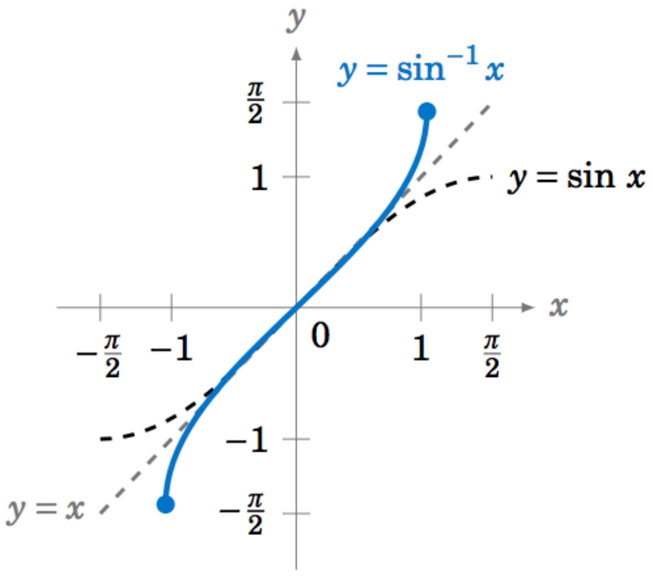
Salahaddin University
College of Engineering
Electrical Department

Chapter Three

Inverse Trigonometric Function



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Subject: Math II
Class: 1st Year 2nd Sem.

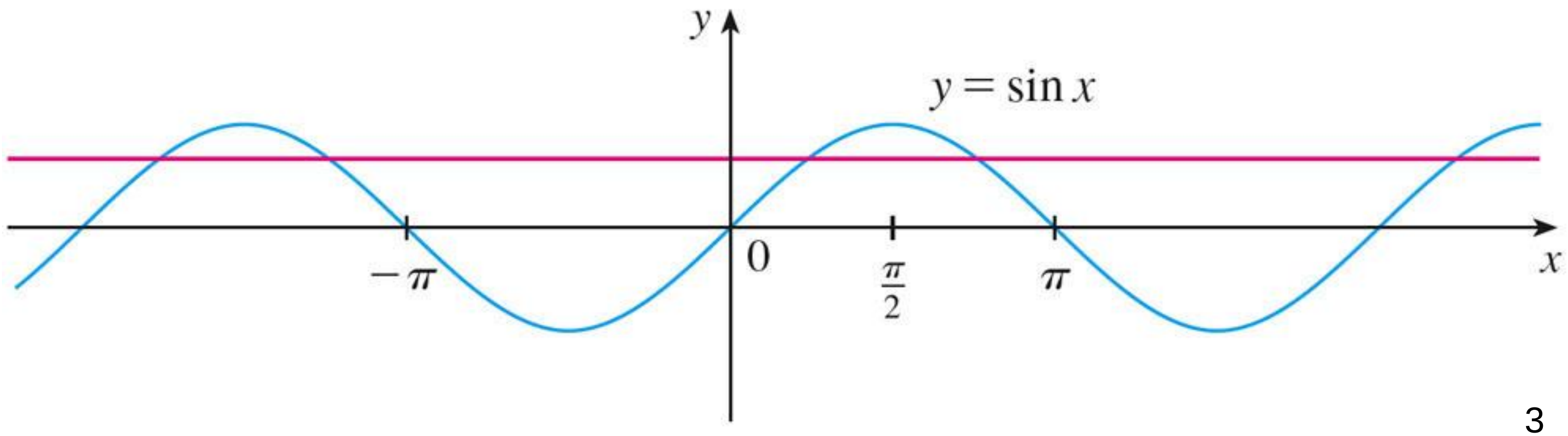
Outline

Inverse Trigonometric Function

3.1 Properties of Inverse Trigonometric Function

- Here, you can see that the sine function ***y* = sin (*x*)** is not one-to-one.

  – Use the Horizontal Line Test.



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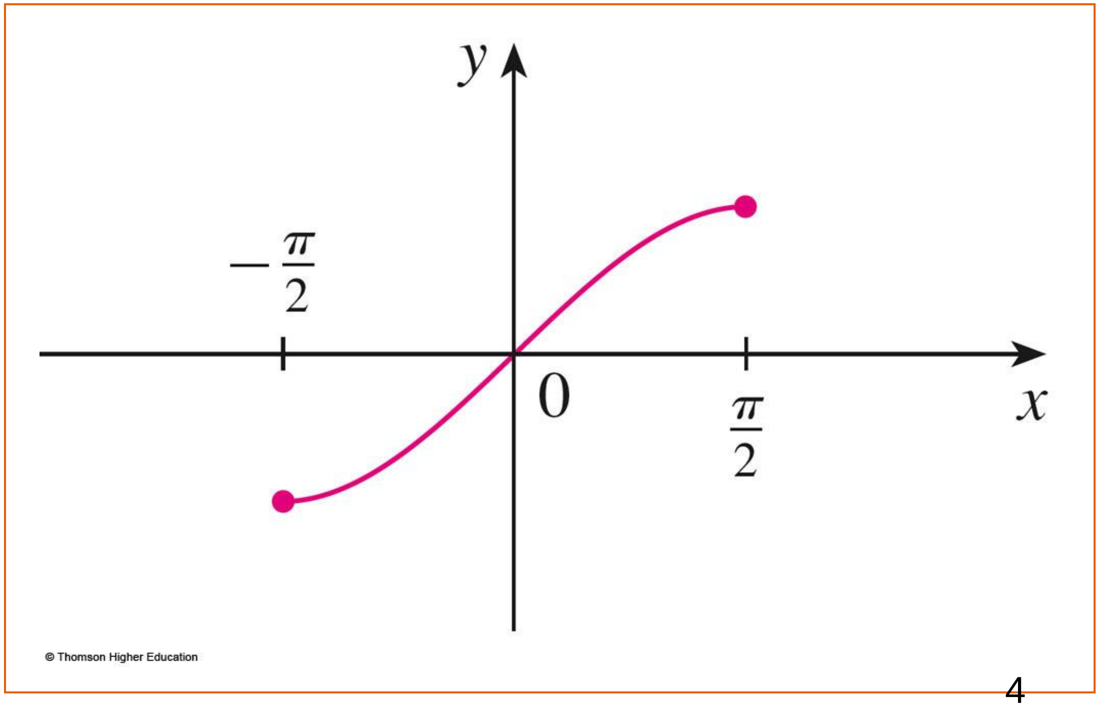
3.1 Properties of Inverse Trigonometric Function

INVERSE TRIGONOMETRIC FUNCTIONS

- However, here, you can see that the function $f(x) = \sin x$, $-\pi/2 \le x \le \pi/2$, is one-to-one.



3.1 Properties of Inverse Trigonometric Function

INVERSE SINE FUNCTIONS

- As the definition of an inverse function states

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

- that
- we have:

- $$\sin^{-1} x = y \Leftrightarrow \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

- Thus, if -1 ≤ *x* ≤ 1, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is *x*.

3.1 Properties of Inverse Trigonometric Function

INVERSE SINE FUNCTIONS

- Evaluate: $\sin^{-1}\left(\frac{1}{2}\right)$
- We have:

$$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

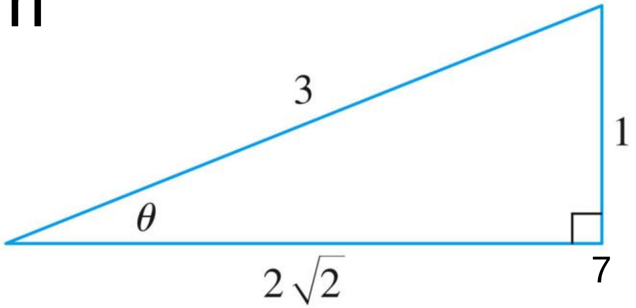
–This is because $\sin(\pi/6)=1/2$,
and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

3.1 Properties of Inverse Trigonometric Function

INVERSE SINE FUNCTIONS

- Evaluate: $\tan(\arcsin\frac{1}{3})$
  - Let $\theta = \arcsin\frac{1}{3}$, so $\sin\theta = \frac{1}{3}$.
    - Then, we can draw a right triangle with angle *θ*.
    - So, we deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$.
  - This enables us to read from the triangle that:

$$\tan(\arcsin\frac{1}{3}) = \tan\theta = \frac{1}{2\sqrt{2}}$$



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INVERSE SINE FUNCTIONS

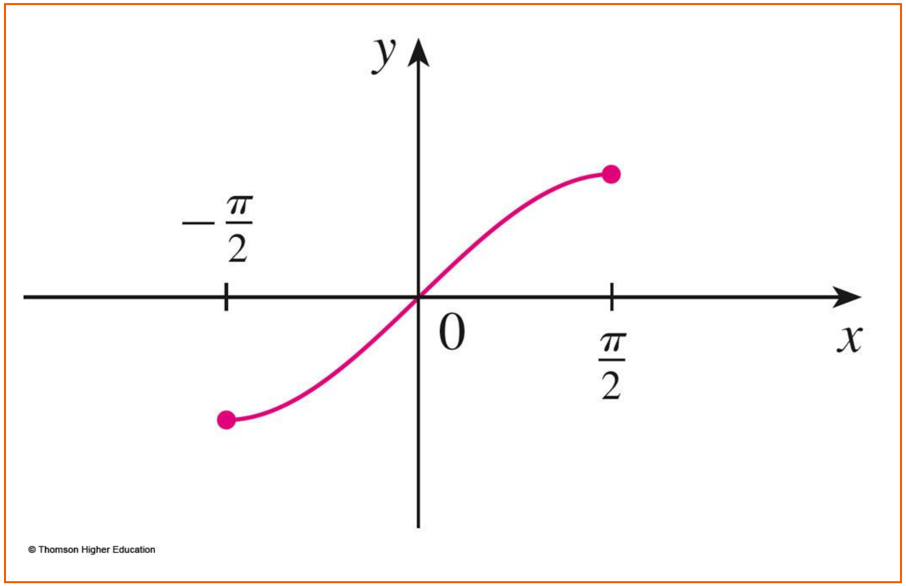
- In this case, the cancellation equations for inverse functions become:

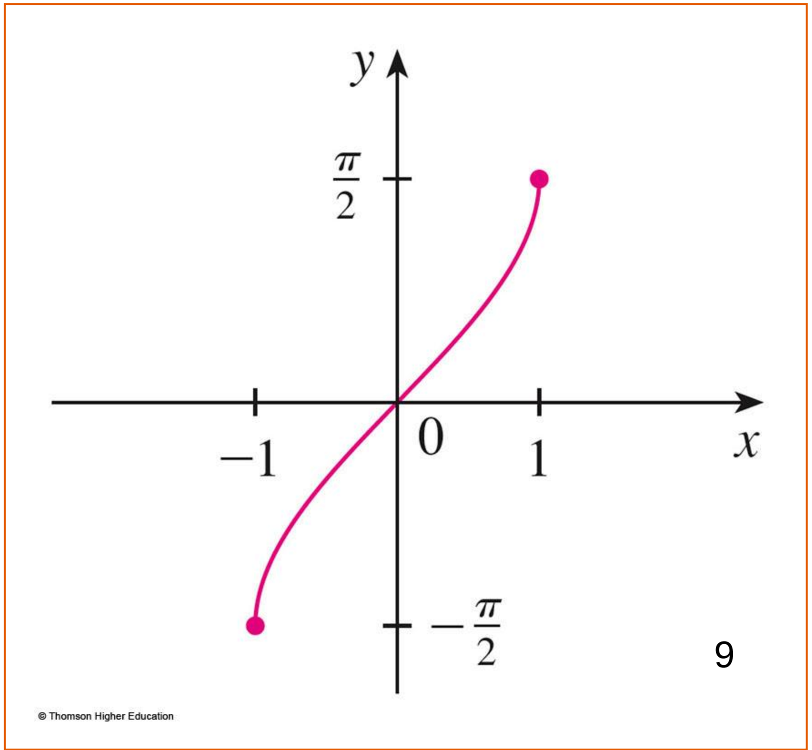
$$\sin^{-1}(\sin x) = x \qquad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \qquad \text{for} \quad -1 \le x \le 1$$

INVERSE SINE FUNCTIONS

- The graph is obtained from that of the restricted sine function by reflection about the line $y = x$.





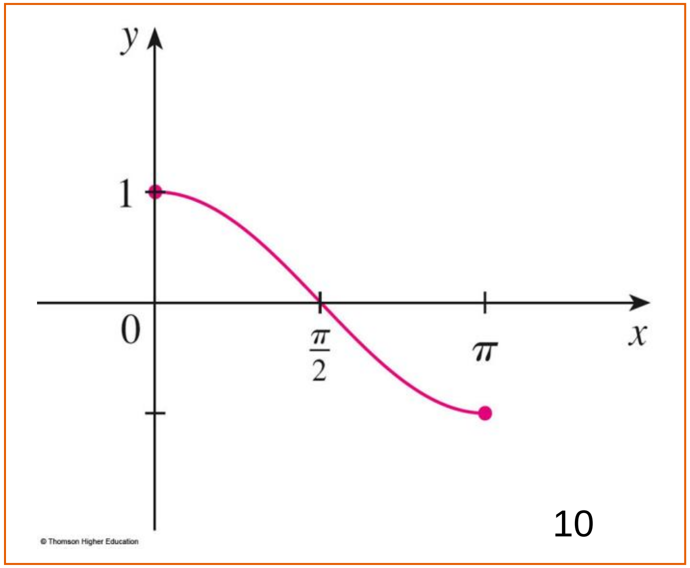
3.1 Properties of Inverse Trigonometric Function

INVERSE COSINE FUNCTIONS

- The inverse cosine function is handled similarly.

- $$\cos^{-1} x = y \quad \Leftrightarrow \cos y = x \;\text{ and }\; 0 \leq y \leq \pi$$

  - The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one.
  - So, it has an inverse function denoted by $\cos^{-1}$ or arccos.



3.1 Properties of Inverse Trigonometric Function

INVERSE COSINE FUNCTIONS

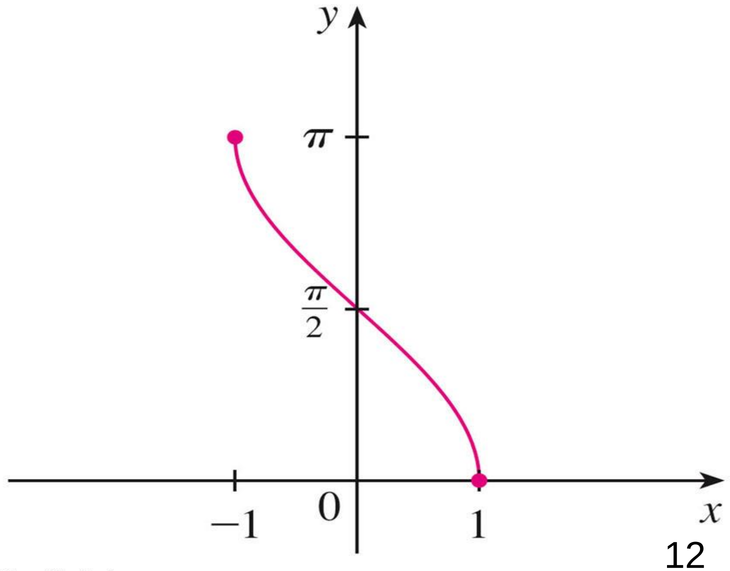
- The cancellation equations are:

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \le x \le \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \le x \le 1$$

INVERSE COSINE FUNCTIONS

- The inverse cosine function, $\cos^{-1}$, has domain [-1, 1] and range $[0, \pi]$, and is a continuous function.



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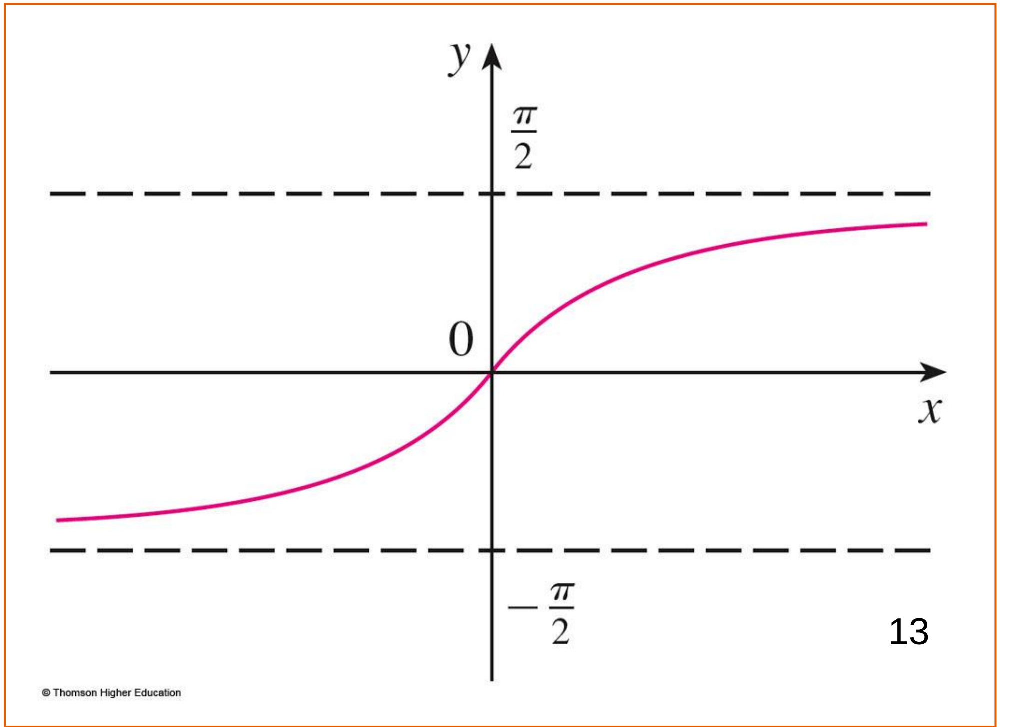
INVERSE TANGENT FUNCTIONS

- The inverse tangent function, $\tan^{-1}$ = arctan, has domain R and range $(-\pi/2, \pi/2)$.



INVERSE TANGENT FUNCTIONS

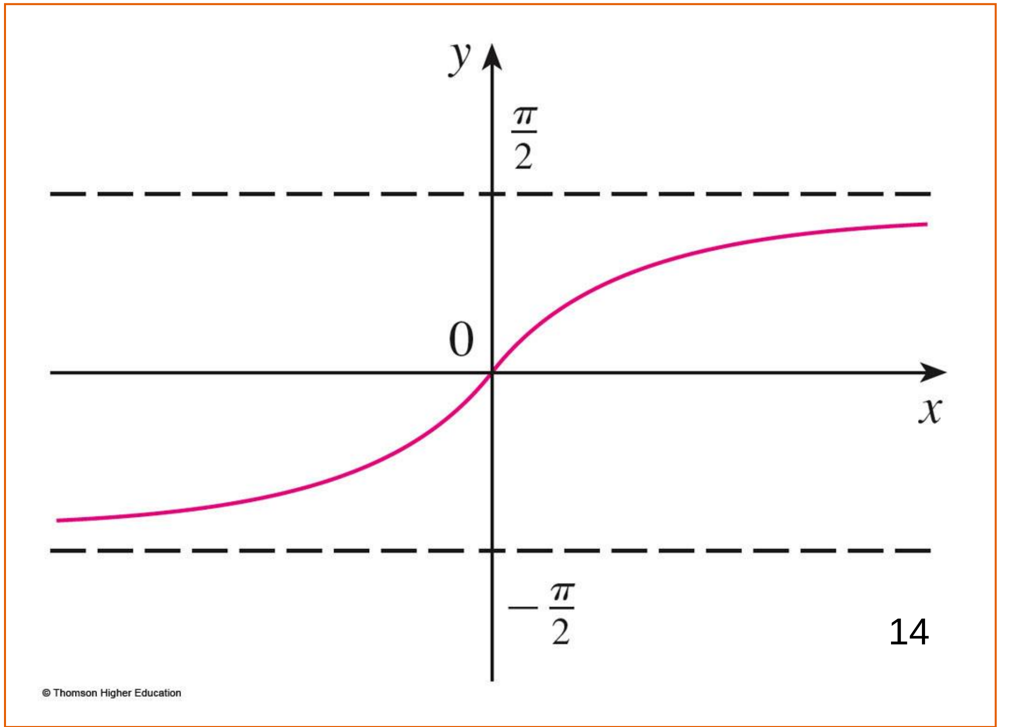
- We know that:

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

  - So, the lines

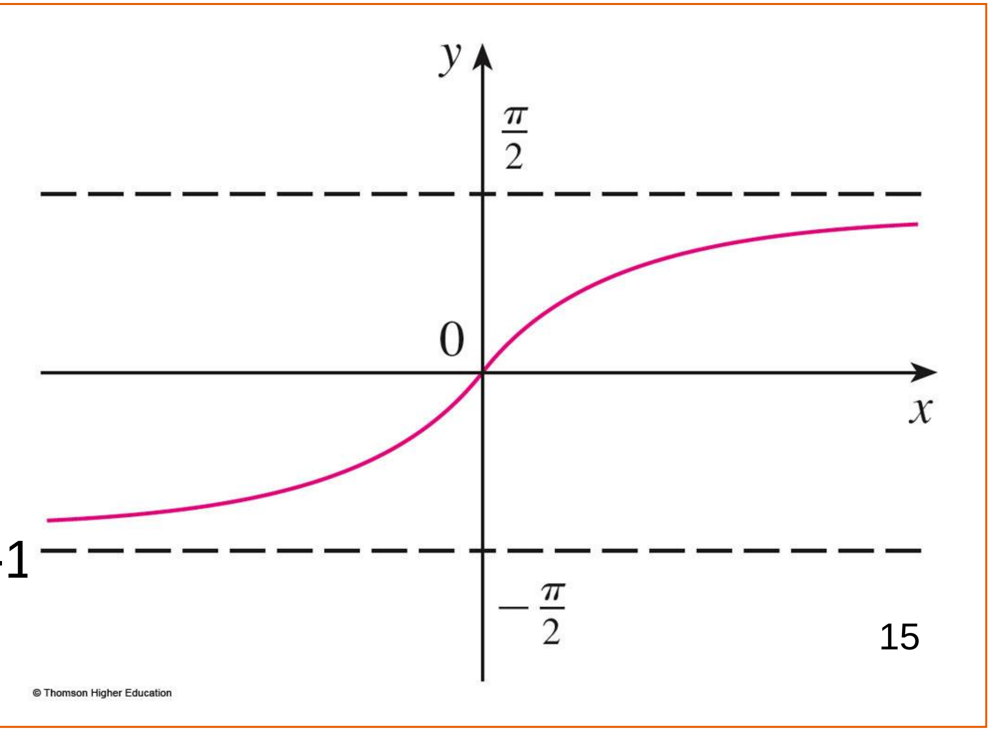
$$x = \pm\pi/2$$

are vertical asymptotes of the graph of tan.



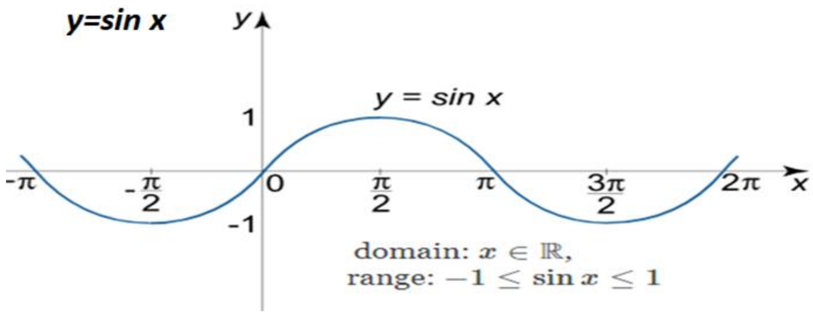
INVERSE TANGENT FUNCTIONS

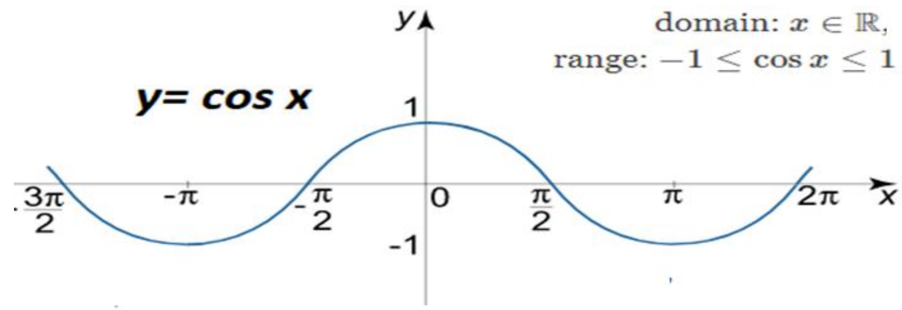
- The graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$.
  - It follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$

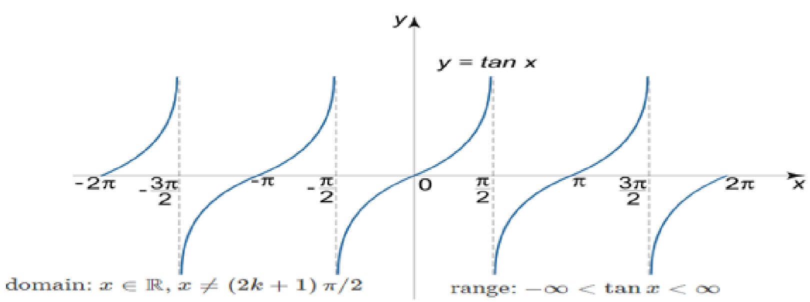


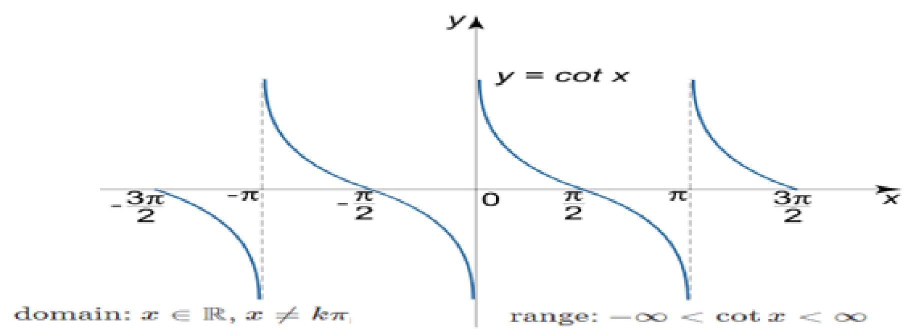
3.1 Properties of Inverse Trigonometric Function

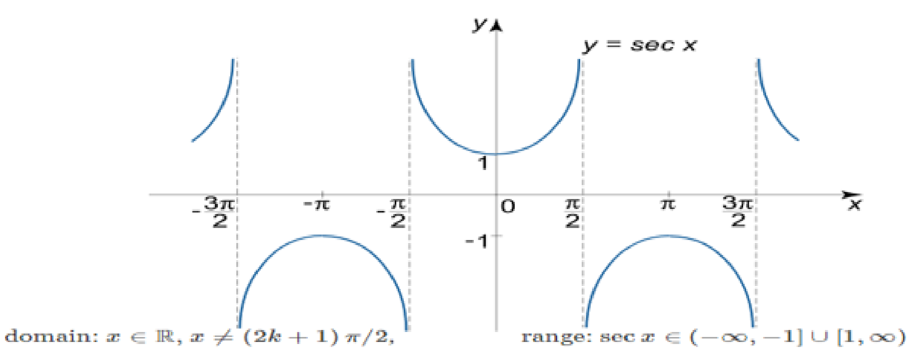
Graph of Trigonometric FUNCTIONS

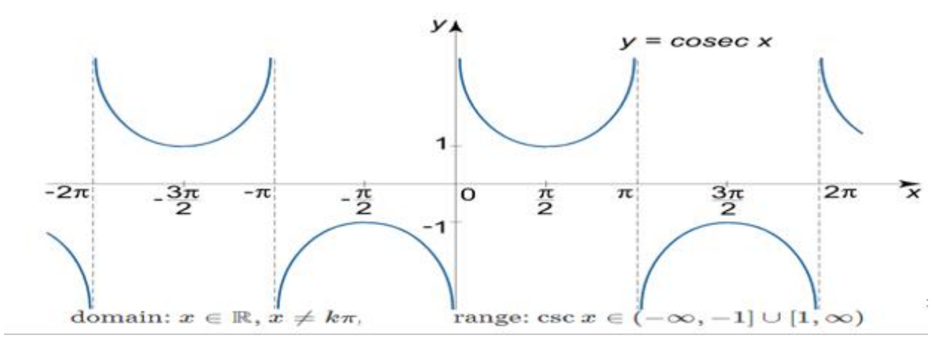






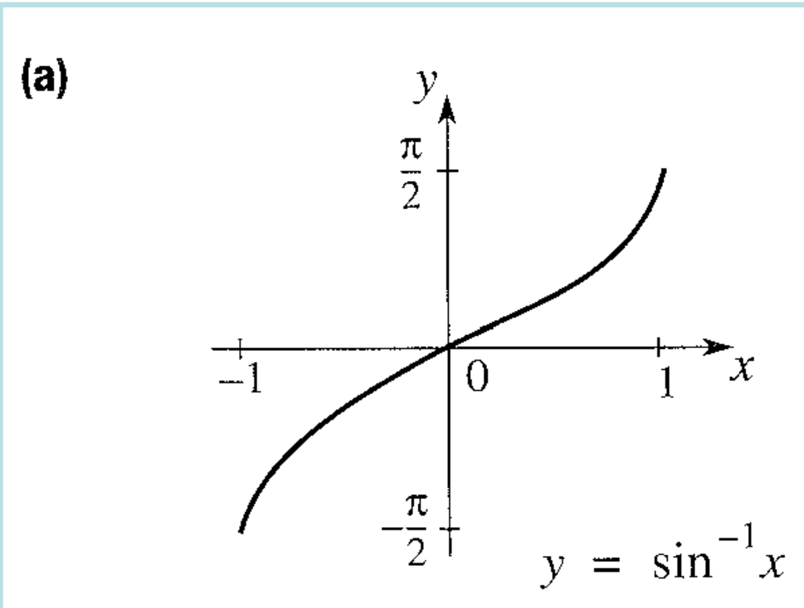


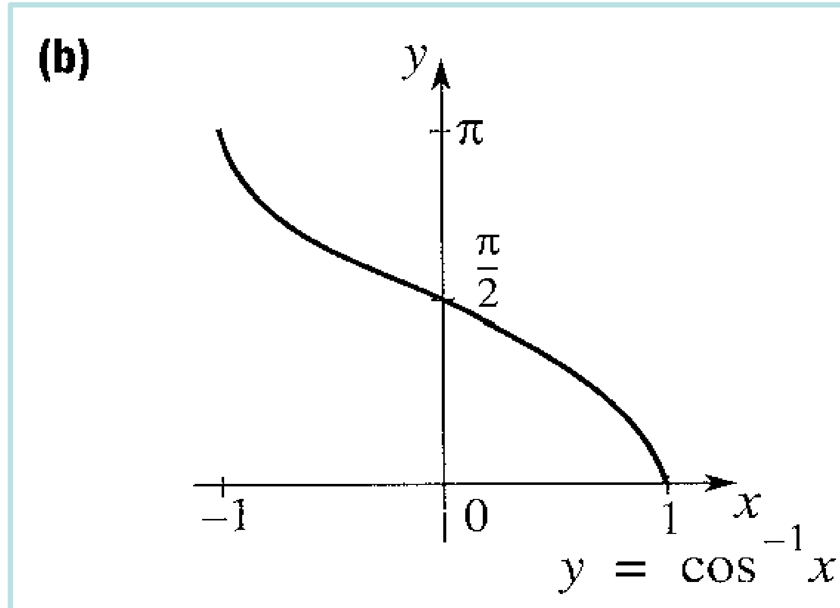




The Inverse Trigonometric Functions

Graphs of **sin & cos** inverse trigonometric function:

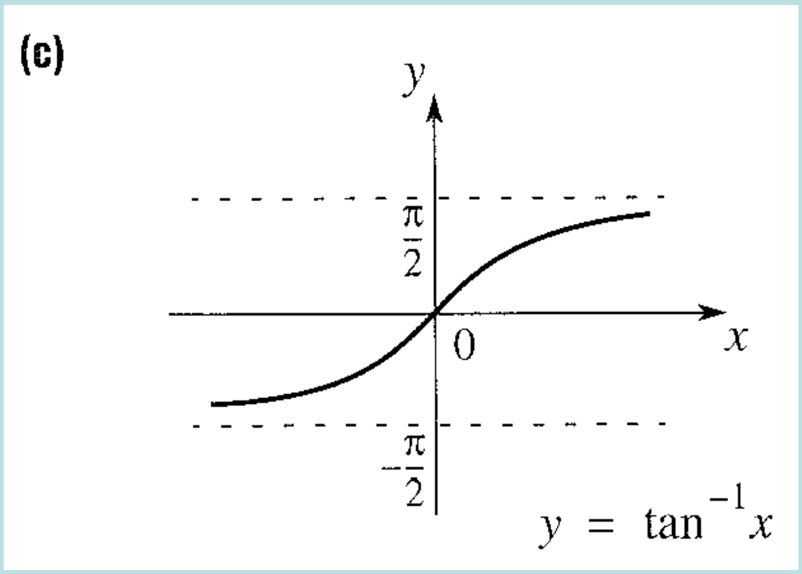


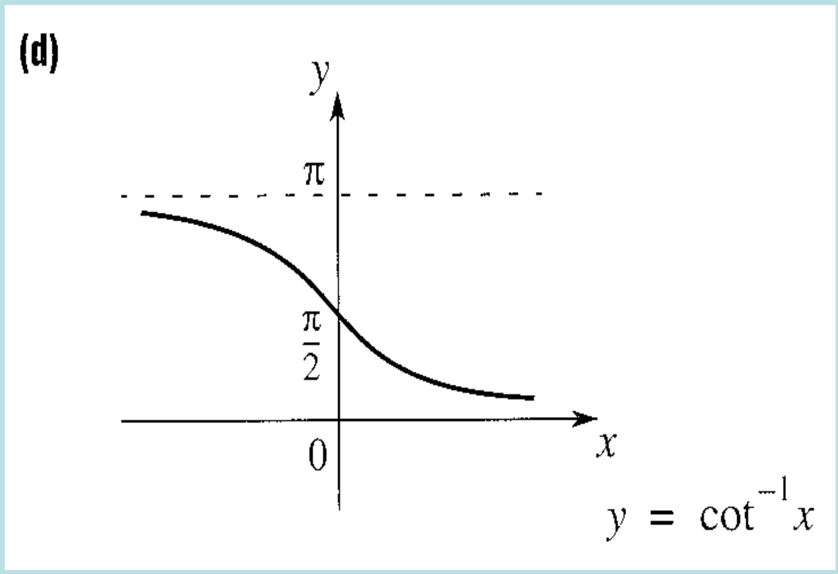


Domain	**Range**	**Domain**	**Range**
$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$-1 \le x \le 1$	$0 \le y \le \pi$

The Inverse Trigonometric Functions

Graphs of **tan & cot** inverse trigonometric function:

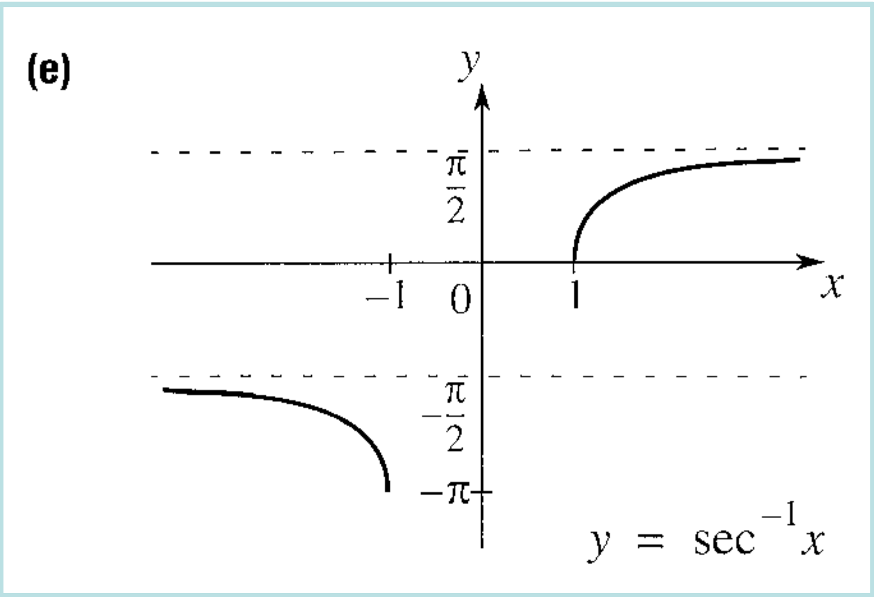


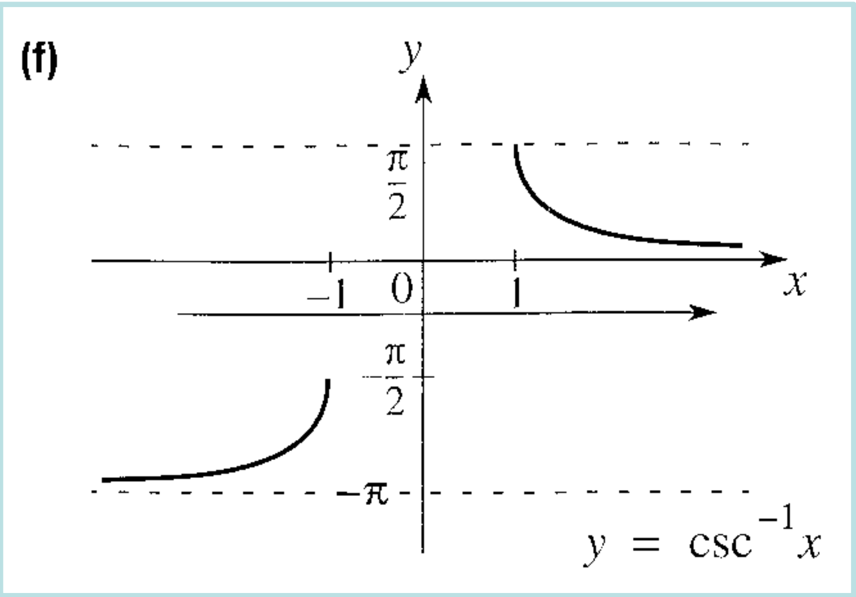


Domain	Range	Domain	Range
$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$-1 \leq x \leq 1$	$0 < y < \pi$

The Inverse Trigonometric Functions

Graphs of **sec & csc** inverse trigonometric function:





Domain	Range	Domain	Range
$x \geq 1$	$0 \leq y < \pi/2$	$x \geq 1$	$0 < y \leq \pi/2$
or	or	*or*	or
$x \leq -1$	$-\pi \leq y < \pi/2$	$x \leq -1$	$-\pi/2 \leq y < \pi$

Inverse Properties

$f\,(f^{-1}(x)) = x$ *and* $f^{-1}(f\,(x)) = x$

Remember that the trig. functions have inverses only in restricted domains.

3.2 Derivative of Inverse Trigonometric Function

DERIVATIVES

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

INVERSE SINE FUNCTIONS

- We know that:
  - The sine function f is continuous, so the inverse sine function is also continuous.
  - The sine function is differentiable, so the inverse sine function is also differentiable.

3.2 Derivative of Inverse Trigonometric Function

INVERSE SINE FUNCTIONS

- since we know that is $\sin^{-1}$ differentiable, we can just as easily calculate it by implicit differentiation as follows:
- Let $y = \sin^{-1}x$.
  - Then, $\sin y = x$ and $-\pi/2 \le y \le \pi/2$.
  - Differentiating **$\sin y = x$** implicitly with respect to x, we obtain:

$$\cos y \,.\, \frac{dy}{dx} = 1$$

$$\text{and}\ \frac{dy}{dx} = \frac{1}{\cos y}$$

INVERSE SINE FUNCTIONS

- Now, cos $y \geq 0$ since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$

3.2 Derivative of Inverse Trigonometric Function

INVERSE SINE FUNCTIONS

- If $f(x) = \sin^{-1}(x^2 - 1)$, find: $f'(x)$.

- we have:

$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{\sqrt{1-(x^4-2x^2+1)}} 2x$$

$$= \frac{2x}{\sqrt{2x^2-x^4}}$$

3.2 Derivative of Inverse Trigonometric Function

- Differentiate: $y = \dfrac{1}{\sin^{-1} x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x)^{-1}$$

$$= -(\sin^{-1} x)^{-2} \frac{d}{dx}(\sin^{-1} x)$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

3.2 Derivative of Inverse Trigonometric Function

- Differentiate: $f(x) = x \arctan\sqrt{x}$

$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}(\tfrac{1}{2}x^{-1/2}) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

3.2 Derivative of Inverse Trigonometric Function

Example 1

Differentiate $\arcsin(2x)$

$$\arcsin(2x)' = \frac{1}{\sqrt{1-(2x)^2}}2$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

Example 3

Differentiate $x\arccos x$

$$(x\arccos x)'$$

$$= \arccos x + x\frac{-1}{\sqrt{1-x^2}}$$

$$= \arccos x - \frac{x}{\sqrt{1-x^2}}$$

Example 2

Differentiate $\arctan\left(\frac{x}{2}\right)$

$$\arctan\left(\frac{x}{2}\right)'$$

$$= \frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{1}{2}$$

$$= \frac{1}{2\left(1+\frac{x^2}{4}\right)}$$

$$= \frac{2}{4+x^2}$$

3.2 Derivative of Inverse Trigonometric Function

Find the derivative of:

$$f(x) = \arctan\sqrt{x} = \arctan(x)^{\frac{1}{2}}$$

Let u = $x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{1+\left(\sqrt{x}\right)^2} = \frac{1}{2\sqrt{x}(1+x)}$$

3.2 Derivative of Inverse Trigonometric Function

Solve: $\frac{d}{dx}\sec^{-1}(x^2 - x)$

Solution:

$u = x^2 - x \qquad \frac{du}{dx} = 2x - 1$

$$= \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$= \frac{2x-1}{|x^2 - x|\sqrt{(x^2-x)^2-1}}$$

Solve: $\frac{d}{dx}\tan^{-1}(\sin x)$

Solution:

$u = \sin x \qquad \frac{du}{dx} = \cos x$

$$= \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$= \frac{\cos x}{1+\sin^2 x}$$

3.2 Derivative of Inverse Trigonometric Function

Example Find the following derivatives.

a. $\tan^{-1}(x^3)$

Set $u = x^3$, so $[\tan^{-1}(x^3)]' = \dfrac{(x^3)'}{1 + (x^3)^2} = \dfrac{3x^2}{1 + x^6}$

b. $\cos^{-1}(e^{x^2})$

Set $u = e^x$, so $[\cos^{-1}(e^{x^2})]' = \dfrac{-(e^{x^2})'}{\sqrt{1 - (e^{x^2})^2}} = \dfrac{-2xe^{x^2}}{\sqrt{1 - e^{2x^2}}}$

c. $\sec^{-1}(\ln(x))$

Set $u = \ln(x)$, so $[\sec^{-1}(\ln(x)]'$

$= \dfrac{1/x}{|\ln(x)|\sqrt{\ln^2(x) - 1}}$

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at x = -1

$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$$

Slope of tangent line

At x = -1

$$\frac{-1}{1+(-1)^2} = \frac{-1}{2}$$

When x = -1, y = $\frac{3\pi}{4}$

$$y - \frac{3\pi}{4} = \frac{-1}{2}(x+1)$$

3.3 Integration of Inverse Trigonometric Function

Rule: Integration Formulas Resulting in **Inverse Trigonometric Functions**

The following integration formulas yield inverse trigonometric functions:

1. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

2. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

3. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

3.3 Integration of Inverse Trigonometric Function

Proof

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now let's use implicit differentiation.

We obtain $\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x) \Rightarrow a\cos y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{a\cos y}$.

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$.

Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$,

we have $\cos y = \sqrt{1 = \sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{a^2 - a^2\sin^2 y}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2 - u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

3.3 Integration of Inverse Trigonometric Function

Example

Find $\displaystyle\int \frac{1}{25+16x^2}\,\mathrm{d}x$.

Here, we take the 16 outside the integral, so that we get

$$\int \frac{1}{25+16x^2}\,\mathrm{d}x = \frac{1}{16}\int \frac{1}{\frac{25}{16}+x^2}\,\mathrm{d}x\,.$$

Now we can see that $a = \sqrt{\frac{25}{16}} = \frac{5}{4}$, so that

$$\begin{aligned}
\int \frac{1}{25+16x^2}\,\mathrm{d}x &= \frac{1}{16}\times\frac{1}{\left(\frac{5}{4}\right)}\tan^{-1}\left(\frac{x}{\left(\frac{5}{4}\right)}\right)+c \\
&= \frac{1}{16}\times\frac{4}{5}\tan^{-1}\left(\frac{4x}{5}\right)+c \\
&= \frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right)+c\,.
\end{aligned}$$

3.3 Integration of Inverse Trigonometric Function

Example Find the integral $\int \frac{dx}{9 + 4x^2}$

Match the form of the integral to the one for $\tan^{-1}(u)$.

Write $9 + 4x^2 = 9\,(1 + \frac{4}{9}x^2) = 9\,[1 + (\frac{2}{3}x)^2]$

Hence $\int \frac{dx}{9 + 4x^2} = \frac{1}{9}\int \frac{dx}{1 + (\frac{2}{3}x)^2}$

$$\frac{1}{9}\int \frac{dx}{1 + (\frac{2}{3}x)^2}$$

$$= \frac{1}{9}\int \frac{1}{1 + u^2}\,\frac{3}{2}\,du$$

$$= \frac{1}{6}\int \frac{1}{1 + u^2}\,du$$

$$= \frac{1}{6}\tan^{-1}(u) + C$$

$$= \frac{1}{6}\tan^{-1}(\frac{2}{3}x) + C$$

Set $u = \frac{2}{3}x \rightarrow \frac{du}{dx} = \frac{2}{3}$ So $dx = \frac{3}{2}du$

substitution method

3.3 Integration of Inverse Trigonometric Function

Example Find the definite integral $\int_{\ln(1/2)}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

$$\int_{\ln(1/2)}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

$$= \int_{1/2}^{1} \frac{e^x}{\sqrt{1-u^2}}\,\frac{du}{e^x}$$

$$= \int_{1/2}^{1} \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1}(u)\Big|_{1/2}^{1}$$

$$= \sin^{-1}(1) - \sin^{-1}(1/2) = \pi/2 - \pi/6 = \pi/3$$

Set $u = e^x \rightarrow \frac{du}{dx} = e^x$

So $dx = du/e^x$

for $x = \ln(1/2) \rightarrow u = 1/2$

$x = 0 \rightarrow u = 1$

substitution method

3.3 Integration of Inverse Trigonometric Function

Example:

Evaluate $$\int \frac{\sin^{-1} t dt}{\sqrt{1-t^2}}$$

Solution:

Substitute $u = \arcsin(t) \longrightarrow \frac{du}{dt} = \frac{1}{\sqrt{1-t^2}}$ (steps) $\longrightarrow dt = \sqrt{1-t^2}\, du$:

$$= \int u \, du$$

Apply power rule:

$$\int u^n \, du = \frac{u^{n+1}}{n+1} \text{ with } n = 1:$$

$$= \frac{u^2}{2}$$

Undo substitution $u = \arcsin(t)$:

$$= \frac{\arcsin^2(t)}{2}$$

3.3 Integration of Inverse Trigonometric Function

Example:

Evaluate $\int \frac{\tan^{-1}(2t)}{1+4t^2} dt$

Solution:

Substitute $u = \arctan(2t) \longrightarrow \frac{du}{dt} = \frac{2}{4t^2+1}$ (steps) $\longrightarrow dt = \frac{4t^2+1}{2} du$:

$$= \frac{1}{2} \int u \, du$$

$$= \frac{u^2}{4}$$

Undo substitution $u = \arctan(2t)$:

$$= \frac{\arctan^2(2t)}{4}$$

3.3 Integration of Inverse Trigonometric Function

Example:

Evaluate $$\int \frac{e^t \cos^{-1}(e^t)}{\sqrt{1-e^{2t}}} dt$$

Solution:

Substitute $u = \arccos(e^t) \longrightarrow \frac{du}{dt} = -\frac{e^t}{\sqrt{1-e^{2t}}}$ (steps) $\longrightarrow dt = -e^{-t}\sqrt{1-e^{2t}}\, du$:

$$= -\int u \, du$$

$$= -\frac{u^2}{2}$$

Undo substitution $u = \arccos(e^t)$:

$$= -\frac{\arccos^2(e^t)}{2}$$