left[\sqrt{\ln(x)}\right]$$

$$= \frac{1}{2}\ln^{\frac{1}{2}-1}(x) \cdot \frac{d}{dx}[\ln(x)]$$

Apply the power rule:

$[u(x)^n]' = n \cdot u(x)^{n-1} \cdot u'(x)$

Note: The chain rule has been applied here:
Multiply by the inner function's derivative $u'(x)$.

$$= \frac{\frac{1}{x}}{2\sqrt{\ln(x)}} \quad = \frac{1}{2x\sqrt{\ln(x)}}$$

1.3 Derivative of Logarithmic (ln)

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

1.3 Derivative of Logarithmic (ln)

- **EXAMPLE**

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

1.3 Derivative of Logarithmic (ln)

➢ **EXAMPLE** Using Logarithmic Differentiation

Find dy/dx if
$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) \qquad \text{Rule 2}$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \text{Rule 1}$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). \qquad \text{Rule 3}$$

take derivatives of both sides with respect to x,
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:
$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:
$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

1.3 Derivative of Logarithmic (ln)

- **Exercises**

I. find the derivative of y with respect to x

$$y = \frac{\ln x}{1 + \ln x}$$

II. use logarithmic differentiation to find the derivative of y with respect to θ

$$y = \frac{\theta + 5}{\theta \cos \theta}$$

1.4 Integration of Logarithmic (ln)

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

EXAMPLE

(a) $$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$u = x^2 - 5,\quad du = 2x\,dx,$
$u(0) = -5,\quad u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

(b) $$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$u = 3 + 2\sin\theta,\quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1,\quad u(\pi/2) = 5$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$

1.4 Integration of Logarithmic (ln)

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc x| + C$$

EXAMPLE

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2} \ln|\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Substitute $u = 2x$,
$dx = du/2$,
$u(0) = 0$,
$u(\pi/6) = \pi/3$