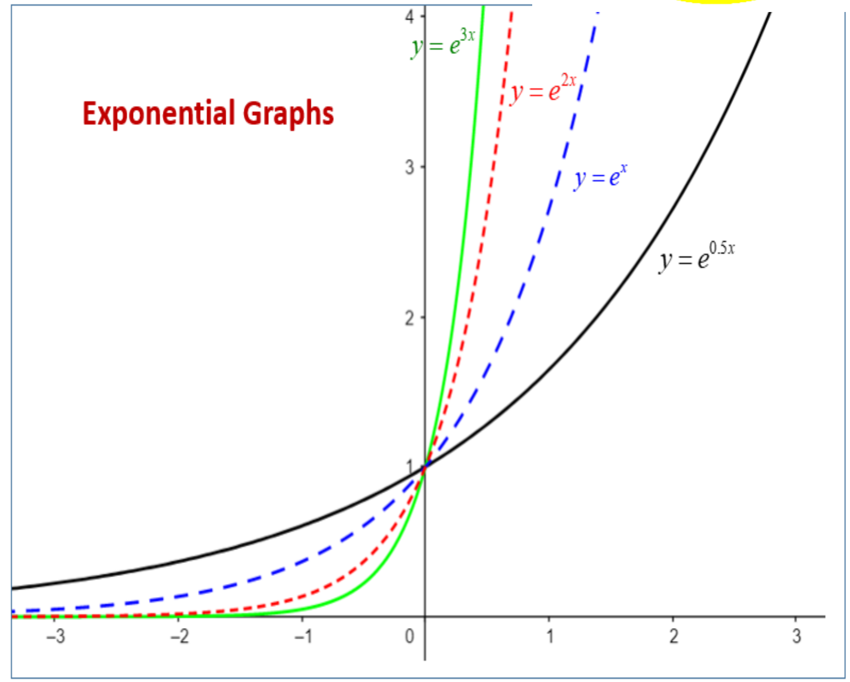
Salahaddin University
College of Engineering
Electrical Department

Chapter Two

Exponential function



Prepared By: Khalid A.Hamed
Khalid.abduljabbar@su.edu.krd

Subject: Math II
Class: 1st Year 2nd Sem.

Outline
Exponential function

2.1 Graph of exponential

2.2 Properties of exponential

2.3 Derivative of exponential

2.4 Integration of exponential

2.1 Graph of exponential

The ***exponential function f*** with base b is defined by

$$f(x) = ab^x$$

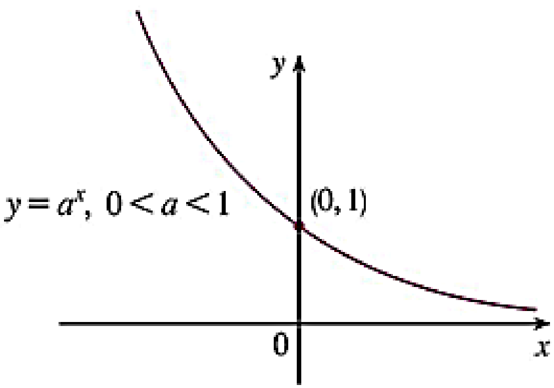
where $b > 0$, $b \neq 1$, and x is any real number.

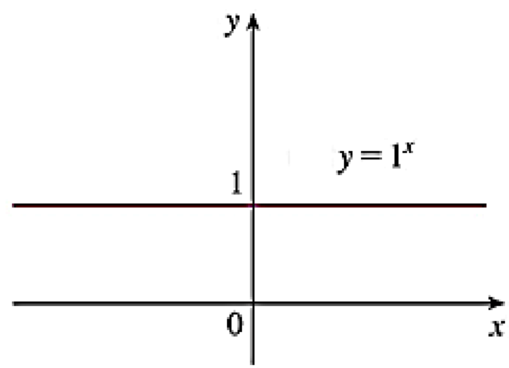
For instance,

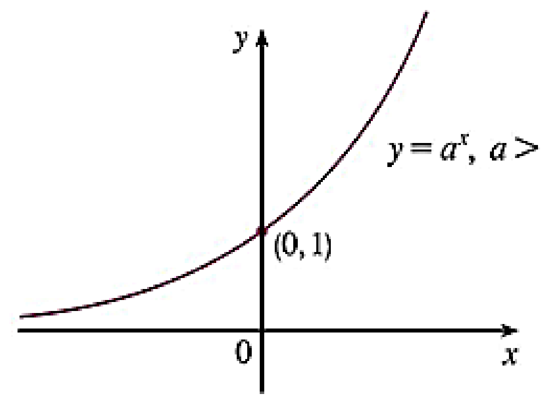
$f(x) = 3^x$ and $g(x) = 0.5^x$

are exponential functions.

** when b> 1; b is considered a growth factor.







2.1 Graph of exponential

The value of $f(x) = 3^x$ when $x = 2$ is

$$f(2) = 3^2 = 9$$

The value of $f(x) = 3^x$ when $x = -2$ is

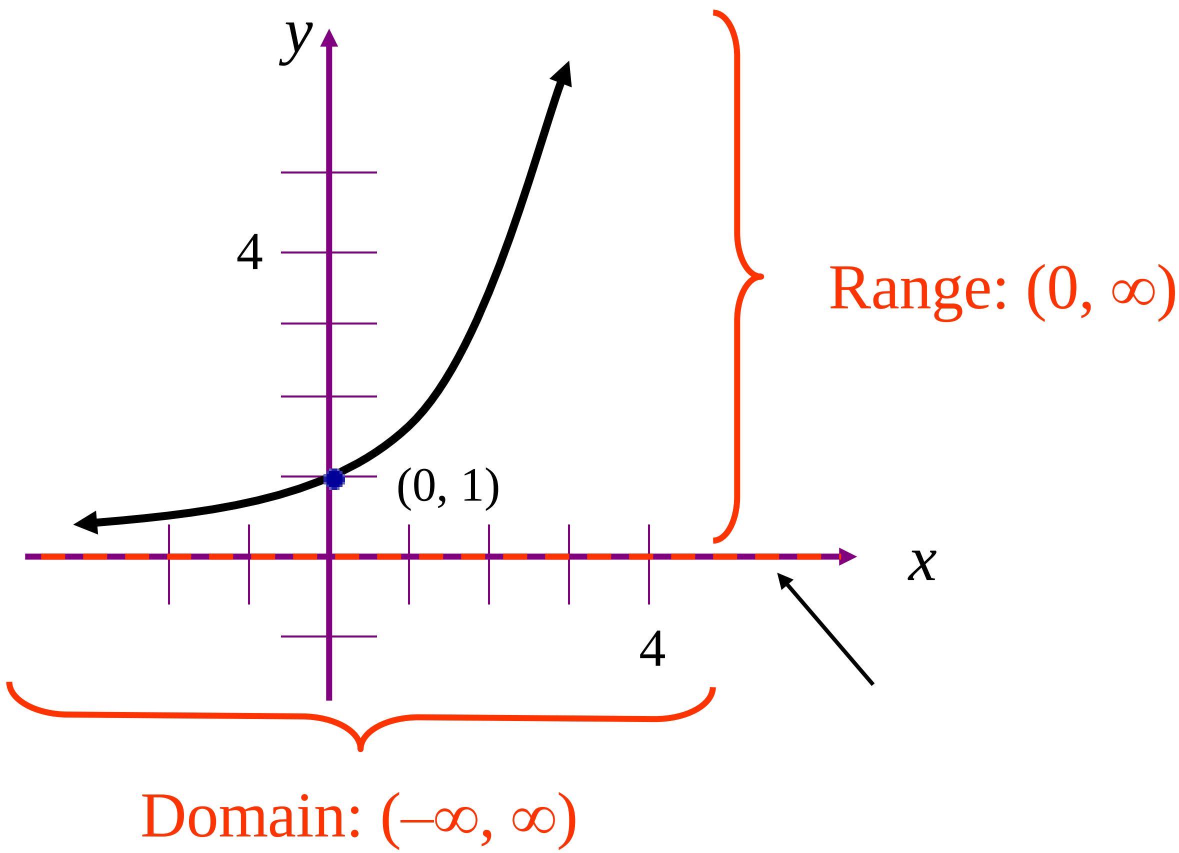
$$f(-2) = 3^{-2} = \frac{1}{9}$$

The value of $g(x) = 0.5^x$ when $x = 4$ is

$$g(4) = 0.5^4 = 0.0625$$

2.1 Graph of exponential

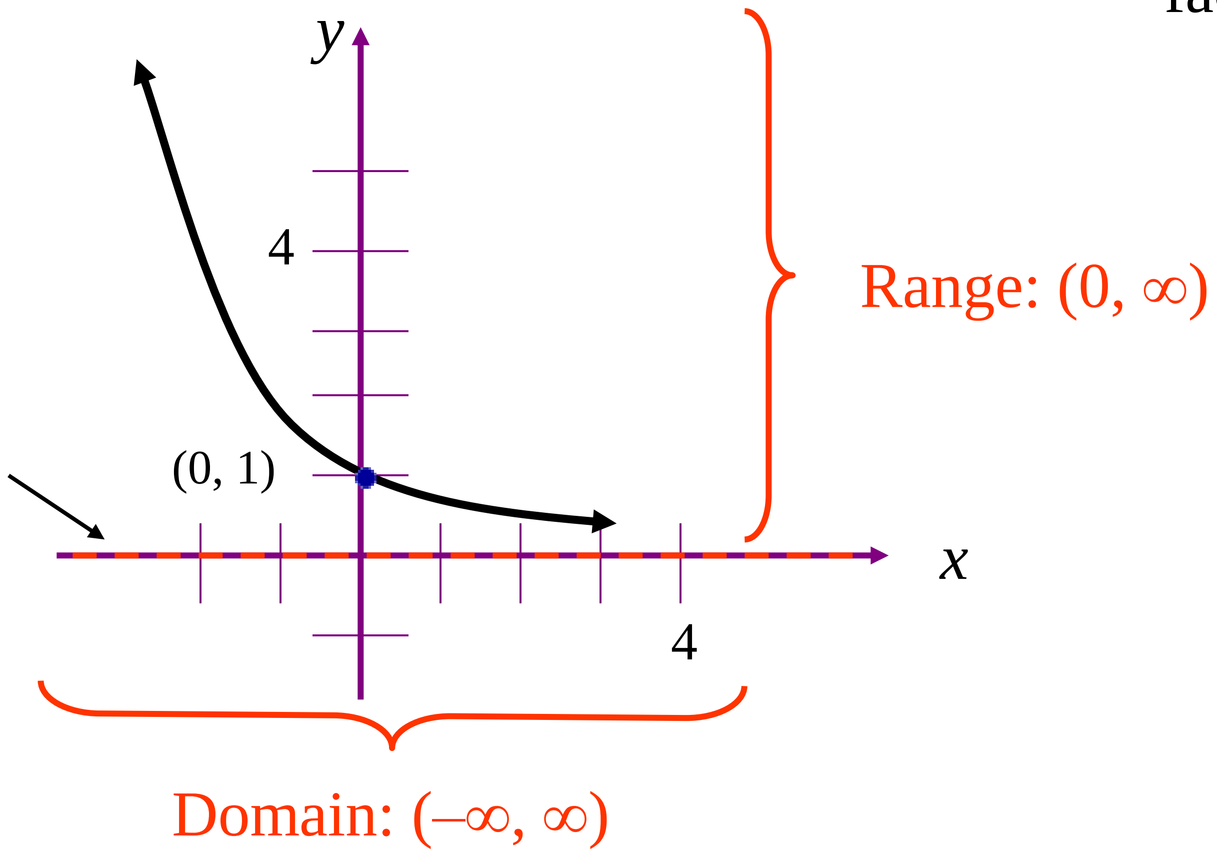
The graph of $f(x) = ab^x$, $b > 1$



2.1 Graph of exponential

The graph of $f(x) = ab^x$, $0 < b < 1$

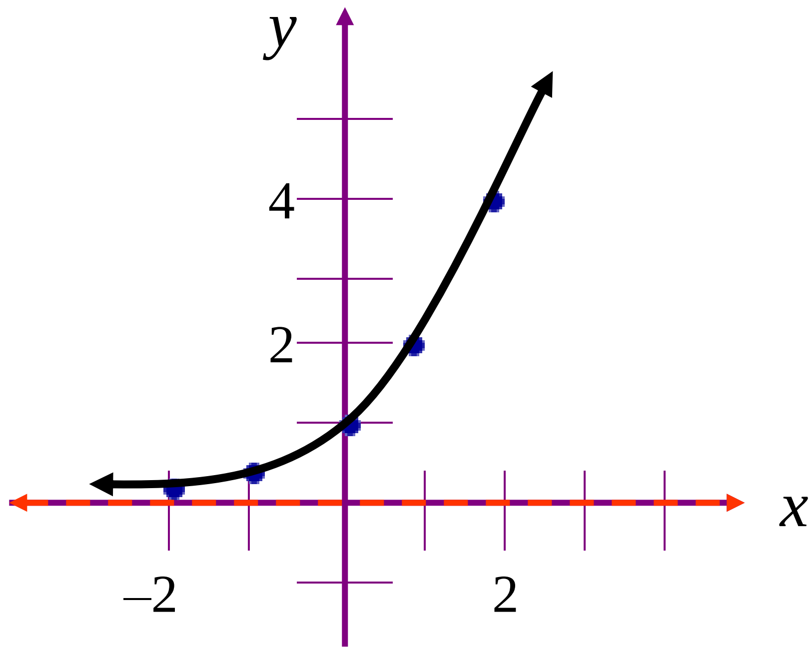
Since a< 1;
a is decay
factor.



2.1 Graph of exponential

Example: Sketch the graph of $f(x) = 2^x$.

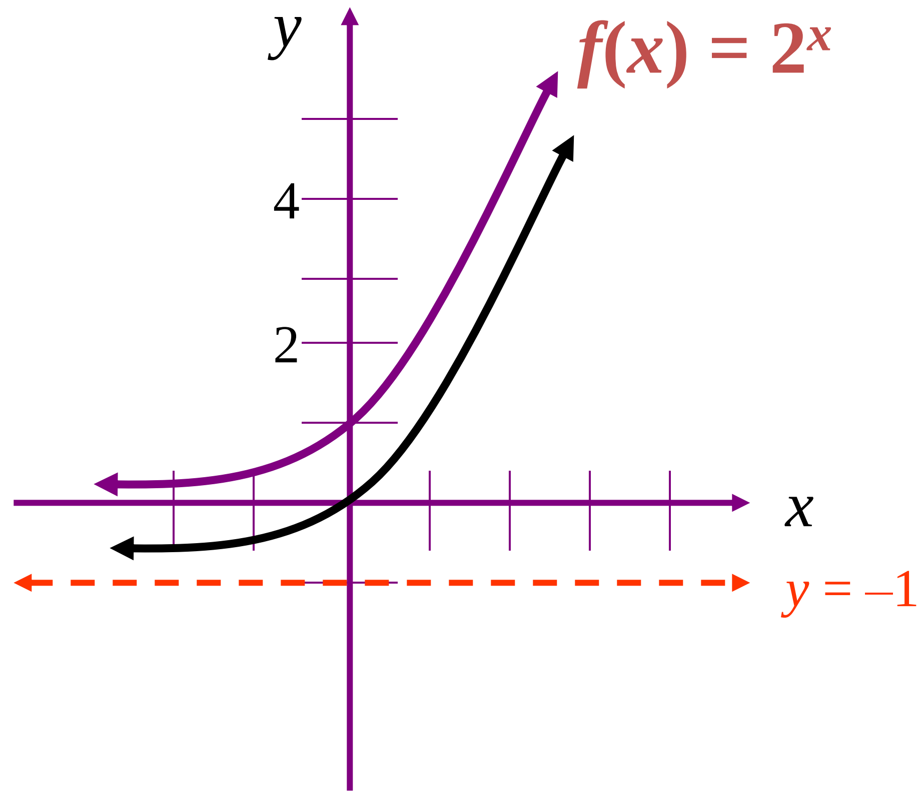
x	f(x)
-2	$2^{-2} = 1/4$
-1	$2^{-1} = 1/2$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$



2.1 Graph of exponential

Example: Sketch the graph of $g(x) = 2^x - 1$. State the domain and range.

The graph of this function is a vertical translation of the graph of $f(x) = 2^x$ down one unit .



Domain: $(-\infty, \infty)$

Range: $(-1, \infty)$

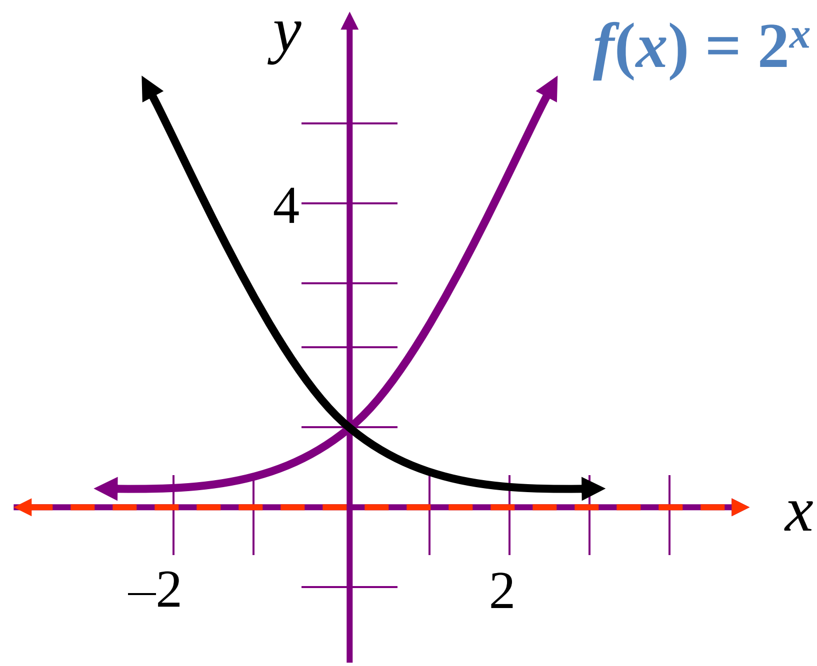
2.1 Graph of exponential

Example: Sketch the graph of $g(x) = 2^{-x}$. State the domain and range.

The graph of this function is a reflection the graph of $f(x) = 2^x$ in the y-axis.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



2.1 Graph of exponential

The Natural Base **e**

The irrational number ***e***, where

$$e \approx 2.718281828\ldots$$

is used in applications involving growth and decay.

Using techniques of calculus, it can be shown that

$$\left(1+\frac{1}{n}\right)^{n} \to e \ \textbf{as}\ n \to \infty$$

2.1 Graph of exponential

The Natural Base **e**

$$\left(1+\frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

- We can experimentally verify that this number exists and is

  $e \approx 2.718281828459045\ldots$

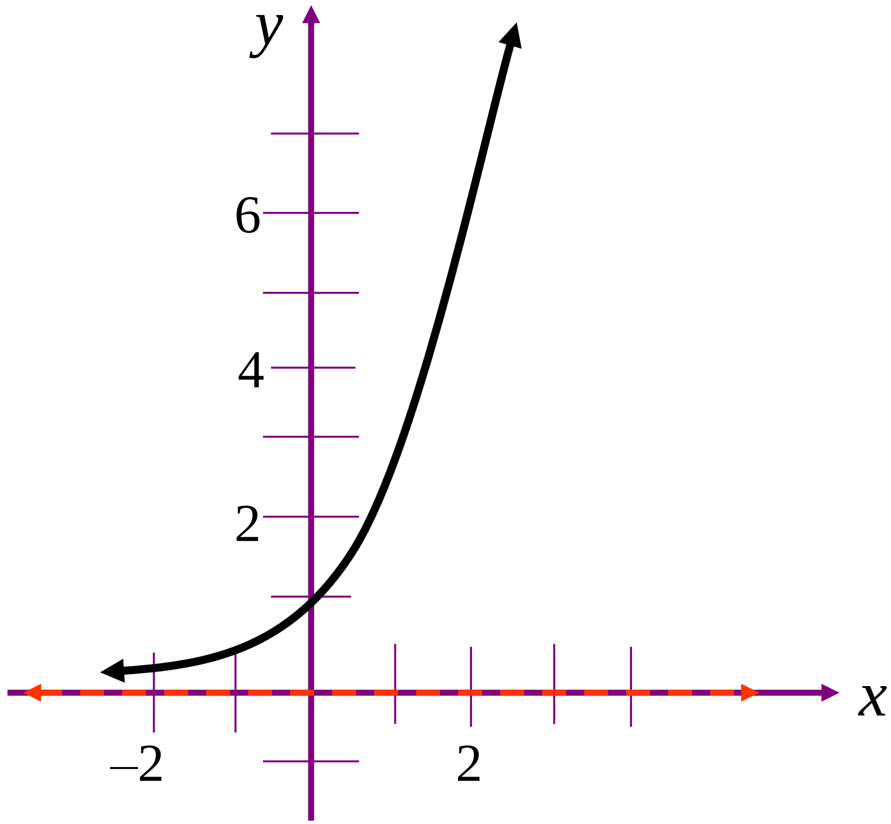
- e is irrational
- e is *transcendental*

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
3	2.37037
10	2.59374
100	2.70481
1000	2.71692
10^6	2.71828

An irrational number is a number that cannot be expressed as a fraction **p/q** for any integers **p** and **q**. Irrational numbers have decimal expansions that neither terminate nor become periodic. Every transcendental number is irrational.

2.1 Graph of exponential

The graph of $f(x) = e^x$



x	$f(x)$
-2	0.14
-1	0.38
0	1
1	2.72
2	7.39

2.2 Properties of exponential

- The exponential function $y = b^x$ $(b > 0, b \neq 1)$ has the following properties:

1. Its **domain** is $(-\infty, \infty)$.
2. Its **range** is $(0, \infty)$.
3. Its graph **passes through** the point **(0, 1)**
4. It is **continuous** on $(-\infty, \infty)$.
5. It is **increasing** on $(-\infty, \infty)$ if $b > 1$ and **decreasing** on $(-\infty, \infty)$ if $b < 1$.

2.2 Properties of exponential

Theorem

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$ *negative exponents mean reciprocals.*
- $(a^x)^y = a^{xy}$ *fractional exponents mean roots*
- $(ab)^x = a^x b^x$

Proof.

- This is true for positive integer exponents by natural definition
- Our conventional definitions make these true for rational exponents

2.2 Properties of exponential

Example

Simplify: $8^{2/3}$

Solution

- $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$
- $Or,\ \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$

Example

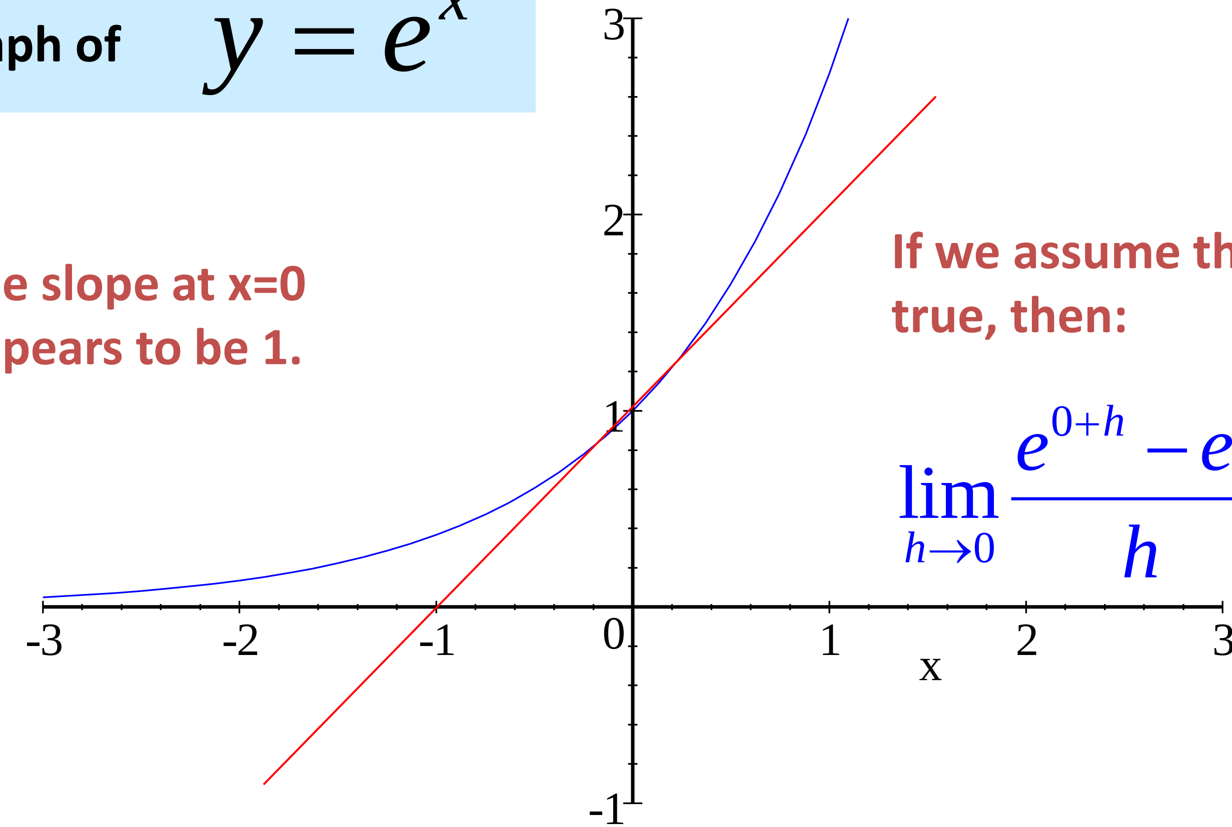
Simplify: $\dfrac{\sqrt{8}}{2^{1/2}}$

Answer

2

2.3 Derivative of exponential

Look at the graph of $y = e^{x}$



The slope at x=0 appears to be 1.

If we assume this to be true, then:

$$\lim_{h\to 0}\frac{e^{0+h}-e^{0}}{h}=1$$

definition of derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x} = \lim_{h\to 0}\frac{e^{x+h}-e^{x}}{h} = \lim_{h\to 0}\frac{e^{x}e^{h}-e^{x}}{h}$$

$$= \lim_{h\to 0}\frac{e^{x}(e^{h}-1)}{h} = e^{x}\lim_{h\to 0}\frac{e^{h}-1}{h} = e^{x} f'(0)$$

2.3 Derivative of exponential

Now we attempt to find a general formula for the derivative of $y = e^x$ using the definition.

$$\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0}\left(e^x \cdot \frac{e^h - 1}{h}\right)$$

$$= e^x \cdot \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)$$

This is the slope at x=0, which we have assumed to be 1.

$$= e^x \cdot 1$$

$$= e^x$$

2.3 Derivative of exponential

e^x **is its own derivative!**

If we incorporate the chain rule:

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

We can now use this formula to find the derivative of a^x

2.3 Derivative of exponential

Derivative of a^x

If $a > 0$ and $a \neq 1$, we can use the properties of logarithms to write a^x in terms of e^x. The formula for doing so is

$$a^x = e^{x \ln a}. \qquad e^{x \ln a} = e^{\ln(a^x)} = a^x$$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

2.3 Derivative of exponential

example:

Differentiate the function $y = e^{\tan x}$

To use the Chain Rule, we let $u = \tan x$.

Then, we have $y = e^u$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \frac{du}{dx} = e^{\tan x} \sec^2 x$$

example:

Find y' if $y = e^{-4x} \sin 5x$.

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4)$$
$$= e^{-4x}(5\cos 5x - 4\sin 5x)$$

2.3 Derivative of exponential

Example: Find the derivative of 2^x.

Solution: Recall $2 = e^{\ln 2}$, so $2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}$.

Thus
$$\frac{d}{dx} 2^x = \frac{d}{dx} e^{(\ln 2)x} = e^{(\ln 2)x} \frac{d}{dx}((\ln 2)x)$$
$$= (\ln 2) \cdot 2^x.$$

Theorem

$$\frac{d}{dx} b^x = (\ln b) \cdot b^x, \quad \textit{for any base } b > 0.$$

2.3 Derivative of exponential

Example: Find $\dfrac{d}{dx} 7^{x^2}$.

Solution: We apply the chain rule with outer function $f(u) = 7^u$ and inner function $g(x) = x^2$:

$$\frac{d}{dx} 7^{x^2} = (\ln 7) \cdot 7^{(x^2)} \cdot \frac{d}{dx} x^2$$
$$= 2 \ln 7 \cdot x \cdot 7^{x^2}$$

Example: Find $\dfrac{d}{dx} 5^{5^x}$.

Solution:

$$\frac{d}{dx} 5^{5^x} = (\ln 5) \cdot 5^{5^x} \cdot \frac{d}{dx} 5^x$$
$$= (\ln 5) \cdot 5^{5^x} \cdot (\ln 5) \cdot 5^x$$
$$= \ln^2 5 \cdot 5^x \cdot 5^{5^x}$$

2.4 Integration of exponential

- Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

$$\int e^x \, dx = e^x + C$$

2.4 Integration of exponential

Example Evaluate $\int e^{(3x+1)}\,dx = \int e^{u}\,dx$

$$\text{Let } u = 3x+1$$
$$du = 3\,dx$$
$$\tfrac{1}{3}du = dx$$

$$= \int e^{u}\cdot\tfrac{1}{3}du$$
$$= \tfrac{1}{3}\int e^{u}\,du$$
$$= \tfrac{1}{3}e^{u} + C$$
$$= \tfrac{1}{3}e^{(3x+1)} + C$$

2.4 Integration of exponential

Evaluate $\int \left(10x^4 + 4x\right) e^{\left(x^5 + x^2 - 1\right)} \, dx$

u = ?

Test: Take the derivative of the choice of u. If you cannot find it elsewhere in the integrand, then use a different expression for u.

Testing Zone:

✗ What if $u = 10x^4 + 4x$?

Then $du = \left(40x^3 + 4\right) dx$

✓ What if $u = x^5 + x^2 - 1$?

Then $du = \left(5x^4 + 2x\right) dx$

This expression is off only by a constant multiple.

2.4 Integration of exponential

Evaluate $\int \left(10x^4 + 4x\right)e^{\left(x^5+x^2-1\right)}\,dx = \int e^u \left(10x^4 + 4x\right)dx$

$$= \int e^u \cdot 2du$$

Let $u = x^5 + x^2 - 1$

$du = \left(5x^4 + 2x\right)dx$

$$= 2\int e^u\,du$$

$2du = \left(10x^4 + 4x\right)dx$

$$= 2e^u + C$$

$$= 2e^{\left(x^5+x^2-1\right)} + C$$