1.1 Definition A set is collection of distinct objects. These objects are called the elements, or members.

Important Sets of Real Numbers

The set $\mathbb{N}$ of natural numbers is define by

$$\mathbb{N} = \{1, 2, 3, \dots \}$$

The set $\mathbb{Z}$ of integers is define by

$$\mathbb{Z} = \{\dots , -2, -1, 0, 1, 2, \dots\}.$$

The set $\mathbb{Q}$ of rational numbers is define by

$$\mathbb{Q} = \{a/b: a, b \in \mathbb{Z}, b \neq 0\}.$$

Non-periodic decimal fractions are called irrational numbers and denoted by Irr.

For example, $\sqrt{2}, \sqrt{3}, \pi$

Real numbers

Real Numbers are made up of rational numbers and irrational numbers and denoted by $\mathbb{R}$.

The Number Line

We may use the number line to represent all the real numbers graphically; each real number corresponds to exactly one point on the number line. ∞ and $-\infty$ are not real numbers because there is no point on the number line corresponding to either of them.

$\mathbb{C}$, denoting the set of all complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

For example, $1 + 2i \in \mathbb{C}$.

Absolute value

Definition The **absolute value** of a number x, denoted by is defined by the formula

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}.$$

Example $|2| = 2$, $|-5| = -(-5) = 5$.

Some properties of the absolute value

Let a, b and x be any real numbers then:

1) $|x| = \sqrt{x^2}$
2) $|ab| = |a||b|$
3) $|a + b| \leq |a| + |b|$
4) $|a - b| \geq ||a| - |b||$
5) $|x| \leq a$ if and only if $x \leq a$ and $x \geq -a$ (or $-a \leq x \leq a$)
6) $|x| > a$ if and only if $x \geq a$ or $x \leq -a$

The Function

Definition Let A and B be any two non empty sets then a function (denoted by f) from A to B is a relation from A to B provided that for each $x \in A$ there exist only a unique $y \in B$ such that $(x, y) \in f$ and f can be written as:

$$f: A \rightarrow B, \ y = f(x) \text{ or } \ y \overset{f}{\rightarrow} x.$$

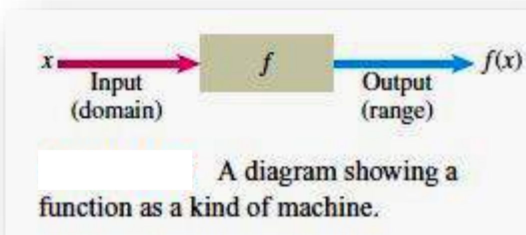
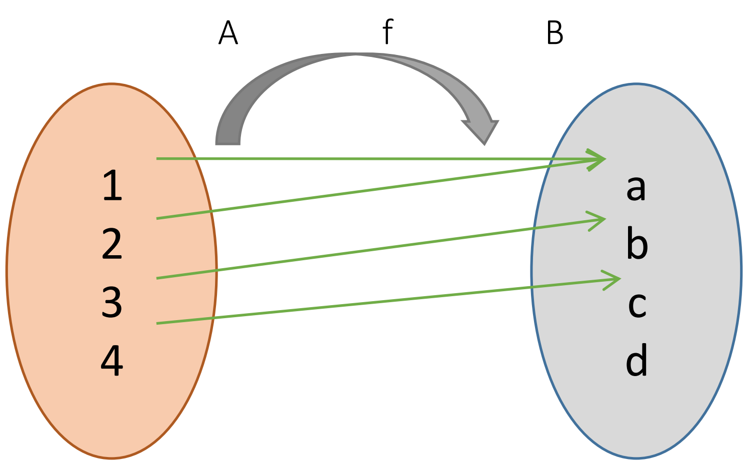


Figure 1.3

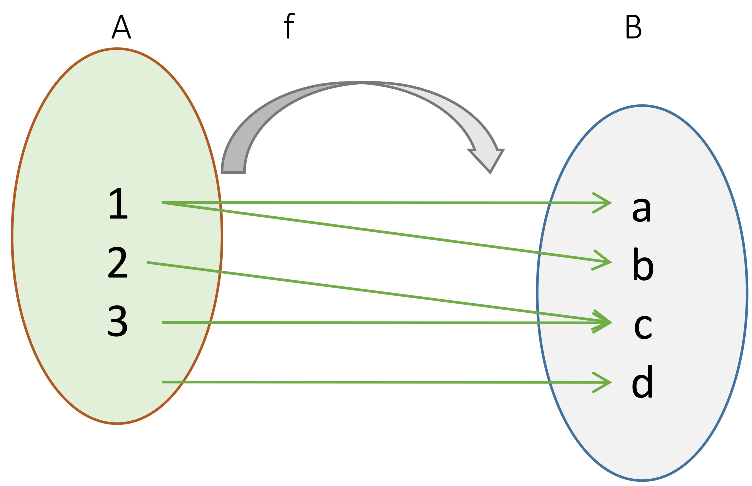
Example



Solution: It's a function, because $\forall\, x \in A, \exists\, y \in B$ such that $(x, y) \in f$

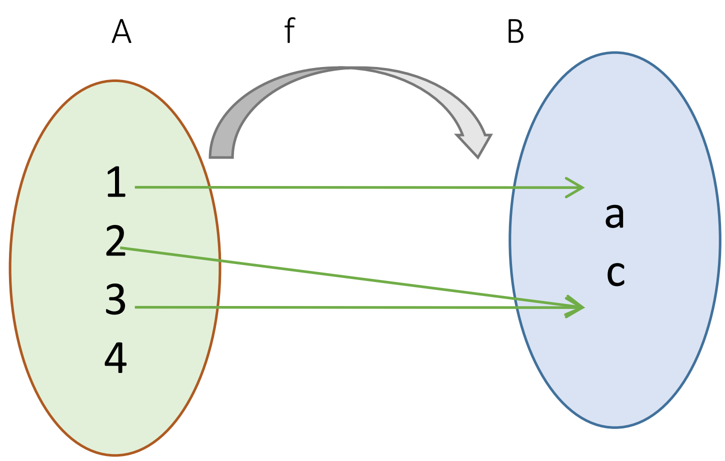
$f = \{(1, a), (2, a), (3, b), (4, c)\}.$

1.23 Example



Solution: It's not function.

Example



Solution: It's not function.

Example Let $f: \mathbb{R} \to \mathbb{R}$. Does the following f are functions or not?

1) $f(x) = \sqrt{x}$ 2) $f(x) = x^2$ 3) $f(x) = 3$

Solution:

1. Is not a function because $\sqrt{-1}$ is undefined.

2. Is a function since for all x there exist y such that $(x, y) \in f$.

3. is a function since for all x there exist y such that $(x, y) \in f$.

Example Let $f: \mathbb{R}^+ \to \mathbb{R}$, $x = y^2$. Is not a function because

$$(4, -2) \in f \text{ and } (4, 2) \in f$$

Definition (The Graph of Function)

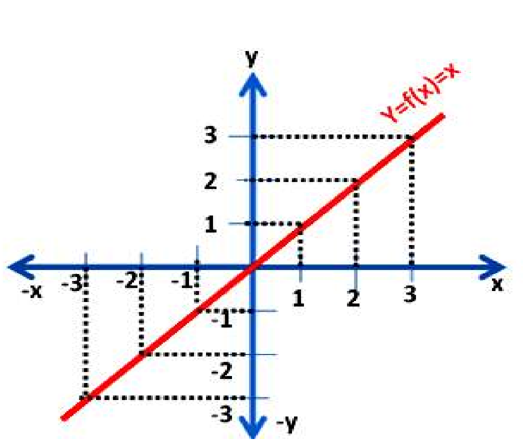
The graph of the function $y = f(x)$ is the set of all points (x, y) in the Cartesian plane $X \times Y$ such that (x, y) satisfies the function $y = f(x)$.

That means the graph is $\{(x, y): y = f(x)\}$.

Example Find the graph of this function $y = f(x) = x$

Solution:

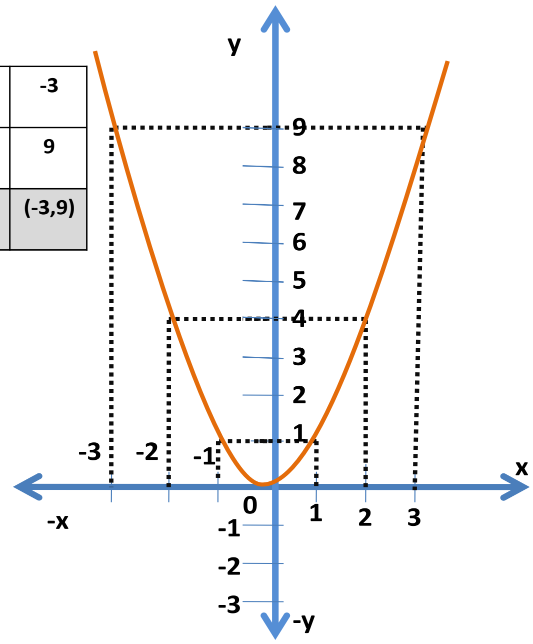
x	1	2	3	0	-1	-2	-3
$y = f(x)$	1	2	3	0	-1	-2	-3
(x, y)	$(1, 1)$	$(2, 2)$	$(3, 3)$	$(0, 0)$	$(-1, -1)$	$(-2, -2)$	$(-3, -3)$



Example : Find the graph of this $y = f(x) = x^2$

Solution:

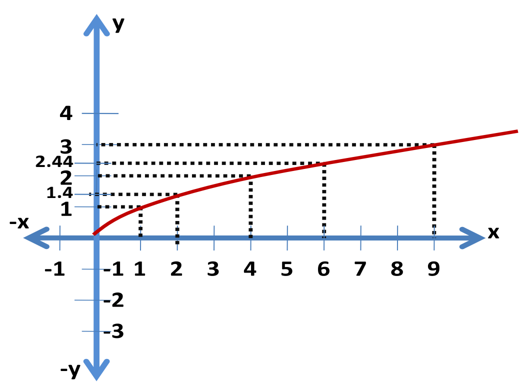
x	1	2	3	0	-1	-2	-3
y=f(x)	1	4	9	0	1	4	9
(x,y)	(1,1)	(2,4)	(3,9)	(0,0)	(-1,1)	(-2,4)	(-3,9)



Example : Graph this function $y = f(x) = \sqrt{x}$

Solution:

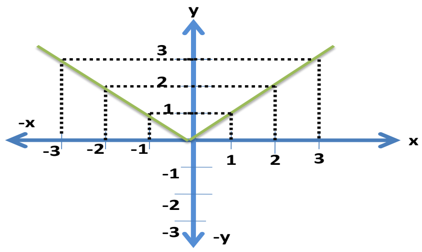
x	y=f(x)	(x, y)
0	0	(0,0)
1	1	(1,1)
2	1.4	(2,1.4)
4	2	(4,2)
6	2.44	(6,2.44)
9	3	(9,3)
0	0	(0,0)



Example : graph of this function $y = f(x) = |x|$

Solution:

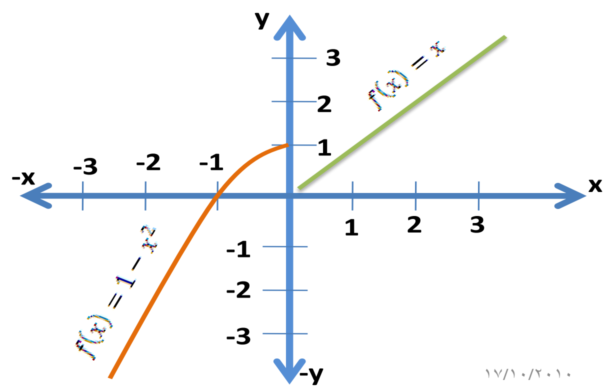
x	y=f(x)	(x, y)
1	1	(1,1)
2	2	(2,2)
3	3	(3,3)
0	0	(0,0)
-1	1	(-1,1)
-2	2	(-2,2)
-3	3	(-3,3)



$$y = f(x) = \begin{cases} 1 - x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$$

Solution:

Domain and the rang of this function is all real numbers and have graph



Definition

You are familiar with the following mathematical expression,

$$\lim_{x \to a} f(x) = L$$

which is read as follows, “The limit of $f(x)$, as x approaches a, is equal to L”. this statement means that “we can make the value of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a), but not equal to a.”

Theorem *Suppose that c is a constant and the limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then*

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ *where n is a positive integer.*
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ *where n is a positive integer.*
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where n is a positive integer, and if n is even, we assume that* $a > 0$.

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where n is a positive integer, and if n is even, we assume that* $\lim_{x\to a} f(x) > 0$.

Example . Evaluate

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$$

Solution

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} &= \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)} \\
&= \frac{\lim_{x\to -2}(x^3)+\lim_{x\to -2}(2x^2)-\lim_{x\to -2}(1)}{\lim_{x\to -2}(5)-\lim_{x\to -2}(3x)} \\
&= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} \\
&= = -\frac{1}{11}
\end{aligned}$$

Note that if we let $f(x)=\frac{x^3+2x^2-1}{5-3x}$, then $f(-2)=-\frac{1}{11}$. In other words, we would have gotten the correct answer by directly substituting -2 for x.

Example . Evaluate

$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$

Solution . Let $f(x)=\frac{x^2-9}{x-3}$. We can not find the limit by substituting $x=3$ because $f(3)$ is not defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{(x-3)(x+3)}{x-3}$$

The numerator and denominator have a common factor of $x-3$. When we take the limit as approaches 3, we have $x\neq 3$ and so $x-3\neq 0$. Therefore we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

Example Find

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

Solution We can not apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \lim_{t \to 0} \frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} \\
&= \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)} \\
&= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Exercise Evaluate

$$\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$$

Example

A)

$$\lim_{x \to 1} \frac{x^3-1}{x^2-3x+2} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+2)} = -3$$

B)

$$\lim_{x \to 1} \frac{x^4-1}{x^3+2x^2-3} = \lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{(x-1)(x-2)} = \lim_{x \to 1} \frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+3x+3)} = 4/7$$

Some limits are best calculated by first finding the left– and right–hand limits as shown in the following examples.

Example If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

Solution Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x) = 8 - 2\times 4 = 0$$

The right– and left–hand limits are equal. Thus the limit exists and

$$\lim_{x\to 4} f(x) = 0$$

Example If

$$f(x) = \frac{|x-2|}{x^2 + x - 6}$$

find: $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2} f(x)$

Solution Observe that

$$|x-2| = \begin{cases} x-2 & \text{if} \quad x \geq 2 \\ -(x-2) & \text{if} \quad x < 2 \end{cases}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 2^+}\frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^+}\frac{x-2}{(x+3)(x-2)}\\
&= \lim_{x\to 2^+}\frac{1}{x+3}\\
&= \frac{1}{5}\\
\lim_{x\to 2^-}\frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^-}\frac{-(x-2)}{(x+3)(x-2)}\\
&= \lim_{x\to 2^-}\frac{-1}{x+3}\\
&= -\frac{1}{5}
\end{aligned}$$

Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, then the limit $\lim_{x\to 2} f(x)$ does not exist.

Example Find $\lim_{x\to 1}\frac{\sqrt[3]{x}+2\sqrt{x+1}-2\sqrt{2x}-1}{x^3+2x^2-3x+4}$

Solution:

$$\lim_{x\to 1}\frac{\sqrt[3]{x}+2\sqrt{x+1}-2\sqrt{2x}-1}{x^3+2x^2-3x+4} = \frac{0}{4}.$$

Example. Find the limit

$$\lim_{x\to 1^+}\frac{x-2}{x-1} = -\infty$$

Example Find $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x^4-1}$

Solution:

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x^4-1} = \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(x-1)(x+1)(x^2+1)}$$

$$= \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)(x+1)(x^2+1)}$$

$$= \frac{1}{12}.$$

Example. $\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6}$

$$\lim_{x\to+\infty} \frac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{3x-6}{|x|}} = \lim_{x\to+\infty} \frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\frac{3x-6}{x}} = \lim_{x\to+\infty} \frac{\sqrt{\frac{x^2+2}{x^2}}}{\frac{3x-6}{x}} = \lim_{x\to+\infty} \frac{\sqrt{1+\frac{2}{x^2}}}{3-6/x} = 1/3.$$

$$\lim_{x\to+\infty} \frac{x^3-2x+1}{3x^3+x^2+5} = \lim_{x\to+\infty} \frac{x^3(1-\frac{2}{x^2}+\frac{1}{x^3})}{x^3(3+\frac{x^2}{x^3}+\frac{5}{x^3})} = 1/3.$$

Exercise (H. W) Evaluate the following limits

1) $\lim_{x\to 2} \frac{(x-3)(x+1)}{(x-2)}$

2) $\lim_{x\to 6} \frac{x+6}{x^2-36}$

3) $\lim_{x\to 4} \frac{3-x}{x^2-2x-8}$

4) $\lim_{x\to 0} \frac{x}{|x|}$

5) $\lim_{x\to 1} \frac{|x^2-1|-3x+3}{x^6-1}$

6 $\lim_{x\to 3} \frac{1-\sqrt{x-2}}{x-3}$

4.7 Definition (The Derivative)

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

The domain of the derivative f' (read "f prime") is the set of numbers x in the domain of f where the graph of f has a *nonvertical* tangent line, and the value $f'(x_0)$ of f' at such a point x_0 is the slope of the tangent line to $y = f(x)$ there. Thus the equation of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Remark The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In the second limit $x_0 + h$ is replaced by x, so that $h = x - x_0$ and $h \to 0$ is equivalent to $x \to x_0$.

EXAMPLE 1 **(The derivative of a linear function)** Show that if $f(x) = ax + b$, then $f'(x) = a$.

Solution The result is apparent from the graph of f (Figure 2.11), but we will do the calculation using the definition:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} = a. \end{aligned}$$

EXAMPLE 2 Use the definition of the derivative to calculate the derivatives of the functions:

$$\text{(a) } f(x) = x^2, \quad \text{(b) } g(x) = \frac{1}{x}, \quad \text{and} \quad \text{(c) } k(x) = \sqrt{x}.$$

Solution Figures 2.12–2.14 show the graphs of these functions and their derivatives.

$$\begin{aligned}
\text{(a)}\quad f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x. \\
\text{(b)}\quad g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}. \\
\text{(c)}\quad k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Note that k is not differentiable at the endpoint $x = 0$.

Exercise Use the definition of the derivative to calculate the derivative of the functions $f(x) = \sqrt[3]{x}$.

Differentiation rules for sums, differences, and constant multiples

If functions f and g are differentiable at x, and if C is a constant, then the functions $f + g$, $f - g$, and Cf are all differentiable at x and

$$\begin{aligned}
(f + g)'(x) &= f'(x) + g'(x), \\
(f - g)'(x) &= f'(x) - g'(x), \\
(Cf)'(x) &= Cf'(x).
\end{aligned}$$

The Product Rule

If functions f and g are differentiable at x, then their product fg is also differentiable at x, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

The Quotient Rule

If f and g are differentiable at x, and if $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Theorem If $n \in \mathbb{N}, then\ \frac{d}{dx}\left[\left(f(x)\right)^n\right] = n(f(x))^{n-1}.f(x)'$.

EXAMPLE 9 Find the derivatives of

(a) $y = \frac{1-x^2}{1+x^2}$, (b) $\frac{\sqrt{t}}{3-5t}$, and (c) $f(\theta) = \frac{a+b\theta}{m+n\theta}$.

Solution We use the Quotient Rule in each case.

(a) $\frac{dy}{dx} = \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2} = -\frac{4x}{(1+x^2)^2}$.

(b) $\frac{d}{dt}\left(\frac{\sqrt{t}}{3-5t}\right) = \frac{(3-5t)\frac{1}{2\sqrt{t}} - \sqrt{t}(-5)}{(3-5t)^2} = \frac{3+5t}{2\sqrt{t}(3-5t)^2}$.

(c) $f'(\theta) = \frac{(m+n\theta)(b) - (a+b\theta)(n)}{(m+n\theta)^2} = \frac{mb-na}{(m+n\theta)^2}$.

Example Find the derivative of $f(x) = \sqrt{1+\sqrt{1+x}}$

Solution: $f'(x) = \frac{1}{2}[1+\sqrt{1+x}]^{-\frac{1}{2}}.\frac{1}{2}(1+x)^{-\frac{1}{2}}$

Trigonometric Function

Consider the circle (see Figure 2.1)

$$x^2 + y^2 = r^2, \ \ r > 0$$

We define

$$\sin\theta = \frac{y}{r} \ \text{ and } \ \cos\theta = \frac{x}{r}$$

$$\Longrightarrow y = r\sin\theta \ \text{ and } \ x = r\cos\theta$$

Now, since $x^2 + y^2 = r^2$

$$\Longrightarrow r^2\sin^2\theta + r^2\cos^2\theta = r^2, \ \ r \neq 0$$

$$\Longrightarrow \qquad \sin^2\theta + \cos^2\theta = 1 \qquad \dots (1)$$

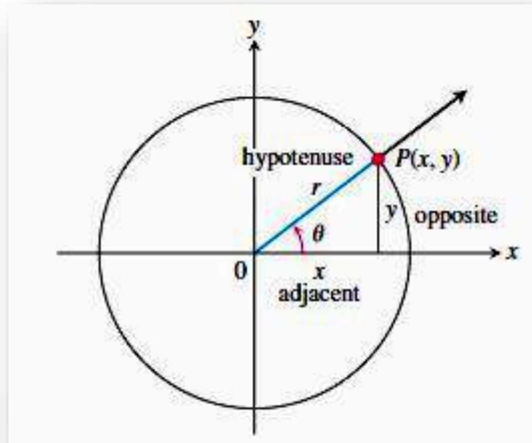


Figure 2.1

1. Sine Function

It is a function $f: \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$.

$\sin x = 0$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \dots = n\pi, n \in \mathbb{Z}$.

$\sin x = \mp 1$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \dots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$.

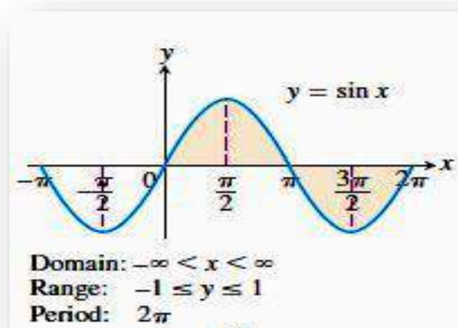
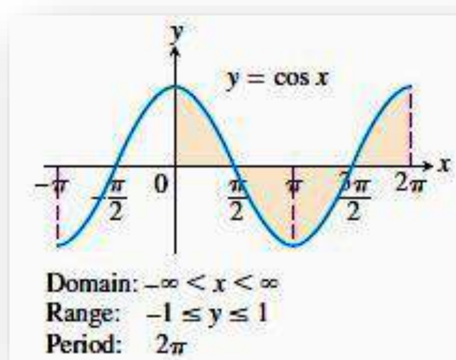


Figure 2.5

2. Cosine Function

It is a function $f: \mathbb{R} \to [-1, 1]$ defined by $f(x) = \cos x$.



$\cos x = 0$ iff $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}.$

$\cos x = \mp 1$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots = n\pi, n \in \mathbb{Z}$

even function.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

EXAMPLE 2 Evaluate the derivatives of the following functions:

(a) $\sin(\pi x) + \cos(3x)$, (b) $x^2 \sin\sqrt{x}$, and (c) $\frac{\cos x}{1 - \sin x}$.

Solution

(a) By the Sum Rule and the Chain Rule:

$$\frac{d}{dx}(\sin(\pi x) + \cos(3x)) = \cos(\pi x)(\pi) - \sin(3x)(3) = \pi\cos(\pi x) - 3\sin(3x).$$

(b) By the Product and Chain Rules:

$$\frac{d}{dx}(x^2 \sin\sqrt{x}) = 2x\sin\sqrt{x} + x^2\left(\cos\sqrt{x}\right)\frac{1}{2\sqrt{x}} = 2x\sin\sqrt{x} + \frac{1}{2}x^{3/2}\cos\sqrt{x}$$

(c) By the Quotient Rule:

$$\frac{d}{dx}\left(\frac{\cos x}{1 - \sin x}\right) = \frac{(1 - \sin x)(-\sin x) - (\cos x)(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}.$$

We used the identity $\sin^2 x + \cos^2 x = 1$ to simplify the middle line.

1. The Natural Logarithmic Functions

It is the logarithmic function with the base $a = e$.

$i.e.\ f(x) = \text{Log}_a x = \ln x$, $\ln: \mathbb{R}^+ \to \mathbb{R}$.

(e is the Euler's number and $e = 2.718281828\ldots$)

Some Properteis

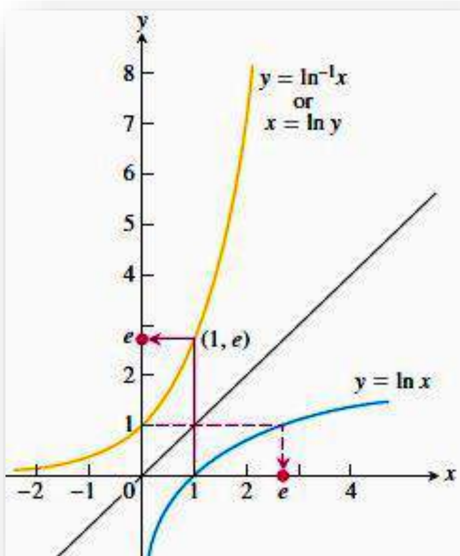
1. $\ln(xy) = \ln x + \ln y$

2. $\frac{\ln x}{\ln y} = \ln x - \ln y$

3. $\ln 1 = 0, \ln e = 1$

4. $\ln x^r = r \ln x$

.



x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Exponential Functions

1. The Natural Exponential Functions

It is the function $\exp: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x, \forall x \in \mathbb{R}$.

Some Properteis

The natural exponential e^x obeys the following laws:

1. $e^x\, e^y = e^{x+y} \;\; \forall\, x, y \in \mathbb{R}$

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^x}{e^y} = e^{x-y}$

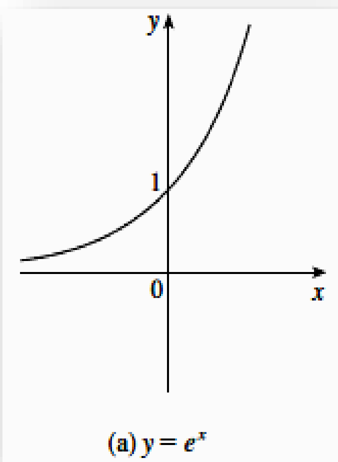
4. $(e^x)^y = e^{xy} = (e^y)^x$

5. $e^0 = 1$

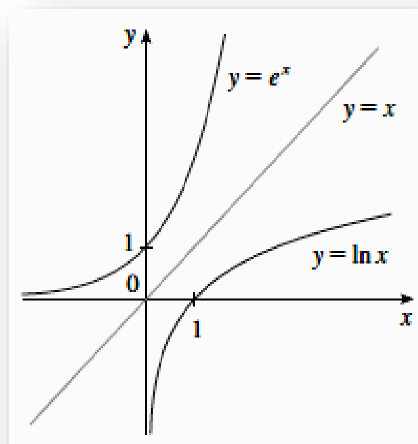
6. $e^{\ln x} = x$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}



(a) $y = e^x$



Derivatives of the Logarithmic Functions

1. The Natural Logarithmic Functions

If $y = \ln u$, u is differentiable with respect to x, then

$$\boxed{\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}}$$

EXAMPLE 2 Find $\frac{d}{dx}\ln(\sin x)$.

SOLUTION Using (3), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

Derivatives of the Exponential Functions

1. The Natural Exponential Functions

If $y = e^u$, u is differentiable with respect to x, then

$$\boxed{\frac{dy}{dx} = e^u \cdot \frac{du}{dx}}$$

EXAMPLE 3 Find the derivatives of

(a) e^{x^2-3x}, (b) $\sqrt{1+e^{2x}}$, and (c) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Solution

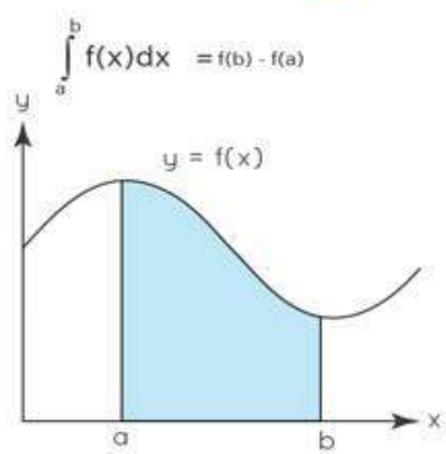
(a) $\dfrac{d}{dx} e^{x^2-3x} = e^{x^2-3x}(2x-3) = (2x-3)e^{x^2-3x}$.

(b) $\dfrac{d}{dx}\sqrt{1+e^{2x}} = \dfrac{1}{2\sqrt{1+e^{2x}}}\left(e^{2x}(2)\right) = \dfrac{e^{2x}}{\sqrt{1+e^{2x}}}$.

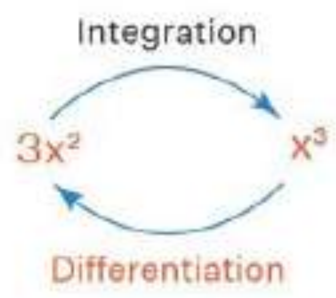
(c)
$$\frac{d}{dx}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})(e^x - (-e^{-x})) - (e^x - e^{-x})(e^x + (-e^{-x}))}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2 - [(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2]}{(e^x + e^{-x})^2}$$
$$= \frac{4e^{x-x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}.$$

Integration

Integration is the process of finding the area of the region under the curve. This is done by drawing as many small rectangles covering up the area and summing up their areas. The sum approaches a limit that is equal to the region under the curve of a function. Integration is the process of finding the antiderivative of a function. If a function is integrable and if its integral over the domain is finite, with the limits specified, then it is the definite integration.



If d/dx(F(x) = f(x), then ∫ f(x) dx = F(x) +C. These are indefinite integrals. For example, let f(x) = x^3 be a function. The derivative of f(x) is f'(x) = $3x^2$ and the antiderivative of $3x^2$ is f(x) = x^3



Example1: $\int x^3 dx = \frac{1}{4}x^4 + c$ because $\frac{d}{dx}\left[\frac{1}{4}x^4\right] = x^3$ In fact,

$\frac{1}{4}x^4, \frac{1}{4}x^4 + 2, \frac{1}{4}x^4 - 4, \frac{1}{4}x^4 + \frac{1}{2}$ are all antiderivatives of x^3, because they all differentiate to x^3.

Teorem: Suppose that $f(x)$ and $g(x)$ has antiderivatives. Then for any constants a and b,

$$\int [a\, f(x) \pm b\, g(x)]\, dx = a\int f(x)\, dx \pm b\int g(x)\, dx$$

Example 2: Evaluate $\int \frac{1}{2x^3} dx$.

Solution $\int \frac{1}{2x^3} dx = \frac{1}{2}\int x^{-3} dx = \frac{1}{2}\left[\frac{x^{-2}}{-2}\right] + c = -\frac{1}{4x^2} + c$

Example 3: Evaluate $\int \left(x^3 - 2x + 7\right) dx$.

Solution

$$\int \left(x^3 - 2x + 7\right) dx = \int x^3 dx - 2\int x dx + 7\int dx = \left[\frac{x^4}{4}\right] - 2\left[\frac{x^2}{2}\right] + 7x + c = \frac{1}{4}x^4 - x^2 + 7x + c$$

Example 4: Evaluate $\int\left(x^{2/3}-4x^{-1/5}+4\right)dx$.

Solution

$$\int\left(x^{2/3}-4x^{-1/5}+4\right)dx=\int x^{2/3}dx-4\int x^{-1/5}dx+4\int dx=\left[\frac{x^{5/3}}{5/3}\right]-4\left[\frac{x^{4/5}}{4/5}\right]+4x+c$$

$$=\frac{3}{5}x^{5/3}-5x^{4/5}+4x+c$$

Example 5: Evaluate $\int\left(\frac{7}{y^{3/4}}-\sqrt[3]{y}+4\sqrt{y}\right)dy$.

Solution

$$\int\left(\frac{7}{y^{3/4}}-\sqrt[3]{y}+4\sqrt{y}\right)dy=7\int y^{-3/4}dy-\int y^{1/3}dy+4\int y^{1/2}=7\left[\frac{y^{1/4}}{1/4}\right]-\left[\frac{y^{4/3}}{4/3}\right]+4\left[\frac{y^{3/2}}{3/2}\right]+c$$

$$=28y^{1/4}-\frac{3}{4}y^{4/3}+\frac{8}{3}y^{3/2}+c$$

Example 6: Evaluate $\int x^{1/3}\left(2-x^{2}\right)dx$.

Solution

$$\int x^{1/3}\left(2-x^{2}\right)dx=\int\left(2x^{1/3}-x^{7/3}\right)dx=2\int x^{1/3}dx-\int x^{7/3}dx=2\left[\frac{x^{4/3}}{4/3}\right]-\left[\frac{x^{10/3}}{10/3}\right]+c$$

$$=\frac{3}{2}x^{4/3}-\frac{3}{10}x^{10/3}+c$$

Example 7: Evaluate $\int\frac{x^{5}+2x^{2}-1}{x^{4}}dx$.

Solution

$$\int\frac{x^{5}+2x^{2}-1}{x^{4}}dx=\int\left(x+\frac{2}{x^{2}}-\frac{1}{x^{4}}\right)dx=\int xdx+2\int x^{-2}dx-\int x^{-4}dx$$

$$=\left[\frac{x^{2}}{2}\right]+2\left[\frac{x^{-1}}{-1}\right]-\left[\frac{x^{-3}}{-3}\right]+c=\frac{1}{2}x^{2}-\frac{2}{x}+\frac{1}{3x^{3}}+c$$

Methods of Integration

Sometimes, the inspection is not enough to find the integral of some functions. There are additional methods to reduce the function in the standard form to find its integral. Prominent methods are discussed below.

The methods of integration are:

- Integration by Substitution
- Integration using Partial Fractions
- Integration by Parts

Integration by Parts

This Integration rule is used to find the integral of two functions.

By product rule of derivatives, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \cdots (1)$$

Integration on both sides of equation (1), we get

$$\int \boldsymbol{u}\frac{\boldsymbol{dv}}{\boldsymbol{dx}}\boldsymbol{dx} = \boldsymbol{uv} - \int \boldsymbol{v}\frac{\boldsymbol{du}}{\boldsymbol{dx}}\boldsymbol{dx} \cdots (\mathbf{2})$$

Example

$$\int x\, e^x \ dx$$

Let $u = x$ and $dv = e^x \ dx$, then

$\int x\, e^x \ dx = x\, e^x - \int \ e^x \ dx = xe^x - e^x + c.$

Example

$$\int \ln x \; dx$$

Let $u = \ln x$, then $du = \frac{1}{x} dx$

and $dv = dx$, so $v = x$.

$\int ln\, x \;\; dx = x\; lnx - \int x\,.\frac{1}{x}\; dx = x\; lnx - x + c.$

Example. Evaluate each of the following

$\int e^x\; sinx \;\; dx, \int x^2 \cos 3x$

Integration by substitution

The method of substitution is a method for algebraically simplifying the form of a function so that its antiderivative can be easily recognized. This method is intimately related to the chain rule for differentiation. For example

$$\int \frac{x^2+1}{x^3+3x} dx$$

Let

$$u = x^3 + 3x$$

Then Go directly to the du part.

$$du = \left(3x^2+3\right)dx = 3\left(x^2+1\right)dx$$

so that

$$dx = \frac{du}{3\left(x^2+1\right)}$$

Make substitutions into the original problem, removing all forms of x , resulting in

$$\int \frac{x^2+1}{x^3+3x}dx = \int \frac{x^2+1}{u}\,\frac{du}{3\left(x^2+1\right)} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u|+c = \frac{1}{3}\ln\left|x^3+3x\right|+c\,.$$

Example 1: Integrate the function $f(x)=2x\sin(x^2+1)$ with respect to x.

Solution:

Observe that the derivative of x^2+1 is $2x$.

So, we will proceed with integration by substitution.

Let $x^2+1=z$

Then, $2x\,dx = dz$

$$\int f(x)dx = \int 2x\sin(x^2+1)dx$$

$$= \int \sin z\,dz$$

$$= -\cos z + C$$

$$= -\cos(x^2+1) + C$$

$$\therefore \int 2x\sin(x^2+1)dx = -\cos(x^2+1) + C$$

Integration using Partial Fractions

Rational Function	Form of Partial Function
$\frac{px+q}{(x-a)(x-b)}$, where $a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+c}{x^2+bx+c}$

Example: Evaluate $\int \frac{1}{x^2+3x+2} dx$.

$$\int \frac{1}{x^2 + 3x + 2} dx = \int \frac{1}{(x+2)(x+1)} dx$$

By using partial fraction we have

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \cdots (1).$$

$A(x+2) + B(x+1) = 1$

$A + B = 0 \; and \; 2A + B = 1$ implies $A = 1$ and $B = -1$.

Now $\int \left(\frac{1}{(x+1)(x+2)}\right) dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$

$$= ln|x+1| - ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C$$

Example. Evaluate $\int \frac{5x-2}{(x+3)^2}dx$.

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2} \Longrightarrow 5x-2 = A(x+3)+B$$

$$\begin{aligned} x=-3: \quad & 5(-3)-2 = A((-3)+3)+B \\ \Longrightarrow \quad & -17 = 0+B \\ \Longrightarrow \quad & -17 = B \end{aligned}$$

$$\begin{aligned} 5x-2 &= A(x+3)-17 \\ &= Ax+3A-17 \\ \Longrightarrow \quad 5x &= Ax \\ \Longrightarrow \quad 5 &= A \end{aligned}$$

$$\int \frac{5x-2}{(x+3)^2}dx = \int \frac{5}{x+3} - \frac{17}{(x+3)^2}dx = \boxed{5\ln|x+3| + \frac{17}{x+3} + C}$$

Example. Compute $\int \frac{-2x+4}{(x^2+1)(x-1)}dx$.

$$\frac{-2x+4}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} = \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}$$

$$\implies -2x+4 = (Ax+B)(x-1)+C(x^2+1).$$

$$\begin{aligned} x=1: \quad & -2(1)+4 = (Ax+B)(1-1)+C((1)^2+1) \\ \implies \quad & 2 = 0+2C \\ \implies \quad & 1 = C \end{aligned}$$

So now we have,

$$\begin{aligned} & -2x+4 = (Ax+B)(x-1)+(x^2+1) \\ \implies \quad & -x^2-2x+3 = (Ax+B)(x-1) \\ \implies \quad & -(x-1)(x+3) = (Ax+B)(x-1) \\ \implies \quad & -(x+3) = Ax+B \\ \implies \quad & -1 = A \\ & -3 = B. \end{aligned}$$

Thus our integral becomes,

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)}dx &= \int \frac{-x-3}{x^2+1} + \frac{1}{x-1}dx \\ &= -\int \frac{x}{x^2+1}dx - 3\int \frac{1}{x^2+1}dx + \int \frac{1}{x-1}dx \\ &= \boxed{-\frac{1}{2}\ln(x^2+1) - 3\tan^{-1}(x) + \ln|x-1| + C} \end{aligned}$$

Example. Compute $\int \frac{18}{x^3-3x^2}dx$.

$$\frac{18}{x^3-3x^2} = \frac{18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Example 6 Evaluate the following integral.

$$\int \frac{x^2}{x^2-1}\,dx$$

Solution

In this case the numerator and denominator have the same degree. As with the last example we'll need to do long division to get this into the correct form. I'll leave the details of that to you to check.

$$\int \frac{x^2}{x^2-1}\,dx = \int 1+\frac{1}{x^2-1}\,dx = \int dx + \int \frac{1}{x^2-1}\,dx$$

So, we'll need to partial fraction the second integral. Here's the decomposition.

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1}$$

Setting numerator equal gives,

$$1 = A(x+1)+B(x-1)$$

Picking value of x gives us the following coefficients.

$$x=-1 \qquad 1=B(-2) \qquad \Rightarrow \qquad B=-\frac{1}{2}$$

$$x=1 \qquad 1=A(2) \qquad \Rightarrow \qquad A=\frac{1}{2}$$

The integral is then,

$$\int \frac{x^2}{x^2-1}\,dx = \int dx + \int \frac{\frac{1}{2}}{x-1}-\frac{\frac{1}{2}}{x+1}\,dx$$

$$= x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+c$$

Example 4: Calculate ∫ $\cos^2 x\,dx$

Using the trigonometric identity, we have:

$$\int \cos^2 x\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)dx = \frac{x}{2}+\frac{1}{4}\sin(2x)+C$$

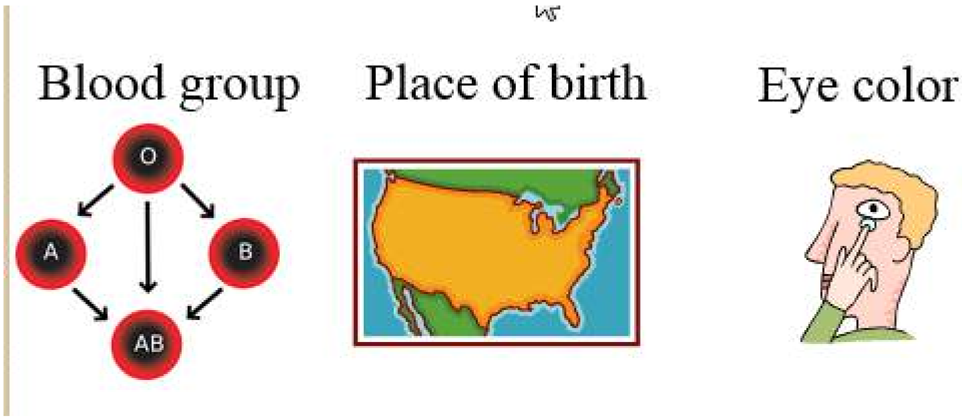
Introduction To Statistics

- **Statistics** consists of conducting studies to collect, organize, summarize and analyze data to draw conclusions
- **Data** are the values (measurements or observations) that the variables can assume.
- A collection of data values forms a **data set**.
- When data is collected in original form, they are called **raw data**.
- A **population** consists of all subjects (human or otherwise) that are being studied.
- A **sample** is a subgroup of the population.
- Descriptive and Inferential Statistics**Descriptive statistics** consists of the collection, organization, summation and presentation of data.
- **Inferential statistics** consists of generalizing from samples to populations, performing hypothesis testing, determining relationships among variables, and making predictions.

Types of Data

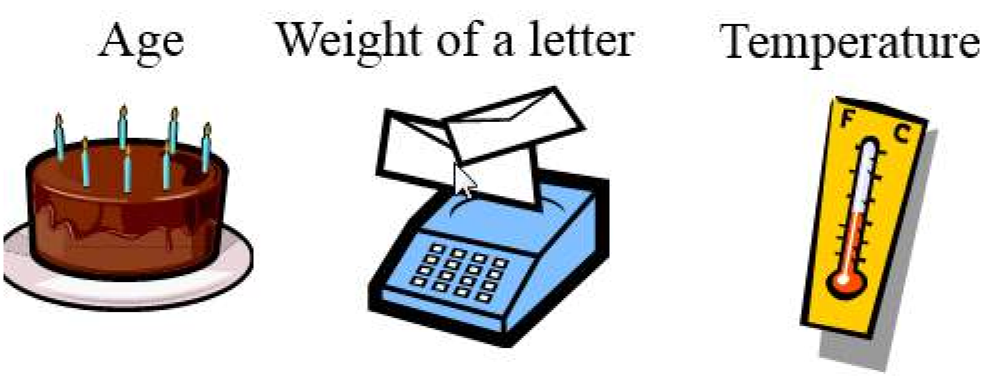
Qualitative Data

Consists of attributes, labels, or nonnumeric entries.



Quantitative data

Numerical measurements or counts.



1-2 Variables and Types of Data

- **Discrete variables** assume values that can be counted.
- **Continuous variables** can assume all values between any two specific values. They are obtained by measuring.

1.3 Level of measurement: Measurement is the assignment of numbers to objects or event according to the rules. With which is compared for measuring is called measurement scale. Variables can be measured under four levels or scales of measurement. The measurement levels are,

i. Nominal Measurement.

Examples: *a) Religion. b) Marital Status. c) Blood Group. d) Nationality*

ii. Ordinal Measurement.

Examples: *a) Economic Status. b) Level of Education. c) Beauty.*

iii. Interval Measurement.

Examples: *a) Temperature. b) I.Q. Score. c) Dates on Calendar*

iv. Ratio Measurement.

Example: *a) Height. b) Weight. c) Age. d) Income. e) Price*

Frequency Distributions

- After the data have been collected, the main tasks a statistician must accomplish are the organization and presentation of the data. The organization must be done in a meaningful way and the presentation should be such that an interested reader of the study can understand the data distribution.

Frequency Distribution

- All the data values obtained are divided into **classes** that must satisfy the following conditions:

1- there is usually between 5 and 20 classes;
2- the classes must be mutually exclusive;
3- the classes must be exhaustive;

- The **frequency** is the number of values in a specific class.

- A **frequency distribution** is the organization of raw data in table form, using classes and frequencies.

The Types of Frequency Distributions

The types of frequency distributions that are used the most are

1- The categorical frequency distribution

is used for data that can be placed in specific categories or represent values of a qualitative variable.

2- The grouped frequency distribution
is used when the data are numerical and their range is large, the data must be grouped into classes that are more than one unit in length.

Constructing a categorical frequency distribution
Example-: Construct a frequency distribution for the data below.

A	**B**	**B**	**AB**	**O**
O	**O**	**B**	**AB**	**B**
B	**B**	**O**	**A**	**O**
A	**O**	**O**	**O**	**AB**
AB	**A**	**O**	**B**	**A**

Class	Frequency (f)	Percent
A	5	20
B	7	28
O	9	36
AB	4	16
	Sum of Frequency (n) = 25	Total percent= 100

Constructing a Grouped Frequency Distribution

In this case we have additional conditions for the classes:

1- The classes must be equal in width.

2- The classes must be continuous.

The procedure for constructing a grouped frequency distribution

1. **Decide on the number of classes.**

- ◦ Usually between 5 and 20; otherwise, it may be difficult to detect any patterns.

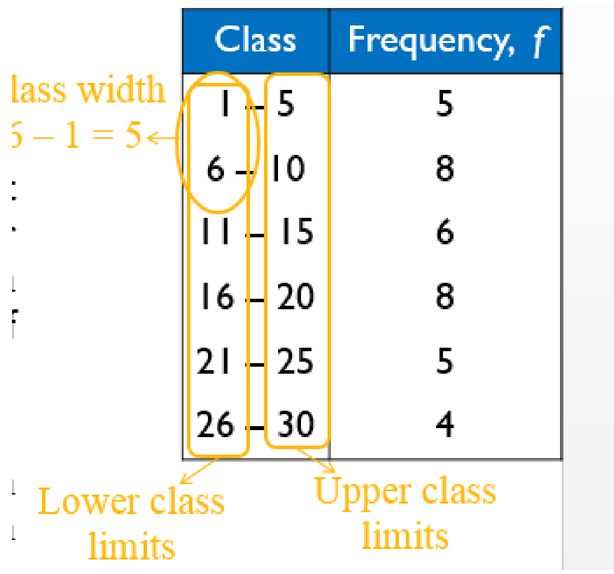
2. Find the class width.

- ◦ Determine the range of the data.
- ◦ Divide the range by the number of classes.
- ◦ *Round up to the next convenient number.*

Frequency Distribution

- **Frequency Distribution:** A table that shows **classes** or **intervals** of data with a count of the number of entries in each class.

The **frequency, *f*,** of a class is the number of data entries in the class.



3. Find the class limits.

- ◦ You can use the minimum data entry as the lower limit of the first class.
- ◦ Find the remaining lower limits (add the class width to the lower limit of the preceding class).
- ◦ Find the upper limit of the first class. Remember that classes cannot overlap.
- ◦ Find the remaining upper class limits.

4. Make a tally mark for each data entry in the row of the appropriate class.

5. Count the tally marks to find the total frequency *f* for each class.

Example: Constructing a Frequency Distribution

The following sample data set lists the number of minutes 50 Internet subscribers spent on the Internet during their most recent session. Construct a frequency distribution that has seven classes.

50 40 41 17 11 7 22 44 28 21 19 23 37 51 54 42 86

41 78 56 72 56 17 7 69 30 80 56 29 33 46 31 39 20

18 29 34 59 73 77 36 39 30 62 54 67 39 31 53 44

1. Number of classes = 7 (given)
2. Find the class width

$$\frac{\max - \min}{\#\text{classes}} = \frac{86-7}{7} \approx 11.29$$

Round up to 12

3. Use 7 (minimum value) as first lower limit. Add the class width of 12 to get the lower limit of the next class.

   7 + 12 = 19

   Find the remaining lower limits.

Class width = 12

Lower limit	Upper limit
7	
19	
31	
43	
55	
67	
79	

Lower limit	Upper limit
7	18
19	30
31	42
43	54
55	66
67	78
79	90

Class width = 12

Class	Tally	Frequency, f
7 – 18	IIII I	6
19 – 30	IIII IIII	10
31 – 42	IIII IIII III	13
43 – 54	IIII III	8
55 – 66	IIII	5
67 – 78	IIII I	6
79 – 90	II	2
		$\Sigma f = 50$

Midpoint of a class

$$\frac{(\text{Lower class limit}) + (\text{Upper class limit})}{2}$$

Class	Midpoint	Frequency, f
7 – 18	$\frac{7+18}{2} = 12.5$	6
19 – 30	$\frac{19+30}{2} = 24.5$	10
31 – 42	$\frac{31+42}{2} = 36.5$	13

Class width = 12

Relative Frequency of a class

- Portion or percentage of the data that falls in a particular class.

$$\text{relative frequency} = \frac{\text{class frequency}}{\text{Sample size}} = \frac{f}{n}$$

Class	Frequency, f	Relative Frequency
7 – 18	6	$\frac{6}{50} = 0.12$
19 – 30	10	$\frac{10}{50} = 0.20$
31 – 42	13	$\frac{13}{50} = 0.26$

Expanded Frequency Distribution

Cumulative frequency of a class

- The sum of the frequency for that class and all previous classes.

Class	Frequency, f	Cumulative frequency
7 – 18	6	6
19 – 30	+ 10	16
31 – 42	+ 13	29

Expanded Frequency Distribution

Class	Frequency, f	Midpoint	Relative frequency	Cumulative frequency
7 – 18	6	12.5	0.12	6
19 – 30	10	24.5	0.20	16
31 – 42	13	36.5	0.26	29
43 – 54	8	48.5	0.16	37
55 – 66	5	60.5	0.10	42
67 – 78	6	72.5	0.12	48
79 – 90	2	84.5	0.04	50

$\Sigma f = 50$ $\qquad$ $\sum \frac{f}{n} = 1$

Class Boundaries

Class boundaries

- The numbers that separate classes without forming gaps between them.

- The distance from the upper limit of the first class to the lower limit of the second class is 19 – 18 = 1.
- Half this distance is 0.5.

Class	Class Boundaries	Frequency, f
7 – 18	6.5 – 18.5	6
19 – 30		10
31 – 42		13

- First class lower boundary = 7 – 0.5 = 6.5
- First class upper boundary = 18 + 0.5 = 18.5

21

Class Boundaries

Class	Class boundaries	Frequency, f
7 – 18	6.5 – 18.5	6
19 – 30	18.5 – 30.5	10
31 – 42	30.5 – 42.5	13
43 – 54	42.5 – 54.5	8
55 – 66	54.5 – 66.5	5
67 – 78	66.5 – 78.5	6
79 – 90	78.5 – 90.5	2

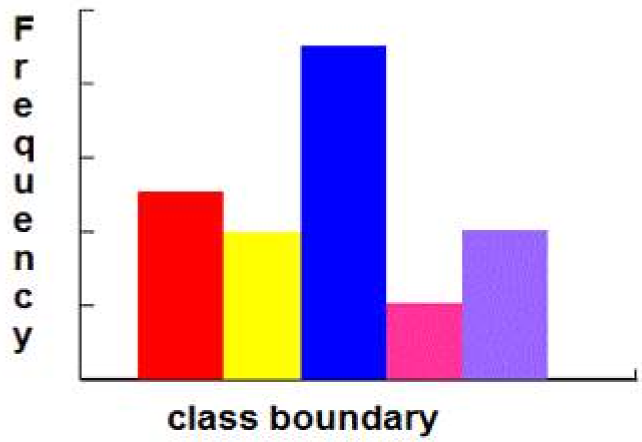
Graphs of Frequency Distributions

Most common graphs are:

1. **Histogram,**
2. **Frequency polygon,**
3. **Cumulative frequency graph or Ogive.**
4. **Pie Chart**

1- Histogram

The histogram is a graph that uses contiguous vertical bars to display the frequency of the data contained in each class. The heights of the bars equal the frequency and the bases of the bars lie on the corresponding class.



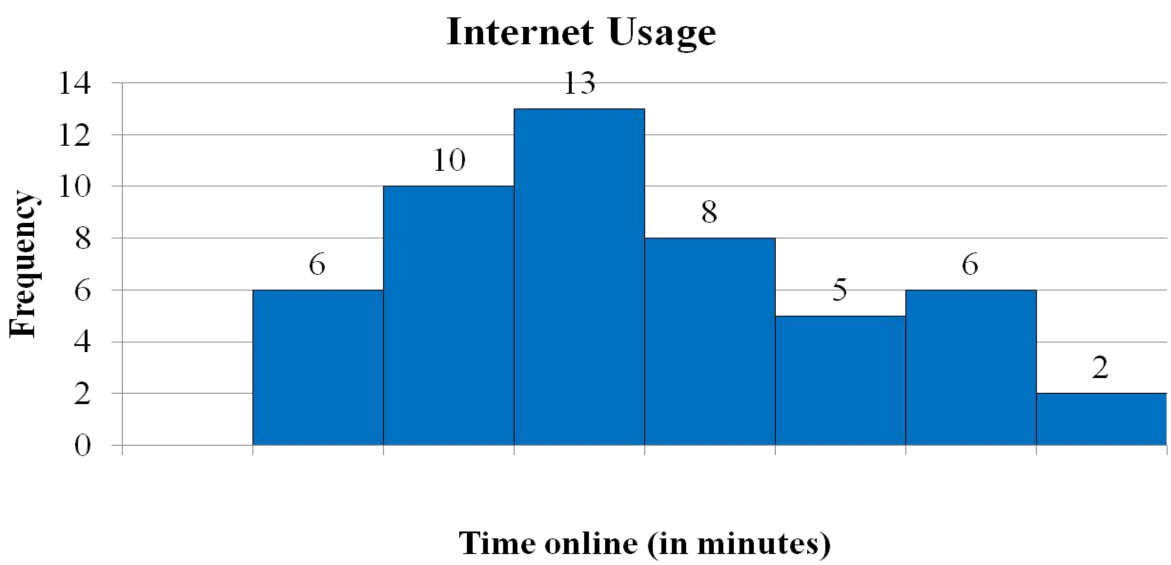
Steps for constructing a histogram:

- Draw and label the x (horizontal) and the y (vertical) axes.
- Represent the frequencies on the y axis and the class boundaries on the x axis.
- Using the frequencies as the heights draw vertical bars for each class.

Note: For the histogram we need the frequencies and the class boundaries.

Example: Construct a frequency histogram for the Internet usage frequency distribution.

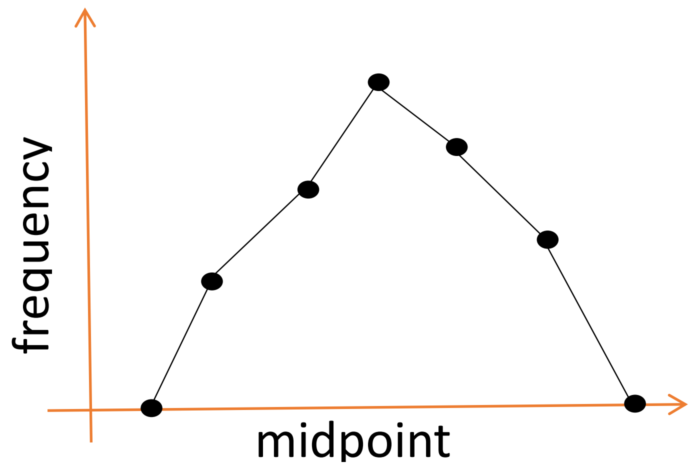
Class	Class boundaries	Midpoint	Frequency, f
7 – 18	6.5 – 18.5	12.5	6
19 – 30	18.5 – 30.5	24.5	10
31 – 42	30.5 – 42.5	36.5	13
43 – 54	42.5 – 54.5	48.5	8
55 – 66	54.5 – 66.5	60.5	5
67 – 78	66.5 – 78.5	72.5	6
79 – 90	78.5 – 90.5	84.5	2



You can see that more than half of the subscribers spent between 19 and 54 minutes on the Internet during their most recent session.

2- Frequency Polygon

A line graph that emphasizes the continuous change in frequencies.



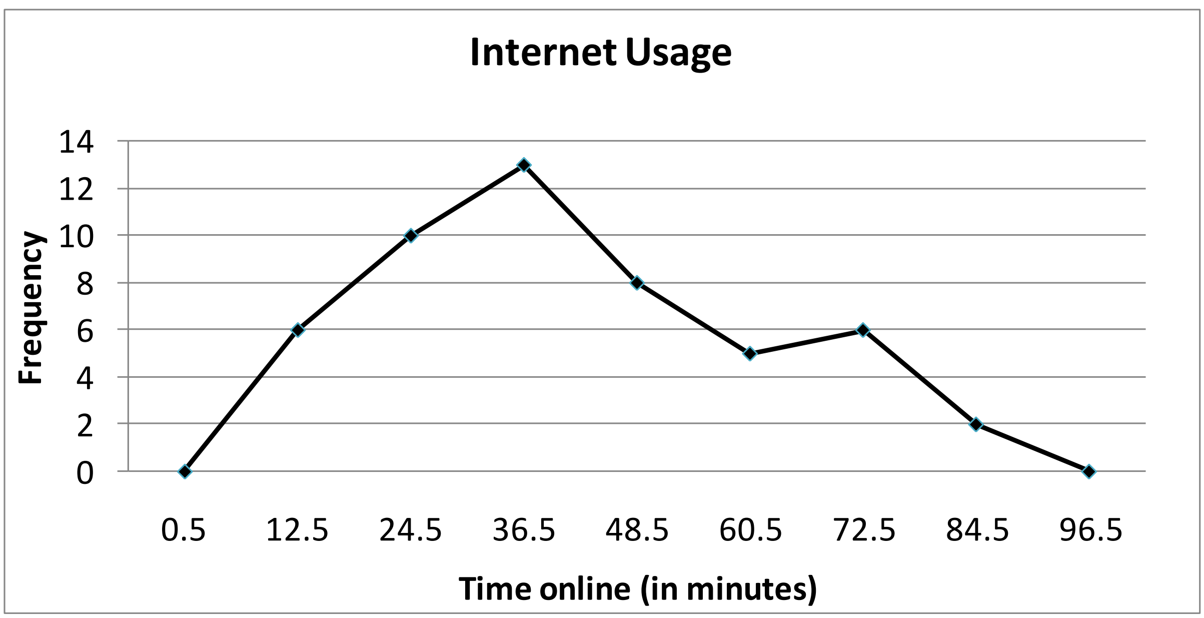
Steps for constructing a frequency polygon:

- Draw and label the x (horizontal) and the y (vertical) axes.
- Represent the frequencies on the y axis and the midpoints on the x axis.
- Plot the vertices of the polygon.
- Connect adjacent points with line segments. Draw a line back to the x axis at the beginning and the end of the graph at the same distance that the previous and the next midpoints would be located.

Note: For the frequency polygon we need the frequencies and the midpoints.

Example. Construct a frequency polygon for the Internet usage frequency distribution.

Class	Midpoint	Frequency, *f*
7 – 18	12.5	6
19 – 30	24.5	10
31 – 42	36.5	13
43 – 54	48.5	8
55 – 66	60.5	5
67 – 78	72.5	6
79 – 90	84.5	2

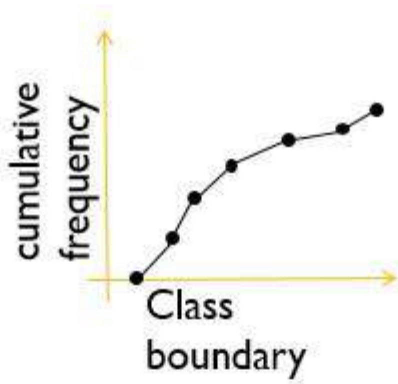


The graph should begin and end on the horizontal axis, so extend the left side to one class width before the first class midpoint and extend the right side to one class width after the last class midpoint.

You can see that the frequency of subscribers increases up to 36.5 minutes and then decreases.

3- Cumulative Frequency Graph or Ogive

An ogive is a graph that represents the cumulative frequencies for the classes in a frequency distribution. It shows how many of values of the data are below certain boundary.



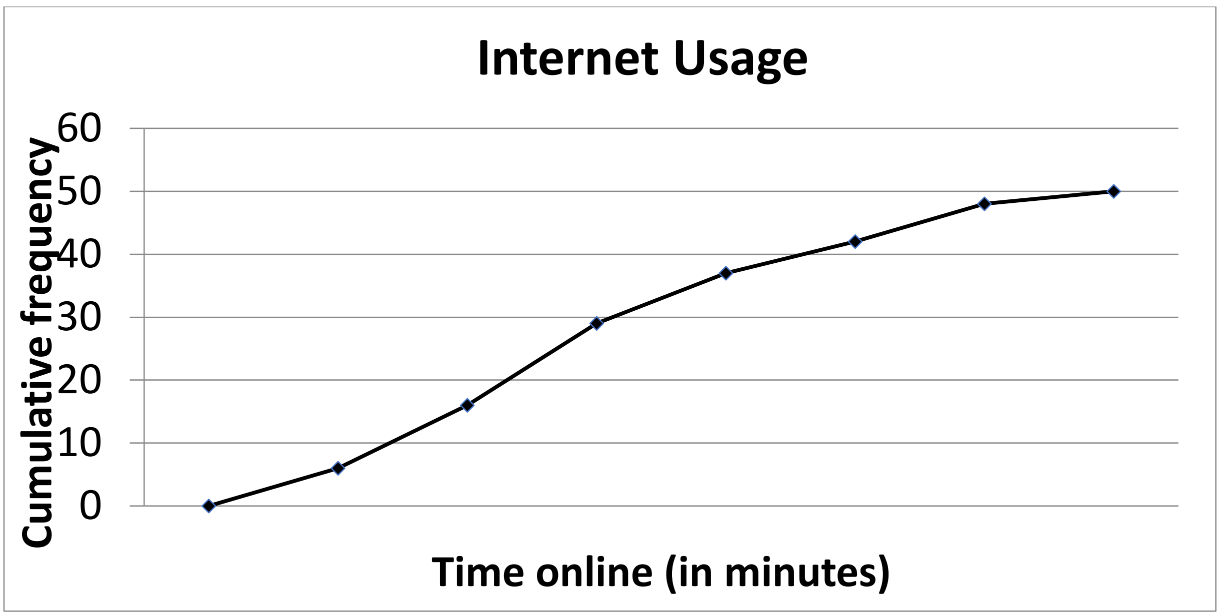
Steps for constructing an ogive:

- Draw and label the x (horizontal) and the y (vertical) axes.
- Represent the cumulative frequencies on the y axis and the class boundaries on the x axis.
- Plot the cumulative frequency at each upper class boundary with the height being the corresponding cumulative frequency.
- Connect the points with segments. Connect the first point on the left with the x axis at the level of the lowest lower class boundary.

Note: For the ogive we need the class boundaries and the cumulative frequencies

Example. Construct an ogive for the Internet usage frequency distribution.

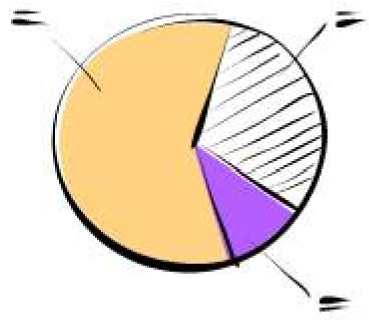
Class	Class boundaries	Frequency, f	Cumulative frequency
7 – 18	6.5 – 18.5	6	6
19 – 30	18.5 – 30.5	10	16
31 – 42	30.5 – 42.5	13	29
43 – 54	42.5 – 54.5	8	37
55 – 66	54.5 – 66.5	5	42
67 – 78	66.5 – 78.5	6	48
79 – 90	78.5 – 90.5	2	50



From the ogive, you can see that about 40 subscribers spent 60 minutes or less online during their last session. The greatest increase in usage occurs between 30.5 minutes and 42.5 minutes.

4- Pie Chart

- A circle is divided into sectors that represent categories.
- The area of each sector is proportional to the frequency of each category.



Steps for constructing a pie chart

- Convert the frequency for each class into a proportional part of the circle using the formula

$$\text{Degrees} = 360\,\frac{f}{n}$$

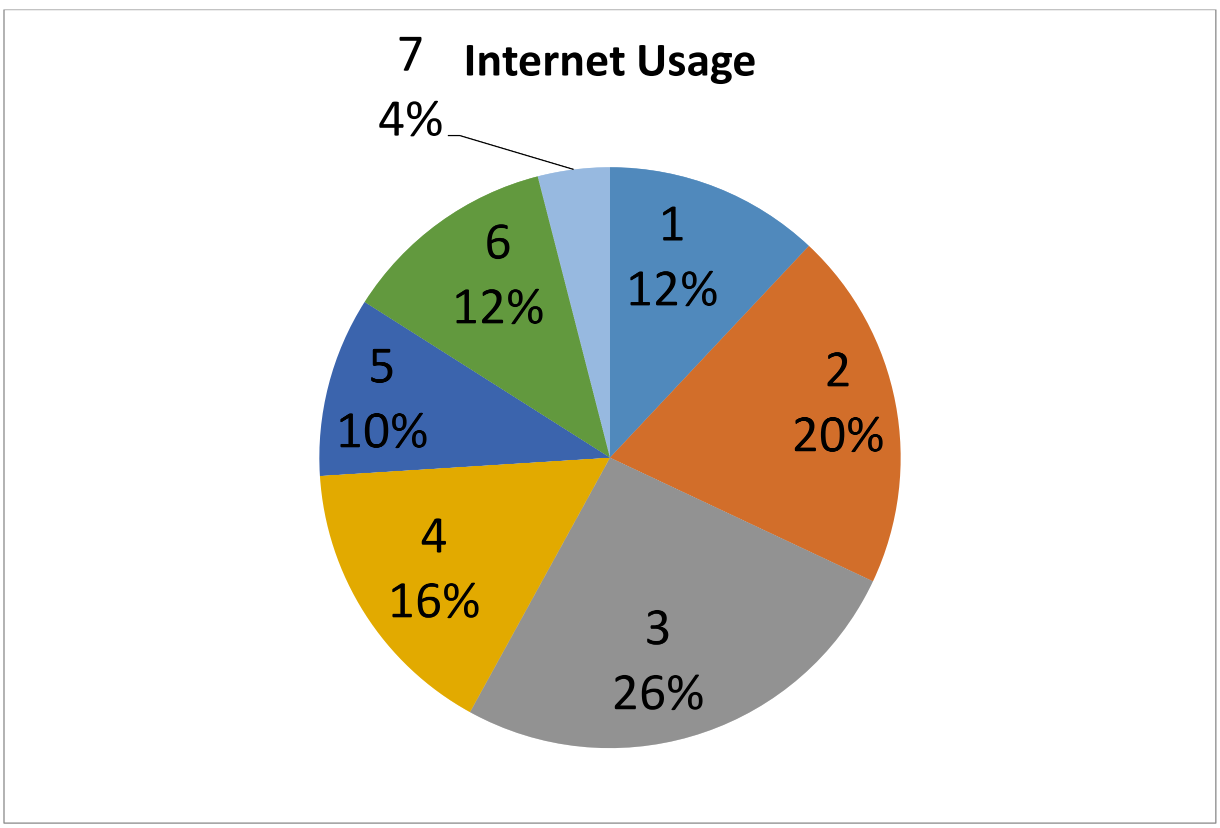
where *f* is the frequency for each class and *n* is the sum of the frequencies.

- Find the percentages corresponding to each class
- Using a protector, graph each section and write its name and corresponding percentage.

Expanded frequency distribution

Class	Frequency, f	Relative frequency	Central angle
7 – 18	6	0.12	43.2°
19 – 30	10	0.20	72°
31 – 42	13	0.26	93.6°
43 – 54	8	0.16	57.6°
55 – 66	5	0.10	36°
67 – 78	6	0.12	43.2°
79 – 90	2	0.04	14.4°

$\Sigma f = 50$ $\quad\quad$ $\Sigma \frac{f}{n} = 1$



Measures of Central Tendency

- The central tendency measure is defined as the number used to represent the center or middle of a set of data values. Most common measures of central tendency are:
    - **Mean**
    - **Median**
    - **Mode**

Mean (average)

The sum of all the data entries divided by the number of entries.

For ungrouped data the mean is:

Sample mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Population mean:

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Example:- if you have the data (4, 7, 2, 1, 7, 5, 4) find the mean.

$$\bar{x} = \frac{4+7+2+1+7+5+4}{7} = \frac{30}{7} = 4.28$$

Mean of grouped data

$$\bar{X} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

- f_i:frequency of each class
- x_i: is midpoint of the classes

In the **example** of internet usages, the sample mean

Class	Midpoint, x	Frequency, f	$(x \cdot f)$
7 – 18	12.5	6	12.5·6 = 75.0
19 – 30	24.5	10	24.5·10 = 245.0
31 – 42	36.5	13	36.5·13 = 474.5
43 – 54	48.5	8	48.5·8 = 388.0
55 – 66	60.5	5	60.5·5 = 302.5
67 – 78	72.5	6	72.5·6 = 435.0
79 – 90	84.5	2	84.5·2 = 169.0
		n = 50	$\Sigma(x \cdot f)$ = 2089.0

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4 + f_5 x_5 + f_6 x_6 + f_7 x_7}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7} = \frac{2089}{50} = 41.78.$$

Median

The **median** is the value that lies in the middle of the data when the data set is **ordered**.

Median for ungrouped data:

Is divided into two parts. If the data set has an

- **odd number of entries**: median is the middle data entry.
- **even number of entries**: median is the mean of the two middle data entries.

Finding the Median

❖ If n is odd

872 432 397 427 388 782 397

- First order the data.

388 397 397 427 432 782 872

- There are seven entries (an odd number), the median is the middle, or fourth, data entry.

The median is 427.

❖ If n is even

872 397 427 388 782 397

- First order the data.

388 397 397 427 782 872

- There are six entries (an even number), the median is the mean of the two middle entries.

$$Median = \frac{397 + 427}{2} = 412$$

The median is 412.

Median for grouped data

- find the median by examining the cumulative frequencies to locate the middle value.
- If n is the sample size, compute $n/2$.

$$\text{Median} = L_m + \left(\frac{\frac{n}{2} - \sum f_{m-1}}{f_m} \right).w$$

Where,

L_m is the lower boundary of the median class

$\sum f_{m-1}$ is the cumulative frequency before the median class

f_m is the frequency of the median class

n is the total number of items in the distribution

w is the class width

The Mode

- *The **mode** is defined to be the value that occurs most often in a data set.*
- *A data set can have more than **one mode**.*
- *A data set is said to have no mode if all values occur with equal frequency.*
- *If two entries occur with the same greatest frequency, each entry is a mode (**bimodal**).*

- *The **mode** for **ungrouped data**,*

data set: 6, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 10, 10, 11, 11, 14, 14, 14.

Mode = 8.

The mode for grouped data

Mode = $L_{mo} + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right).w$

Where,

L_{mo} *is the lower boundary of the modal class*

Δ_1 *is the difference between frequency of the modal class and the frequency before the modal class*

Δ_2 *is the difference between frequency of the modal class and the frequency after the modal class*

W *is the class width*

The **example** of internet usages, the mode is

Class	Class boundaries	Frequency, f
7 – 18	6.5 – 18.5	6
19 – 30	18.5 – 30.5	10
31 – 42	30.5 – 42.5	13
43 – 54	42.5 – 54.5	8
55 – 66	54.5 – 66.5	5
67 – 78	66.5 – 78.5	6
79 – 90	78.5 – 90.5	2

$$Mode = 30.5 + \frac{3}{3+5}(12)$$

$=35$

Measure of Dispersion or Variation

- A statistic that tells us how the data values are dispersed or spread out is called the measure of dispersion. A simple measure of dispersion is the range. The range is equivalent to the difference between the highest and least data values. Another measure of dispersion is the standard deviation, representing the expected difference (or deviation) among a data value and the mean. For the variability of a data set, three measures are commonly used: **Range, Variance and Standard deviation.**

Range

- The difference between the maximum and minimum data entries in the set.
- The data must be quantitative.

Range = Largest value - smallest value.

Example:- Find the range of the following data

41 38 39 45 47 41 44 41 37 42

Solution: **Range = 47 – 37 = 10**

Variance and Standard Deviation

Variance :- It is the average of the squared deviations from the mean.

Standard Deviation :- It is the square root of the variance

For ungrouped data,

Sample Variance and Standard Deviation

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \qquad s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

Population Variance and Standard Deviation

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N} \qquad \sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}$$

Variance and Standard Deviation

For grouped data,

Sample Variance and Standard Deviation

$$s^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n-1} \qquad s = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n-1}}$$

Where,

X_i is midpoint and f_i : is frequency

class	frequency	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
7-18	6	12.5	-29.28	857.3184	5143.91
19-30	10	24.5	-17.28	298.5984	2985.984
31-42	13	36.5	-5.28	27.8784	362.4192
43-54	8	48.5	6.72	45.1584	361.2672
55-66	5	60.5	18.72	350.4384	1752.192
67-78	6	72.5	30.72	943.7184	5662.31
79-90	2	84.5	42.72	1824.998	3649.997

$$s^2 = \frac{\sum_{i=0}^{n} f_i(x_i - \bar{x})}{n-1} = \frac{19918.08}{49} = 406.4914$$

$$s = 20.16163$$