Estimation Theory

Department of Statistics & Informative

Fourth Stage

Second Semester (2022-2023)

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** Same References as First Semester

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Chapter One

Methods of Estimation

First: Maximum Likelihood Estimation (MLE)

Let $X_1, X_2, \dots, X_n$ be a rss$n$ from a dist^n with a p.d.f. $f(x;\theta)$, the joint p.d.f. of $X_1, X_2, \dots, X_n$ denote L(θ) is called the likelihood function, and the value of $\hat{\theta}$ which maximizes the likelihood function is called Maximum Likelihood Estimator (MLE) for θ, or the m.l.e is solution of:

$$\begin{aligned} j.p.d.f.(x_1, x_2, \dots, x_n) &= f(x_1, x_2, \dots, x_n; \theta) \\ &= L(x_1, x_2, \dots, x_n; \theta) \\ &= \prod_{i=1}^{n} f(x_i; \theta) \end{aligned}$$

$$\left(\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \quad , \quad with \quad \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} < 0 \right)$$

Note: If the second derivative less than zero that were the maximum**.**

The Steps of Maximum Likelihood Estimation

1) $Find \;\; L(x_1, x_2, \dots, x_n; \theta) = L(\underline{x}; \theta) = \prod_{i=1}^{n} f(x_i; \theta).$

2) $Find \;\; \ln\left(L(\underline{x}; \theta)\right).$

3) $\dfrac{\partial \ln\left(L(\underline{x}; \theta)\right)}{\partial \theta} = 0.$

4) $Find \;\; \hat{\theta}.$

Ex: Let $X_1, X_2, \dots, X_n$ denote a random sample from Bernoulli dist^n Ber(θ), find the m.l.e for θ.

Sol:

$\because$ X ~ Ber(θ)

$f(x;\theta) = \theta^x (1-\theta)^{1-x} \qquad , \qquad x = 0, 1$

$\because X's \;\; are \; indep.$

$$L(\theta) = f(x_1, x_1, \dots, x_1;\ \theta) = \Pi f(x_i;\theta)$$

$$= \theta^{\Sigma x_i} (1-\theta)^{n-\Sigma x_i}$$

$$\ln L(\theta) = \Sigma x_i \ln(\theta) + (n - \Sigma x_i) \ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{\Sigma x_i}{\theta} - \frac{n - \Sigma x_i}{1-\theta} \qquad , \qquad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{\Sigma x_i}{\theta} - \frac{n - \Sigma x_i}{1-\theta} = 0$$

$$\frac{(1-\ \theta)\Sigma x_i -\ \theta(n-\Sigma x_i)}{\theta\text{^}(1-\theta\text{^})} = 0$$

$$\Sigma x_i - \theta\text{^}\Sigma x_i -\ n\theta\text{^} +\ \theta\text{^}\Sigma x_{i_i} = 0$$

$$\Sigma x_i -\ n\theta\text{^} = 0$$

$$\Sigma x_i =\ n\theta\text{^} \qquad\qquad \theta\text{^}_{m.l.e} = \frac{\Sigma X_i}{n} = \overline{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} = -\ \frac{\Sigma x_i}{\theta^2} - \frac{n-\Sigma x_i}{(1-\theta)^2} <\ 0$$

$\therefore\ \theta\text{^} = \overline{X}$ is m.l.e for θ .

Ex: Let $X_1, X_2, \ldots, X_n$ denote a random sample from Poisson dist^n Poi(θ), find the m.l.e for θ.

Sol:

$\because$ X ~ Poi(θ)

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad , \qquad x = 0\,,1,\ldots\ldots$$

$\because X's\ \ are\ indep.$

$$L(\theta) = f(x_1, x_1, \ldots, x_1;\ \theta) =\ \Pi\, f(x_i;\theta)$$

$$= \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod(x_i)!}$$

$$\ln L(\theta) =\ -n\ \theta + \Sigma x_i \ln(\theta) -\ \prod(x_i)!$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} =\ -n + \frac{\Sigma x_i}{\theta} \qquad\qquad , \qquad \frac{\partial \ln L(\theta)}{\partial\theta}\ =\ \ 0$$

$$-n + \frac{\Sigma x_i}{\theta\text{^}} = 0$$

$$\frac{\Sigma x_i}{\theta\text{^}} = n$$

$$\Sigma x_i =\ n\theta\text{^}$$

$$\theta\text{^}_{m.l.e} = \frac{\Sigma X_i}{n} = \overline{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} = -\ \frac{\Sigma x_i}{\theta^2}\ <\ 0$$

$\therefore\ \theta\text{^} = \overline{X}$ is m.l.e for θ .

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from normal dist[n] $N(\theta, 1)$, find the m.l.e for θ.

Sol:

$$X \sim N(\theta, 1)$$

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(x-\theta)^2} \quad , \quad -\infty < x < \infty$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i-\theta)^2}$$

$$\ln L(\theta) = n \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum(x_i-\theta)^2$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = zero - \frac{2}{2}\sum(x_i-\theta)(-1) = \sum(x_i-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\sum(x_i-\theta) = 0 \quad \Rightarrow \sum x_i - n\,\theta = 0 \quad \Rightarrow \hat{\theta} = \frac{\sum X_i}{n} = \overline{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -n < 0 \quad \Rightarrow \quad \therefore \hat{\theta} = \overline{X} \text{ is m.l.e for } \theta .$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from Binomial dist[n] $Bin(m, \theta)$, find the m.l.e for θ.

Sol:

$$X \sim Bin(m, \theta)$$

$$f(x;\theta) = C_x^m\ \theta^x (1-\theta)^{m-x} \quad , \; x = 0, 1, \ldots, m$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} C_{x_i}^m\ \theta^{\sum x_i}(1-\theta)^{\sum(m-x_i)}$$

$$\ln L(\theta) = \ln\prod_{i=1}^{n} C_{x_i}^m + \sum x_i \ln(\theta) + \sum(m-x_i)\ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = zero + \frac{\sum x_i}{\theta} + \frac{\sum(m-x_i)}{(1-\theta)} \times (-1) = \frac{\sum x_i}{\theta} - \frac{\sum(m-x_i)}{(1-\theta)}$$

$$= \frac{\sum x_i(1-\theta) - \theta\sum(m-x_i)}{\theta(1-\theta)} = \frac{\sum x_i - \theta\sum x_i - nm\theta + \theta\sum x_i}{\theta(1-\theta)} = \frac{\sum x_i - nm\theta}{\theta(1-\theta)}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{\sum x_i - nm\theta}{\theta(1-\theta)} = 0 \quad \Rightarrow \sum x_i - nm\theta = 0 \quad \Rightarrow \sum x_i = nm\theta \;\Rightarrow \hat{\theta} = \frac{\sum X_i}{nm}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-\sum x_i}{\theta^2} - \frac{\sum(m-x_i)}{(1-\theta)^2} = -\frac{\sum x_i}{\theta^2} - \frac{nm - \sum x_i}{(1-\theta)^2} < 0$$

$$\therefore \hat{\theta} \text{ is m.l.e for } \theta .$$

Remarks:

1) The m.l.e. $\hat{\theta}$ is a function of the sufficient estimator.
2) The m.l.e. $\hat{\theta}$ is not always unbiased estimator for θ.

Invariance Property of the (m.l.e)

In a rss$n$ from a dist^n with p.d.f. $f(x;\theta)$, let $\hat{\theta}$ be a m.l.e. for the parameter θ, and $u(\theta)$ be a (one-to-one) function of θ, then $u(\hat{\theta})$ is a m.l.e. for $u(\theta)$.

Ex: In a rss$n$ from exponential dist^n Exp($1/\theta$), find the m.l.e for:

1) $u_1(\theta) = \dfrac{1}{\theta}$ 2) $u_2(\theta) = \dfrac{\ln(\theta)}{\theta}$

Sol:

$X \sim Exp(1/\theta)$

$$f(x;\theta) = \theta\ e^{-\theta x}\ , \qquad x > 0$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^n e^{-\theta \sum x_i}$$

$$\ln L(\theta) = n \ln(\theta) - \theta \sum x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} - \sum x_i \qquad , \qquad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{n}{\theta} - \sum x_i = 0 \qquad \Rightarrow \frac{n}{\theta} = \sum x_i \qquad \Rightarrow \hat{\theta} = \frac{n}{\sum X_i} = \frac{1}{\overline{X}}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-n}{\theta^2} < 0 \qquad \Rightarrow \therefore \hat{\theta} \ \ is\ m.l.e \ \ for \ \ \theta .$$

1) $u_1(\hat{\theta}) = \dfrac{1}{\hat{\theta}} = \dfrac{1}{\frac{1}{\overline{x}}} = \overline{x}$ 2) $u_2(\hat{\theta}) = \dfrac{\ln(\hat{\theta})}{\hat{\theta}} = \dfrac{\ln\left(\frac{1}{\overline{x}}\right)}{\frac{1}{\overline{x}}} = \overline{x} \ln\left(\dfrac{1}{\overline{x}}\right)$

Ex: In a rss$n$ from exponential dist^n Exp(θ), find the m.l.e for θ:

Sol: X ~ Exp(θ)

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad , \qquad x > 0$$

$\because X's\ are\ indep.$

$$L(\theta) = f(x_1, x_1, \dots, x_1;\ \theta) = \Pi f(x_i;\theta)$$

$$= \frac{1}{\theta^n} e^{-\Sigma x_i/\theta}$$

$$\ln L(\theta) = -n \ln(\theta) - \frac{\Sigma x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} \qquad , \qquad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$-\frac{n}{\theta^\wedge} + \frac{\Sigma x_i}{\theta^{\wedge 2}} = 0$$

$$\frac{-n\theta^\wedge + \Sigma x_i}{\theta^{\wedge 2}} = 0$$

$$-n\theta^\wedge + \Sigma x_i = 0$$

$$\Sigma x_i = n\theta^\wedge \qquad \theta^\wedge_{m.l.e} = \frac{\Sigma X_i}{n} = \overline{X}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} \quad \} \times \theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -n + \frac{\Sigma x_i}{\theta}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\Sigma x_i}{\theta^2} < 0$$

$\therefore \theta^\wedge = \overline{X}$ is m.l.e for θ .

Ex: In a rssn from Geometric dist^n Geo(θ), with p.d.f ; $f(x;\theta) = \theta(1-\theta)^{x-1}$, $x = 1,2,\ldots..$, find the m.l.e for θ:

Sol: X ~ Geo(θ)

$$f(x;\theta) = \theta\,(1-\theta)^{x-1} \qquad , \qquad x = 1,2,\ldots..$$

$\because X's$ *are indep.*

$$L(\theta) = f(x_1, x_1, \ldots, x_1;\ \theta) = \Pi f(x_i;\theta)$$

$$= \theta^n (1-\theta)^{\Sigma x_i - n}$$

$$\ln L(\theta) = n\ln(\theta) + (\Sigma x_i - n)\ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} - \frac{(\Sigma x_i - n)}{(1-\theta)} \qquad , \qquad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{n}{\theta} - \frac{(\Sigma x_i - n)}{(1-\theta)} = 0$$

$$\frac{n(1-\theta^\wedge) - \theta^\wedge(\Sigma x_i - n)}{\theta^\wedge(1-\theta^\wedge)} = 0$$

$$n - n\theta^\wedge - \theta^\wedge \Sigma x_i + n\theta^\wedge = 0$$

$$n - \theta^\wedge \Sigma x_i = 0$$

$$\theta^\wedge \Sigma x_i = n \qquad \theta^\wedge_{m.l.e} = \frac{n}{\Sigma X_i} = \frac{1}{\overline{X}}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} - \frac{(\Sigma x_i - n)}{(1-\theta)^2} < 0$$

$\therefore \theta^\wedge = 1/\overline{X}$ is m.l.e for θ .

H.W: In a rss$n$ from Geometric dist^n Geo(θ), with p.d.f ; $f(x;\theta) = \theta(1-\theta)^x$, $x = 0,1,2,\ldots$, find the m.l.e for θ:

Ex: In a rss$n$ taken from a dist^n with p.d.f ; $f(x;\theta) = e^{-(x-\theta)}$, $\theta \le x < \infty$, find the m.l.e for θ.

$$f(x;\theta) = e^{-(x-\theta)}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = e^{-\sum(x_i-\theta)}$$

$$\ln L(\theta) = -\sum(x_i-\theta) = -\sum x_i + n\theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = zero + n \quad = n$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \qquad \Rightarrow n \ne 0 \ \ (n : sample\ size)$$

$$\theta \le x_i \quad (y_1, y_2, \ldots, y_n)$$

$$\theta \le Min(X_i) \qquad \Rightarrow \hat{\theta} = Y_1$$

Ex: Let X_1, X_2, …, X_n be a rss$n$ from normal dist^n N(θ, σ^2), **1)** find m.l.e for parameters θ and σ^2. **2)** If S^2 is m.l.e. for σ^2, then find m.l.e. for σ.

Sol: 1)

$$X \sim N(\theta, \sigma^2)$$

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \quad , \quad -\infty < x < \infty$$

$$L(\theta,\sigma^2) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

$$= (2\pi)^{-n/2}\,(\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

1) *For* θ?

$$\ln L(\theta,\sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i-\theta)^2$$

$$\frac{\partial \ln L(\theta,\sigma^2)}{\partial \theta} = zero - zero - \frac{2}{2\sigma^2}\sum(x_i-\theta)\,(-1) = \frac{\sum(x_i-\theta)}{\sigma^2} = \frac{\sum x_i - n\theta}{\sigma^2}$$

$$\frac{\partial \ln L(\theta,\sigma^2)}{\partial \theta} = 0$$

$$\frac{\sum x_i - n\theta}{\sigma^2} = 0 \quad \Rightarrow \sum x_i - n\,\theta = 0 \quad \Rightarrow \sum x_i = n\,\theta \quad \Rightarrow \hat{\theta}_{m.l.e} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial^2 \ln L(\theta,\sigma^2)}{\partial \theta^2} = \frac{-n}{\sigma^2} < 0$$

$\therefore\ \hat{\theta} = \bar{x}$ *is m.l.e for* θ .

2) *For* σ^2?

$$\frac{\partial \ln L(\theta,\sigma^2)}{\partial \sigma^2} = zero - \frac{n}{2\sigma^2} + \frac{2\sum(x_i-\theta)^2}{4\sigma^4} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\theta)^2}{2\sigma^4}$$

$$= \frac{-n\,\sigma^2 + \sum(x_i-\theta)^2}{2\sigma^4} = 0$$

$$-n\,\sigma^2 + \sum(x_i-\theta)^2 = 0 \qquad \Rightarrow\ n\,\sigma^2 = \sum(x_i-\theta)^2$$

$$\Rightarrow\ \hat{\sigma}^2 = \frac{\sum(x_i-\hat{\theta})^2}{n} = \frac{\sum(X_i-\overline{X})^2}{n} = S^2$$

$$\because \frac{\partial \ln L(\theta,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\theta)^2}{2\sigma^4} \qquad \} \times 2\sigma^2$$

$$= -n + \frac{\sum(x_i-\theta)^2}{\sigma^2}$$

$$\frac{\partial^2 \ln L(\theta,\sigma^2)}{\partial (\sigma^2)^2} = zero - \frac{\sum(x_i-\theta)^2}{\sigma^4} = -\frac{\sum(x_i-\theta)^2}{\sigma^4} < 0$$

$\therefore S^2$ *is m.l.e for* σ^2

2)

$$u(\sigma^2) = u(\hat{\sigma}^2)$$

$$u(\sigma^2) = \sqrt{\sigma^2} = \sigma$$

$$u(\hat{\sigma}^2 = S^2) = \sqrt{S^2} = \sqrt{\frac{\sum(X_i-\overline{X})^2}{n}} = S$$

$\therefore\ S$ *is m.l.e. for* σ

Second: Moments Estimation Method (MEM)

Let $X_1, X_2, \ldots, X_n$ be a rss*n* from a distn with a p.d.f. $f(x;\theta)$, the average value of the k^{th} powers of $(X_1, X_2, \ldots, X_n)$; $m_k = \frac{\sum X_i^k}{n}$ is the k^{th} sample moment , $M_k = E(X^k)$ is the k^{th} population moment about origin. The moment's method estimator is the value of the unknown parameter $\hat{\theta}$ that makes:

$$m_k = M_k$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss*n* from normal dist^n $N(\theta, \sigma^2)$, estimate the parameters θ and σ^2 using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \qquad , M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} \quad \Rightarrow M_1 = E(X) = \theta$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \theta \quad \Rightarrow \therefore \hat{\theta} = \overline{X}$$

$$m_2 = \frac{\sum X_i^2}{n} \quad \Rightarrow M_2 = E(X^2)$$

$$M_2 = E(X^2) = V(X) + (E(X))^2 = \sigma^2 + \theta^2$$

$$m_2 = M_2$$

$$\frac{\sum X_i^2}{n} = \sigma^2 + \overline{X}^2$$

$$\therefore \hat{\sigma}^2 = \frac{\sum X_i^2}{n} - \overline{X}^2$$

Ex: In a rss*n* from a dist^n with p.d.f.; $f(x;\theta) = (\theta + 1)\, x^{\theta}$, $0 < x < 1$, estimate the parameter θ using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \qquad , M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n}$$

$$M_1 = E(X)$$

$$E(X) = \int_0^1 x\, f(x;\theta)\, dx = \int_0^1 x\, (\theta + 1)\, x^{\theta}\, dx = \int_0^1 (\theta + 1)\, x^{\theta + 1}\, dx = (\theta + 1) \left. \frac{x^{\theta + 2}}{\theta + 2} \right|_0^1 = \frac{\theta + 1}{\theta + 2}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{\theta + 1}{\theta + 2} \quad \Rightarrow \overline{X} = \frac{\theta + 1}{\theta + 2} \quad \Rightarrow (\theta + 2)\, \overline{X} = \theta + 1 \quad \Rightarrow \theta\, \overline{X} + 2\, \overline{X} = \theta + 1$$

$$\Rightarrow \theta\, \overline{X} - \theta = 1 - 2\, \overline{X} \quad \Rightarrow \theta(\overline{X} - 1) = 1 - 2\, \overline{X}$$

$$\Rightarrow \therefore \hat{\theta} = \frac{1 - 2\, \overline{X}}{\overline{X} - 1}$$

Ex: Estimate the parameters of Γ(α, 1/*θ*), using moment method.

Sol:

$$When\ X \sim \Gamma(\alpha,\beta) \quad , \ E(X)=\alpha\beta \quad , V(X)=\alpha\beta^2$$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \qquad , M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} \quad \Rightarrow M_1 = E(X) = \alpha\beta = \frac{\alpha}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{\alpha}{\theta} \quad \Rightarrow \therefore\ \overline{X} = \frac{\alpha}{\theta} \quad ..... (1)$$

$$k = 2$$

$$m_2 = \frac{\sum X_i^2}{n} \quad \Rightarrow M_2 = E(X^2)$$

$$M_2 = E(X^2) = V(X) + (E(X))^2 = \frac{\alpha}{\theta^2} + \frac{\alpha^2}{\theta^2} \quad ...... (2)$$

$$m_2 = M_2$$

$$\frac{\sum X_i^2}{n} = \frac{\alpha}{\theta^2} + \frac{\alpha^2}{\theta^2} \qquad , \ put\ (1)\ in\ (2)$$

$$\frac{\sum X_i^2}{n} = \frac{\overline{X}}{\theta} + \overline{X}^2 \quad \Rightarrow \quad \frac{\sum X_i^2}{n} - \overline{X}^2 = \frac{\overline{X}}{\theta} \quad \Rightarrow \quad S^2 = \frac{\overline{X}}{\theta} \quad \Rightarrow \quad \hat{\theta} = \frac{\overline{X}}{S^2} \quad ..... (3)$$

$$From\ (1)$$

$$\overline{X} = \frac{\alpha}{\theta} \quad \Rightarrow \alpha = \theta\overline{X} \qquad put\ (3)\ in\ (1)$$

$$\alpha = \frac{\overline{X}}{S^2}\overline{X} = \frac{\overline{X}^2}{S^2}$$

$$\Rightarrow \therefore\ \hat{\alpha}_{moment} = \frac{\overline{X}^2}{S^2} \qquad , \qquad \hat{\theta}_{moment} = \frac{\overline{X}}{S^2}$$

Ex: Estimate the parameter by using moment method for:

1) Ber(*θ*). **2)** Exp(1/*θ*). **3)** Geo(*θ*).

Sol:

$$1) \quad X \sim Ber(\theta) \quad , \quad E(X) = \theta$$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \qquad , \ M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \overline{X} \quad \Rightarrow M_1 = E(X) = \theta$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \theta \quad \Rightarrow \therefore\ \overline{X} = \theta \qquad , \quad \therefore\ \hat{\theta}_{moment} = \overline{X}$$

2) $X \sim Exp(1/\theta)$, $E(X) = 1/\theta$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \quad , \; M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \overline{X} \;\Rightarrow\; M_1 = E(X) = \frac{1}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{1}{\theta} \;\Rightarrow \therefore\; \overline{X} = \frac{1}{\theta} \quad , \;\therefore\; \hat{\theta}_{moment} = \frac{1}{\overline{X}}$$

3) $X \sim Geo(\theta)$, $E(X) = \frac{1-\theta}{\theta}$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \quad , \; M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \overline{X} \;\Rightarrow\; M_1 = E(X) = \frac{1-\theta}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{1-\theta}{\theta}$$

$$\Rightarrow \therefore\; \overline{X} = \frac{1-\theta}{\theta} \;\Rightarrow\; \overline{X}\,\theta = 1-\theta \;\Rightarrow\; \overline{X}\,\theta + \theta = 1 \;\Rightarrow\; \theta(\overline{X}+1) = 1 \Rightarrow \;\therefore\; \hat{\theta}_{moment} = \frac{1}{\overline{X}+1}$$

Ex: Find an estimate the parameter θ from; $f(x;\theta) = \theta\, x^{\theta-1}$, $0 < x < 1$, by using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \quad , \; M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \overline{X}$$

$$M_1 = E(X) = \int_0^1 x\, f(x;\theta)\, dx = \int_0^1 x\,\theta\, x^{\theta-1}\, dx = \int_0^1 \theta\, x^{\theta}\, dx = \theta \left. \frac{x^{\theta+1}}{\theta+1} \right|_0^1 = \frac{\theta}{\theta+1}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{\theta}{\theta+1} \;\Rightarrow \therefore\; \overline{X} = \frac{\theta}{\theta+1} \;\Rightarrow\; (\theta+1)\overline{X} = \theta \;\Rightarrow\; \theta\overline{X} + \overline{X} = \theta \;\Rightarrow\; \theta\overline{X} - \theta = \overline{X}$$

$$\Rightarrow \theta(\overline{X} - 1) = \overline{X}$$

$$\therefore\; \hat{\theta}_{moment} = \frac{\overline{X}}{\overline{X}-1}$$

Third: Minimum Variance Method (MVM)

Let L(θ) be the likelihood function of a rssn with p.d.f. $f(x;\theta)$, then the parameter θ has minimum variance unbiased estimator (m.v.u.e.) if it is possible to express $\left(\frac{\partial}{\partial\theta}\ln L(\theta)\right)$ in the following form;

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\hat{\theta}-\theta}{V(\hat{\theta})}$$

Where; $\hat{\theta}$: is (m.v.e.). , V($\hat{\theta}$): is variance of $\hat{\theta}$.

Ex: In a rssn, find m.v.e. for the parameters of; **1)** Ber(θ). **2)** N(θ, σ^2).

Sol:

1) $X \sim Ber(\theta)$

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \quad , x = 0,1$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\ln L(\theta) = \sum x_i \ln(\theta) + (n - \sum x_i)\ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{(1-\theta)} = \frac{(1-\theta)\sum x_i - \theta(n-\sum x_i)}{\theta(1-\theta)} = \frac{\sum x_i - \theta\sum x_i - n\theta + \theta\sum x_i}{\theta(1-\theta)}$$

$$= \frac{\sum x_i - n\theta}{\theta(1-\theta)} \qquad (\div n)$$

$$= \frac{\bar{x}-\theta}{\frac{\theta(1-\theta)}{n}} = \frac{\hat{\theta}-\theta}{V(\hat{\theta})} \qquad , \quad \hat{\theta} = \bar{X} \quad , \quad V(\hat{\theta}) = V(\bar{X}) = \frac{V(X)}{n} = \frac{\theta(1-\theta)}{n}$$

$\therefore\ \bar{X}$ *is m.v.e. for* θ.

2) $X \sim N(\theta, \sigma^2)$

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \quad , \quad -\infty < x < \infty$$

$$L(\theta,\sigma^2) = \prod_{i=1}^{n} f(x_i;\theta) = (2\pi\sigma^2)^{-\frac{n}{2}}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

$$\ln L(\theta,\sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i-\theta)^2$$

$$\frac{\partial \ln L(\theta,\sigma^2)}{\partial\theta} = zero - \frac{2}{2\sigma^2}\sum(x_i-\theta)(-1) = \frac{\sum(x_i-\theta)}{\sigma^2} = \frac{\sum x_i - n\theta}{\sigma^2} \qquad \}\div n$$

$$\Rightarrow \frac{\bar{x}-\theta}{\sigma^2/n} = \frac{\hat{\theta}-\theta}{V(\hat{\theta})} \qquad \Rightarrow \quad \hat{\theta} = \bar{X} \quad \Rightarrow V(\hat{\theta}) = \frac{\sigma^2}{n}$$

$$\frac{\partial \ln L(\theta,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{2\pi}{2\pi\sigma^2} + \frac{2\sum(x_i-\theta)^2}{4\sigma^4} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\theta)^2}{2\sigma^4}$$

$$= \frac{-n\sigma^2 + \sum(x_i-\theta)^2}{2\sigma^4} \quad \} \div n$$

$$= \frac{\frac{\sum(x_i-\theta)^2}{n} - \sigma^2}{\frac{2\sigma^4}{n}} = \frac{S^2-\sigma^2}{V(S^2)} = \frac{\hat{\sigma}^2-\sigma^2}{V(\hat{\sigma}^2)}$$

$$\because \hat{\theta} = \overline{X} \quad , \quad S^2 = \frac{\sum(X_i-\overline{X})^2}{n} \quad , \quad V(S^2) = \frac{2\sigma^4}{n}$$

Fourth: Bayesian Estimation Method (BEM)

Philosophy: Observed data X is fixed, and the unknown generating parameter θ is random. (Certainty about θ depends on both empirical information X and prior knowledge about θ).
In Bayesian estimation method the parameters treats as a random variable with prior probability p(θ), or we have prior informative about the parameter θ.

Let A and B be two events, then the conditional probability of A given B is;

$$p(A \mid B) = \frac{p(A\,B)}{p(B)} = \frac{p(B \mid A)\; p(A)}{p(B)}$$

Let; A = θ and B = x, then in a rssn with p.d.f. $f(x;\theta)$ and prior probability p(θ);

$$p(\theta \mid x) = \frac{p(x \mid \theta)\; p(\theta)}{p(x)}$$

p(x) does not contain θ, we can write it as;

$$\begin{aligned} p(\theta \mid x) &\propto p(x \mid \theta)\; p(\theta) \\ &\propto L(\theta)\; p(\theta) \end{aligned}$$

Where;
p(θ | x): is called posterior probability and Bayes estimator denote $\hat{\theta}_{Bayes}$ is the mean of posterior probability E(θ | X).
L(θ): is likelihood function.
p(θ): is prior probability.

We have two types of prior probability:
1) Non Informative prior probability (Jeffery's rule).
2) Informative prior probability.

First: Non Informative prior probability (Jeffery's rule)

It is proportional to the square root of Fisher information;

$$p(\theta) \propto \left(I_S(\theta)\right)^{1/2} \qquad , \; I_S = F.I.$$

Ex: Find Bayes estimator for parameter of; **1)** Ber(θ). **2)** Poisson(θ), using non informative prior probability.

Sol:

1) $X \sim Ber(\theta)$

$$f(x;\theta) = \theta^{x}(1-\theta)^{1-x}$$

$$p(\theta \mid x) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = \theta^{\sum x_i}\ (1-\theta)^{n-\sum x_i}$$

$$p(\theta) \propto \left(I_s(\theta)\right)^{1/2} \qquad , \ F.I = -E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)$$

$$\ln f(x;\theta) = x \ln(\theta) + (1-x)\ln(1-\theta)$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{-x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$F.I. - E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{E(X)}{\theta^2} + \frac{E(1-X)}{(1-\theta)^2}$$

$$= \frac{E(X)}{\theta^2} + \frac{1-E(X)}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{(1-\theta)} = \frac{1}{\theta(1-\theta)}$$

$$p(\theta) \propto \left(\frac{1}{\theta(1-\theta)}\right)^{1/2}$$

$$\propto \theta^{-1/2}\ (1-\theta)^{-1/2}$$

$$p(\theta \mid x) \propto L(\theta)\ p(\theta)$$

$$\propto \theta^{\sum x_i}\ e^{n-\sum x_i}\ \theta^{-1/2}\ (1-\theta)^{-1/2}$$

$$\propto \theta^{\sum x_i - \frac{1}{2}}\ (1-\theta)^{n-\sum x_i - \frac{1}{2}}$$

$$\alpha - 1 = \sum x_i - \frac{1}{2} \qquad \Rightarrow \alpha = \sum x_i + \frac{1}{2}$$

$$\beta - 1 = n - \sum x_i - \frac{1}{2} \qquad \Rightarrow \beta = n - \sum x_i + \frac{1}{2}$$

$$p(\theta \mid x) \sim Beta(\alpha = \sum x_i + \frac{1}{2},\ \beta = n - \sum x_i + \frac{1}{2})$$

When; $X \sim Beta(\alpha, \beta)$,

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\ x^{\alpha-1}(1-x)^{\beta-1} \quad , \ E(X) = \frac{\alpha}{\alpha+\beta}$$

then the complete p.d.f. of the posterior probability is;

$$p(\theta \mid x) = \frac{\Gamma(n+1)}{\Gamma\left(\sum x_i + \frac{1}{2}\right)\Gamma\left(n - \sum x_i + \frac{1}{2}\right)}\ \theta^{\sum x_i - \frac{1}{2}}\ (1-\theta)^{n - \sum x_i - \frac{1}{2}}$$

$$\therefore E(\theta \mid X_1,\dots, X_n) = \hat{\theta}_{Bayes} = \frac{\alpha}{\alpha + \beta}$$

$$= \frac{\sum X_i + \frac{1}{2}}{n+1} = \frac{\sum X_i}{n+1} + \frac{1}{2n+2}$$

2) $X \sim Poi(\theta)$

$$f(x;\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \quad , x = 0,1,\dots\dots$$

$$\ln\ f(x;\theta) = -\,\theta + x\ln(\theta) - \ln(x!)$$

$$\frac{\partial \ln\ f(x;\theta)}{\partial\ \theta} = -1 + \frac{x}{\theta}$$

$$\frac{\partial^2 \ln\ f(x;\theta)}{\partial\ \theta^2} = -\ \frac{x}{\theta^2}$$

$$-E\left(\frac{\partial^2 \ln\ f(x;\theta)}{\partial\ \theta^2}\right) = \frac{E(X)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$p(\theta) \propto \left(I_s(\theta)\right)^{1/2}$$

$$\propto \left(\frac{1}{\theta}\right)^{1/2} = \theta^{-1/2}$$

$$L(\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{(\prod_{i=1}^{n} x_i)!}$$

$$L(\theta) \propto e^{-n\theta}\theta^{\sum x_i}$$

$$p(\theta \mid x_1, x_2, \dots, x_n) \propto L(\theta)\ p(\theta)$$

$$\propto e^{-n\theta}\theta^{\sum x_i}\ \theta^{-1/2}$$

$$\propto e^{-n\theta}\theta^{\sum x_i - \frac{1}{2}}$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \sim \Gamma(\alpha = \sum x_i + \tfrac{1}{2}, \beta = n)$$

$$when\ X \sim \Gamma(\alpha, \beta), \quad f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad , \quad E(X) = \frac{\alpha}{\beta}$$

$$\alpha - 1 = \sum x_i - \tfrac{1}{2} \quad \Rightarrow \alpha = \sum x_i + \tfrac{1}{2}$$

$$\beta = n$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) = \frac{n^{\sum x_i + \frac{1}{2}}}{\Gamma\left(\sum x_i + \frac{1}{2}\right)} \theta^{\sum x_i - \frac{1}{2}} e^{-n\theta}$$

$$\therefore \hat{\theta}_{Bayes} = E(\theta \mid x_1, x_2, \ldots, x_n) = \frac{\sum X_i + \frac{1}{2}}{n} = \overline{X} + \frac{1}{2n}$$

Ex: Find Bayes estimator for parameters of; **1)** Exp($1/\theta$), **2)** N(θ, σ^2) , using non informative prior probability.

Sol:

1) $X \sim Exp(1/\theta)$

$$f(x;\theta) = \theta\ e^{-\theta x}, \quad x > 0$$

$$p(\theta \mid x) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = \theta^n\ e^{-\theta \sum x_i}$$

$$p(\theta) \propto (I_s(\theta))^{1/2} \quad , \quad F.I = -E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)$$

$$\ln f(x;\theta) = \ln(\theta) - \theta x$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = \frac{1}{\theta} - x$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{-1}{\theta^2}$$

$$F.I = -E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{1}{\theta^2}$$

$$p(\theta) \propto \left(\frac{1}{\theta^2}\right)^{1/2}$$

$$\propto \theta^{-1}$$

$$p(\theta \mid x) \propto L(\theta)\ p(\theta)$$
$$\propto \theta^{n}\ e^{-\theta \sum x_i}\ \theta^{-1}$$
$$\propto \theta^{n-1}\ e^{-\theta \sum x_i}$$
$$p(\theta \mid x) \sim \Gamma(\alpha = n\ ,\ \beta = \sum x_i)$$
$$when;\ f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\ x^{\alpha-1}\ e^{-\beta x} \Rightarrow E(X) = \frac{\alpha}{\beta}$$

then the complete p.d.f. of the posterior probability is;

$$p(\theta \mid x) = \frac{\left(\sum x_i\right)^{n}}{\Gamma(n)}\ \theta^{n-1}\ e^{-\theta \sum x_i}$$
$$\therefore E(\theta \mid X_1,....,X_n) = \hat{\theta}_{Bayes} = \frac{\alpha}{\beta} = \frac{n}{\sum X_i} = \frac{1}{\overline{\overline{X}}}$$

1) $X \sim N(\theta,\ \sigma^2)$

$$f(x;\theta,\ \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$
$$L(\theta,\sigma^2) = (2\pi\sigma^2)^{-n/2}\ e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

1) $For\ \theta$

$$p(\theta \mid x_1, x_2, \dots, x_n) \propto L(\theta)\ p(\theta)$$
$$L(\theta) \propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$
$$\propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta+\bar{x}-\bar{x})^2} \quad \} \quad \mp \bar{x}$$
$$\propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\bar{x})^2}\ e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}$$
$$\therefore L(\theta) \propto e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}$$
$$\ln f(x;\theta,\sigma^2) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}$$
$$\frac{\partial \ln f(x;\theta,\sigma^2)}{\partial\theta} = zero - \frac{2(x-\theta)(-1)}{2\sigma^2} = \frac{(x-\theta)}{\sigma^2}$$
$$\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial\theta^2} = -\frac{1}{\sigma^2}$$

$$- E\left(\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial\theta^2}\right) = \frac{1}{\sigma^2} = F.I.$$

$$p(\theta) \propto (I_s(\theta))^{1/2}$$
$$\propto \left(\frac{1}{\sigma^2}\right)^{1/2}$$
$$\propto Cons\tan t$$
$$\propto 1$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto L(\theta)\; p(\theta)$$
$$\propto e^{-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2} \times (1)$$
$$\propto e^{-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2}$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \sim N(\overline{X}, \frac{\sigma^2}{n})$$

$$\because \; mean = \overline{X} \quad , \quad \text{var}iance = \frac{\sigma^2}{n}$$

$$\therefore \; p.d.f. \Rightarrow \; p(\theta \mid x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-\frac{n}{2\sigma^2}(\theta - \bar{x})^2}$$

$$\therefore \hat{\theta}_{Bayes} = E\left(p(\theta \mid x_1, x_2, \ldots, x_n)\right) = \overline{X}$$

2) $For \;\sigma^2$

$$p(\sigma^2 \mid x_1, x_2, \ldots, x_n) \propto L(\sigma^2)\; p(\sigma^2)$$

$$L(\sigma^2) \propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

$$p(\sigma^2) \propto \left(I_s(\sigma^2)\right)^{1/2}$$

$$\ln f(x, \sigma^2) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(x;\sigma^2)}{\partial\sigma^2} = -\frac{1}{2}\,\frac{2\pi}{2\pi\sigma^2} + \frac{(x_i - \theta)^2\;(2)}{4\sigma^4} = -\frac{1}{2\sigma^2} + \frac{(x_i-\theta)^2}{2\sigma^4}$$

$$\frac{\partial^2 \ln f(x;\;\sigma^2)}{\partial(\sigma^2)^2} = \frac{2}{4(\sigma^2)^2} - \frac{(x_i-\theta)^2\;(4\sigma^2)}{4(\sigma^2)^4} = \frac{1}{2(\sigma^2)^2} - \frac{(x_i-\theta)^2}{(\sigma^2)^3}$$

$$-E\left(\frac{\partial^2 \ln f(x;\ \sigma^2)}{\partial(\sigma^2)^2}\right) = \frac{-1}{2(\sigma^2)^2} + \frac{E(x_i-\theta)^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{\sigma^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}$$

$$= \frac{-1+2}{2(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} = F.I.$$

$$p(\sigma^2) \propto \left(I_s(\theta)\right)^{1/2}$$

$$\propto \left(\frac{1}{(\sigma^2)^2}\right)^{1/2}$$

$$\propto (\sigma^2)^{-1}$$

$$p(\sigma^2 \mid x_1, x_2, ..., x_n) \propto L(\sigma^2)\ p(\sigma^2)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2} \times (\sigma^2)^{-1}$$

$$\propto (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

$$p(\sigma^2 \mid x_1, x_2, ..., x_n) \sim \Gamma^{-1}\left(\alpha = \frac{n}{2},\ \beta = \frac{\sum(x_i-\theta)^2}{2}\right)$$

$$\text{when } X \sim \Gamma^{-1}(\alpha, \beta),\quad f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x},\quad E(X) = \frac{\beta}{\alpha-1}$$

$$\therefore\ p.d.f. \Rightarrow\ p(\sigma^2 \mid x_1, x_2, ..., x_n) = \frac{\left(\frac{\sum(x_i-\theta)^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\left(\frac{\sum(x_i-\theta)^2}{2\sigma^2}\right)}$$

$$\therefore \hat{\sigma}^2_{Bayes} = E\left(\sigma^2 \mid x_1, x_2, ..., x_n\right) = \frac{\frac{\sum(X_i-\theta)^2}{2}}{\frac{n}{2}-1} = \frac{\sum(X_i-\theta)^2}{n-2}$$

Second: Informative prior probability

The form of prior probability for parameters of some dist^n as follows:

ID	Probability Distribution	Informative Prior Probability
1	Bernoulli ~ Ber(θ)	Beta ~ Beta(α_o, β_o)
2	Binomial ~ Bin(n ,θ)	Beta ~ Beta(α_o, β_o)
3	Geometric ~ Geo(θ)	Beta ~ Beta(α_o, β_o)
4	Poisson ~ Poi(θ)	Gamma ~ $\Gamma(\alpha_o, \beta_o)$
5	Exponential ~ Exp(1/θ)	Gamma ~ $\Gamma(\alpha_o, \beta_o)$
6	Exponential ~ Exp(θ)	Inverse Gamma ~ $\Gamma^{-1}(\alpha_o, \beta_o)$
7	Normal ~ N(θ, σ^2) (θ known)	Inverse Gamma ~ $\Gamma^{-1}(\alpha_o/2, \beta_o/2)$
8	Normal ~ N(θ, σ^2) (σ^2 known)	Normal ~ N(θ_o, σ_o^2)

Ex: Estimate the parameters of; **1)** Geo(θ). **2)** Poisson (θ). **3)** Exp(θ). **4)** N(θ, σ^2) (θ known) and (σ^2 known)., using Bayesian informative prior probability.

Sol:

1) $X \sim Geo(\theta)$

$$f(x;\theta) = \theta(1-\theta)^x$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = \theta^n (1-\theta)^{\sum x_i}$$

$$p(\theta) \sim Beta(\alpha_o\ , \beta_o)$$

$$p(\theta) = \frac{\Gamma(\alpha_o + \beta_o)}{\Gamma(\alpha_o)\ \Gamma(\beta_o)}\ \theta^{\alpha_o - 1}(1-\theta)^{\beta_o - 1}$$

$$\propto\ \theta^{\alpha_o - 1}(1-\theta)^{\beta_o - 1}$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto \theta^n (1-\theta)^{\sum x_i}\ \theta^{\alpha_o - 1}(1-\theta)^{\beta_o - 1}$$

$$\propto \theta^{(\alpha_o + n) - 1}(1-\theta)^{(\sum x_i + \beta_o) - 1}$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \sim Beta(\alpha = \alpha_o + n\ , \beta = \sum x_i + \beta_o)$$

The complete p.d.f. of the posterior probability is;

$$\therefore\ p(\theta \mid x_1, x_2, \ldots, x_n) = \frac{\Gamma\left(\alpha_o + n + \sum x_i + \beta_o\right)}{\Gamma\left(\alpha_o + n\right)\Gamma\left(\sum x_i + \beta_o\right)}\ \theta^{(\alpha_o + n) - 1}(1-\theta)^{(\sum x_i + \beta_o) - 1}$$

$$\therefore\ E(\theta \mid X) = \hat{\theta}_{Bayes} = \frac{\alpha}{\alpha + \beta} = \frac{\alpha_o + n}{\alpha_o + n + \sum X_i + \beta_o}$$

2) $X \sim Poi(\theta)$

$$f(x;\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \quad , x = 0,1,.......$$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{(\prod_{i=1}^{n} x_i)!}$$

$$\propto e^{-n\theta}\theta^{\sum x_i}$$

$$p(\theta) \sim Gamma(\alpha_o, \beta_o)$$

$$p(\theta) = \frac{\beta_o^{\alpha_o}}{\Gamma(\alpha_o)} \theta^{\alpha_o - 1}\ e^{-\beta_o \theta}$$

$$\propto \theta^{\alpha_o - 1}\ e^{-\beta_o \theta}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto e^{-n\theta}\theta^{\sum x_i}\ \theta^{\alpha_o - 1}\ e^{-\beta_o \theta}$$

$$\propto \theta^{(\sum x_i + \alpha_o) - 1}\ e^{-(\beta_o + n)\theta}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \sim \Gamma(\alpha = \sum x_i + \alpha_o\ , \beta = \beta_o + n)$$

The complete p.d.f. of the posterior probability is;

$$p(\theta \mid x_1, x_2, ..., x_n) = \frac{(\beta_o + n)^{\sum x_i + \alpha_o}}{\Gamma(\sum x_i + \alpha_o)} \theta^{\sum x_i + \alpha_o - 1}\ e^{-\theta(\beta_o + n)}$$

$$\therefore E(\theta \mid X) = \hat{\theta}_{Bayes} = \frac{\alpha}{\beta} = \frac{\sum X_i + \alpha_o}{\beta_o + n}$$

3) $X \sim Exp(\theta)$

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = \theta^{-n}\theta^{\sum x_i / \theta}$$

$$p(\theta) \sim \Gamma^{-1}(\alpha_o, \beta_o)$$

$$p(\theta) = \frac{\beta_o^{\alpha_o}}{\Gamma(\alpha_o)} \theta^{-(\alpha_o + 1)}\ e^{-\beta_o / \theta}$$

$$\propto \theta^{-(\alpha_o + 1)}\ e^{-\beta_o / \theta}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto \theta^{-n} e^{-\sum x_i/\theta} \quad \theta^{-(\alpha_o+1)} \ e^{-\beta_o/\theta}$$
$$\propto \theta^{-((n+\alpha_o)+1)} \ e^{-(\sum x_i + \beta_o)/\theta}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \sim \Gamma^{-1}(\alpha = n + \alpha_o \ , \beta = \sum x_i + \beta_o)$$

The complete p.d.f. of the posterior probability is;

$$p(\theta \mid x_1, x_2, ..., x_n) = \frac{(\sum x_i + \beta_o)^{n+\alpha_o}}{\Gamma(n+\alpha_o)} \theta^{-((n+\alpha_o)+1)} \ e^{-(\sum x_i + \beta_o)/\theta}$$

$$\therefore E(\theta \mid X_1, X_2, ..., X_n) = \hat{\theta}_{Bayes} = \frac{\beta}{\alpha - 1} = \frac{\sum X_i + \beta_o}{n + \alpha_o - 1}$$

4) $X \sim N(\theta, \sigma^2)$ $\quad$ *, When (θ known)*

$$f(x;\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \ e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$p(\sigma^2 \mid x_1, x_2, ..., x_n) \propto L(\sigma^2) \ p(\sigma^2)$$

$$L(\sigma^2) = (2\pi\sigma^2)^{-n/2} \ e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$$

$$\propto (\sigma^2)^{-n/2} \ e^{-\frac{n S^2}{2\sigma^2}}$$

$$p(\sigma^2) \sim \Gamma^{-1}(\frac{\alpha_o}{2}, \frac{\beta_o}{2})$$

$$p(\sigma^2) = \frac{\left(\frac{\beta_o}{2}\right)^{\alpha_o/2}}{\Gamma\left(\frac{\alpha_o}{2}\right)} (\sigma^2)^{-\left(\frac{\alpha_o}{2}+1\right)} \ e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$\propto (\sigma^2)^{-\left(\frac{\alpha_o}{2}+1\right)} \ e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$p(\sigma^2 \mid x_1, x_2, ..., x_n) \propto (\sigma^2)^{-n/2} \ e^{-\frac{n S^2}{2\sigma^2}} \ (\sigma^2)^{-\left(\frac{\alpha_o}{2}+1\right)} \ e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$\propto (\sigma^2)^{-\left(\frac{\alpha_o+n}{2}+1\right)} \ e^{-\left(\frac{\beta_o + n S^2}{2\sigma^2}\right)}$$

$$p(\sigma^2 \mid x_1, x_2, ..., x_n) \sim \Gamma^{-1}\left(\alpha = \frac{\alpha_o + n}{2}, \beta = \frac{\beta_o + n S^2}{2}\right)$$

$$\therefore\ p(\sigma^2 \mid x_1, x_2, \ldots, x_n) = \frac{\left(\frac{\beta_o + n S^2}{2}\right)^{\frac{\alpha_o + n}{2}}}{\Gamma\left(\frac{\alpha_o + n}{2}\right)} (\sigma^2)^{-\left(\frac{\alpha_o + n}{2} + 1\right)} e^{-\left(\frac{\beta_o + n S^2}{2\sigma^2}\right)}$$

$$\therefore\ E(\sigma^2 \mid X) = \hat{\sigma}^2_{Bayes} = \frac{\beta}{\alpha - 1} = \frac{\left(\beta_o + n S^2\right)/2}{\frac{\alpha_o + n}{2} - 1} = \frac{\beta_o + n S^2}{\alpha_o + n - 2}$$

$X \sim N(\theta, \sigma^2)$ *When* σ^2 *known.*

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto L(\theta)\ p(\theta)$$

$$L(\theta) = (2\pi\sigma^2)^{-n/2}\ e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$$

$$= (2\pi\sigma^2)^{-n/2}\ e^{-\frac{1}{2\sigma^2}\sum(x_i - \bar{x})^2}\ e^{-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2} \quad , \ \mp \bar{x}$$

$$\propto\ e^{-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2}$$

$$p(\theta) \sim N(\theta_o\ , \sigma_o^2)$$

$$p(\theta) = \frac{1}{\sqrt{2\pi\sigma_o^2}}\ e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$\propto\ e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto L(\theta, \sigma^2)\ p(\theta)$$

$$\propto\ e^{-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2}\ e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$\propto\ e^{-\frac{1}{2}\left(\frac{n}{\sigma^2}(\theta - \bar{x})^2 + \frac{1}{\sigma_o^2}(\theta - \theta_o)^2\right)}$$

$$A(Z - a)^2 + B(Z - b)^2 = D(Z - c)^2$$

$A = \frac{n}{\sigma^2}$, $a = \bar{x}$, $Z = \theta$ (var*iable*)

$B = \frac{1}{\sigma_o^2}$, $b = \theta_o$

$D = A + B$, $c = \frac{1}{A + B}(Aa + Bb)$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto e^{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_o^2}\right)\left(\theta - \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\sigma_o^2}}\left(\frac{n\,\bar{x}}{\sigma^2}+\frac{\theta_o}{\sigma_o^2}\right)\right)^2}$$

$$Let; \quad \sigma'^2 = \frac{1}{\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_o^2}\right)} \quad , \; \theta' = \frac{1}{\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_o^2}\right)}\left(\frac{n\,\bar{x}}{\sigma^2}+\frac{\theta_o}{\sigma_o^2}\right)$$

$$p(\theta \mid x_1, x_2, ..., x_n) \propto e^{-\frac{1}{2\,\sigma'^2}(\theta - \theta')^2}$$

$$p(\theta \mid x_1, x_2, ..., x_n) \sim N(\theta', \sigma'^2)$$

The complete p.d.f. of the posterior probability is;

$$p(\theta \mid x_1, x_2, ..., x_n) = \frac{1}{\sqrt{2\,\pi\,\sigma'^2}}\; e^{-\frac{1}{2\,\sigma'^2}(\theta - \theta')^2}$$

$$\therefore E(\theta \mid X) = \hat{\theta}_{Bayes} = \theta' = \frac{1}{\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_o^2}\right)}\left(\frac{n\,\bar{X}}{\sigma^2}+\frac{\theta_o}{\sigma_o^2}\right)$$

Chapter Two

Interval Estimation

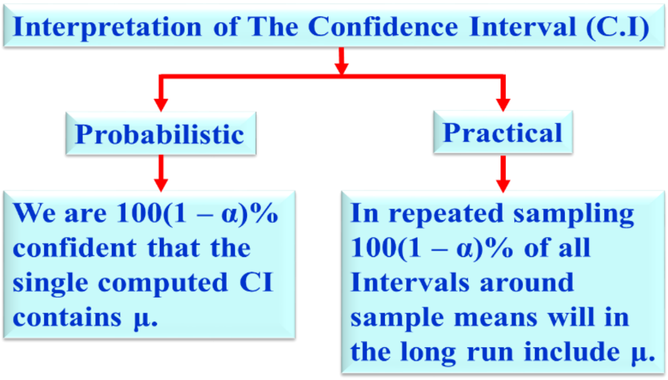
Definition:

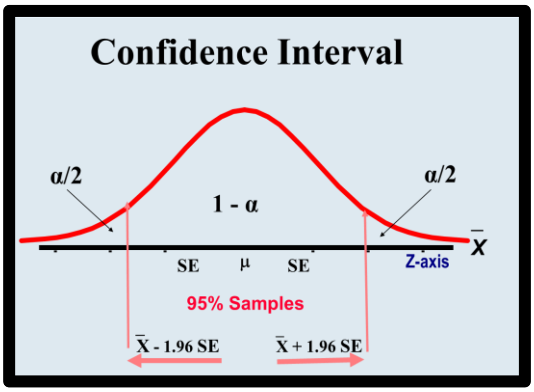
In a rssn taken from a dist^n with p.d.f. $f(x, \theta)$, let L_1 and L_2 be two statistics, then the confidence interval (CI) of parameter θ is;

$$p(L_1 \leq \theta \leq L_2) = 1 - \alpha$$

With $100(1 - \alpha)\%$ confidence coefficient, where; L_1: is lower confidence limit. L_2: is upper confidence limit.

Confidence Interval (CI)





1) Confidence Interval for Mean (When the Variance is known)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with unknown θ, and known variance σ^2, then the sample mean $\bar{X}$ is distributed with mean θ and the variance $\frac{\sigma^2}{n}$ and $Z = \frac{\bar{X} - \theta}{\sigma / \sqrt{n}}$ has standard normal dist^n or: $\bar{X} \sim N(\theta, \frac{\sigma^2}{n})$

$$p\left(-z_{\alpha/2} \leq \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \leq z_{\alpha/2} \right) = 1 - \alpha$$

$$p\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \theta \leq \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) = 1 - \alpha$$

or;

$$\left(L_1 = \bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} , L_2 = \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) \text{ is } 100(1-\alpha)\% \ CI \ \text{ for } \theta.$$

Note:

$\alpha = 5\% \quad \rightarrow 1-\alpha = 95\% \quad \Rightarrow Z_{0.025} = 1.96$

$\alpha = 10\% \quad \rightarrow 1-\alpha = 90\% \quad \Rightarrow Z_{0.05} = 1.645$

$\alpha = 1\% \quad \rightarrow 1-\alpha = 99\% \quad \Rightarrow Z_{0.005} = 2.58$

$\alpha = 2\% \quad \rightarrow 1-\alpha = 98\% \quad \Rightarrow Z_{0.01} = 2.326$

2) Confidence Interval for Mean when the Variance is unknown

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with unknown θ, and unknown variance σ^2, we have two cases:

a) If a sample size $n \geq 30$, $Z = \dfrac{\overline{X} - \theta}{S/\sqrt{n}}$ then:

$$p\left(\overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}} \leq \theta \leq \overline{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

or;

$$\left(L_1 = \overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}} \; , \; L_2 = \overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}}\right) \text{ is } 100(1-\alpha)\ \% \ CI \ \text{ for } \ \theta.$$

b) If a sample size $n < 30$, $T = \dfrac{\overline{X} - \theta}{S/\sqrt{n}}$ has t-distribution with $(n-1)$ df, then;

$$p\left(-t_{(\alpha/2,\, n-1)} \leq T \leq t_{(\alpha/2,\, n-1)}\right) = 1-\alpha$$

$$p\left(-t_{(\alpha/2,\, n-1)} \leq \frac{\overline{X} - \theta}{S/\sqrt{n}} \leq t_{(\alpha/2,\, n-1)}\right) = 1-\alpha$$

$$p\left(\overline{X} - t_{(\alpha/2,\, n-1)}\frac{S}{\sqrt{n}} \leq \theta \leq \overline{X} + t_{(\alpha/2,\, n-1)}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

Ex: In arss100 taken from normal dist^n with mean θ and variance ($\sigma^2 = 225$), and found that $\overline{X}$ of the sample is (125). Find (95%) confidence interval for θ.

Sol:

$1-\alpha = 95\%$

$1-\alpha = 0.95 \quad , \quad \alpha = 0.05 \quad , \quad z_{\alpha/2} = z_{0.025} = 1.96$

$$p\left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \theta \leq \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$p\left(125 - (1.96)\frac{15}{\sqrt{100}} \leq \theta \leq 125 + (1.96)\frac{15}{\sqrt{100}}\right) = 1-0.05$$

$$p(125 - 2.94 \leq \theta \leq 125 + 2.94) = 0.95$$

$\therefore (122.06 \leq \theta \leq 127.94)$

Ex: Let $X_1, X_2, \ldots, X_9$ be a rss9 from a distribution with mean θ and variance σ^2, and ($\bar{X} = 19.74$, $S^2 = 0.65$). Find (99%) confidence interval (CI) for θ.

Sol:

$1-\alpha = 99\%$

$1-\alpha = 0.99 \quad , \quad \alpha = 0.01 \quad , \quad t_{(\alpha/2,\, n-1)} = t_{(0.005,\, 8)} = 3.355$

$$p\left(\bar{X} - t_{(\alpha/2,\, n-1)}\frac{S}{\sqrt{n}} \le \theta \le \bar{X} + t_{(\alpha/2,\, n-1)}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$$p\left(19.74 - (3.355)\frac{0.806}{\sqrt{9}} \le \theta \le 19.74 + (3.355)\frac{0.806}{\sqrt{9}}\right) = 1-0.01$$

$$p(19.74 - 0.9 \le \theta \le 19.74 + 0.9) = 0.99$$

$\therefore (18.84 \le \mu \le 20.64)$

Ex: A rss(50) taken from normal population with mean (θ) and variance σ^2, and ($\bar{X} = 5.67$, $S = 1.94$). Find (95%) confidence interval (CI) for θ.

Sol:

$1-\alpha = 95\%$

$1-\alpha = 0.95 \quad , \quad \alpha = 0.05 \quad , \quad z_{\alpha/2} = z_{0.025} = 1.96$

$$p\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \theta \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$p\left(5.67 - (1.96)\frac{1.94}{\sqrt{50}} \le \theta \le 5.67 + (1.96)\frac{1.94}{\sqrt{50}}\right) = 1-0.05$$

$$p(5.67 - 0.538 \le \theta \le 5.67 + 0.538) = 0.95$$

$\therefore (5.132 \le \mu \le 6.208)$

Ex: An epidemiologist studied the blood glucose level of a random sample of 100 patients. The mean was 170, with a SD of 10. Find (95%) confidence interval for θ.

Sol:

$1-\alpha = 95\%$

$1-\alpha = 0.95 \quad , \quad \alpha = 0.05 \quad , \quad z_{\alpha/2} = z_{0.025} = 1.96$

$$p\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \theta \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$p\left(170 - (1.96)\frac{10}{\sqrt{100}} \le \theta \le 170 + (1.96)\frac{10}{\sqrt{100}}\right) = 1-0.05$$

$$p(170 - 1.96 \le \theta \le 170 + 1.96) = 0.95$$

$\therefore (168.04 \le \mu \le 171.96)$

3) Confidence Interval for Difference Between two Means

Let $\overline{X}$ be a sample mean for a rssn from a normal population with mean μ_X and unknown variance σ_X^2 and $\overline{Y}$ be a sample mean for a rss$n$ from a normal population with mean μ_Y and unknown variance σ_Y^2, then:

$$\overline{X} \sim N(\mu_X, \frac{\sigma_X^2}{n}) \quad , \quad \overline{Y} \sim N(\mu_Y, \frac{\sigma_Y^2}{m})$$

$$(\overline{X} - \overline{Y}) \sim N\left(\mu_X - \mu_Y , \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)$$

$$Z = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

$$p(-Z_{\alpha/2} \le Z \le Z_{\alpha/2}) = 1 - \alpha$$

$$p\left(-Z_{\alpha/2} \le \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \le Z_{\alpha/2}\right) = 1 - \alpha$$

$$p\left((\overline{X} - \overline{Y}) - Z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \le (\mu_X - \mu_Y) \le (\overline{X} - \overline{Y}) - Z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right) = 1 - \alpha$$

Ex: Let $\overline{X}$ be a sample mean for a rss15 from a normal population with mean μ_X and known variance $\sigma_X^2 = 60$ and $\overline{Y}$ be a sample mean for a rss18 from a normal population with mean μ_Y and known variance $\sigma_Y^2 = 40$, we find that $(\overline{X} = 70.1)$, $(\overline{Y} = 75.3)$, find 90% CI for $(\mu_X - \mu_Y)$.

Sol:

$$\alpha = 0.1 \quad , \quad \frac{\alpha}{2} = 0.05 \quad , \quad Z_{\alpha/2} = Z_{0.05} = 1.645$$

$$\overline{X} - \overline{Y} = 70.1 - 75.3 = -5.2$$

$$p\left((\overline{X} - \overline{Y}) - Z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \le (\mu_X - \mu_Y) \le (\overline{X} - \overline{Y}) - Z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right) = 1 - \alpha$$

$$p\left(-5.2 - 1.645\sqrt{\frac{60}{15} + \frac{40}{18}} \le (\mu_X - \mu_Y) \le -5.2 + 1.645\sqrt{\frac{60}{15} + \frac{40}{18}}\right) = 1 - 0.1$$

$$p(-9.303 \le (\mu_X - \mu_Y) \le -1.097) = 0.9$$

$$(-9.303 \le (\mu_X - \mu_Y) \le -1.097)$$

4) Confidence Interval for the Variance

Let $X_1, X_2, \ldots, X_n$ be a random sample from normal population with unknown mean, and unknown variance, then;

$$\chi^2 = \frac{(n-1)\,S^2}{\sigma^2} \quad \textit{is distributed as } \chi^2 \textit{ with } (n-1)\ d.f.$$

$$p\left(\chi^2_{\frac{\alpha}{2},\,n-1} \le \chi^2 \le \chi^2_{1-\frac{\alpha}{2},\,n-1}\right) = 1-\alpha$$

$$p\left(\chi^2_{\frac{\alpha}{2},\,n-1} \le \frac{(n-1)\,S^2}{\sigma^2} \le \chi^2_{1-\frac{\alpha}{2},\,n-1}\right) = 1-\alpha$$

$$p\left(\frac{1}{\chi^2_{\frac{\alpha}{2},\,n-1}} \ge \frac{\sigma^2}{(n-1)\,S^2} \ge \frac{1}{\chi^2_{1-\frac{\alpha}{2},\,n-1}}\right) = 1-\alpha$$

$$p\left(\frac{(n-1)\,S^2}{\chi^2_{1-\frac{\alpha}{2},\,n-1}} \le \sigma^2 \le \frac{(n-1)\,S^2}{\chi^2_{\frac{\alpha}{2},\,n-1}}\right) = 1-\alpha$$

Ex: Let $X_1, X_2, \ldots, X_{20}$ be a random sample from normal population with unknown mean, and unknown variance, we found that $(\overline{X} = 76.1,\ S^2 = 88.36)$, find 99% CI for σ^2.

Sol:

$\alpha = 0.01 \quad , \frac{\alpha}{2} = 0.005 \quad , \chi^2_{\frac{\alpha}{2},\,n-1} = \chi^2_{0.005,19} = 6.84$

$, \chi^2_{1-\frac{\alpha}{2},\,n-1} = \chi^2_{0.995,19} = 38.6$

$$p\left(\frac{(n-1)\,S^2}{\chi^2_{1-\frac{\alpha}{2},\,n-1}} \le \sigma^2 \le \frac{(n-1)\,S^2}{\chi^2_{\frac{\alpha}{2},\,n-1}}\right) = 1-\alpha$$

$$p\left(\frac{19\,(88.36)}{38.6} \le \sigma^2 \le \frac{19\,(88.36)}{6.84}\right) = 1-0.01$$

$$p\left(45.87 \le \sigma^2 \le 245.22\right) = 0.995$$

$$\therefore\ (45.87 \le \sigma^2 \le 245.22)$$

Ex: Let

$$X \sim N(\theta_X , \sigma_X^2) \ , \ Y \sim N(\theta_Y , \sigma_Y^2)$$

$$n = 10 \quad , \ \overline{X} = 4.2 \qquad , \ \sigma_X^2 = 49$$

$$m = 7 \quad , \ \overline{Y} = 3.4 \qquad , \ \sigma_Y^2 = 32$$

Find 90 % CI for $(\theta_X - \theta_Y)$.

Sol:

$$1 - \alpha = 0.9 \quad \Rightarrow \ \alpha = 0.1, \ Z_{\alpha/2} = Z_{0.05} = 1.6450.005 \qquad , \ \chi^2_{\frac{\alpha}{2}, n-1} = \chi^2_{0.005,19} = 6.84$$

$$p\left((\overline{X} - \overline{Y}) - Z_{\alpha/2} \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \le (\theta_X - \theta_Y) \le (\overline{X} - \overline{Y}) - Z_{\alpha/2} \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \right) = 1 - \alpha$$

$$p\left((4.2 - 3.4) - 1.645 \sqrt{\frac{49}{10} + \frac{32}{7}} \le (\theta_X - \theta_Y) \le (4.2 - 3.4) - 1.645 \sqrt{\frac{49}{10} + \frac{32}{7}} \right) = 1 - 0.1$$

$$p(0.8 - 5.063 \le (\theta_X - \theta_Y) \le 0.8 + 5.063) = 0.9$$

$$p(-4.263 \le (\theta_X - \theta_Y) \le 5.863) = 0.9$$

$$\therefore \ (-4.263 \le (\theta_X - \theta_Y) \le 5.863)$$

Ex: from $N(\theta, \sigma^2)$, we have ($n = 9$, $S^2 = 7.63$), find 95 % CI for σ^2.

Sol:

$$1 - \alpha = 0.95 \quad \to \alpha = 0.05 \qquad , \ \chi^2_{\frac{\alpha}{2}, n-1} = \chi^2_{0.025,8} = 2.18$$

$$, \ \chi^2_{1-\frac{\alpha}{2}, n-1} = \chi^2_{0.975,8} = 17.5$$

$$p\left(\frac{(n-1)\ S^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}} \le \sigma^2 \le \frac{(n-1)\ S^2}{\chi^2_{\frac{\alpha}{2}, n-1}} \right) = 1 - \alpha$$

$$p\left(\frac{8\ (7.63)}{17.5} \le \sigma^2 \le \frac{8\ (7.63)}{2.18} \right) = 1 - 0.05$$

$$p(3.488 \le \sigma^2 \le 28) = 0.95$$

$$\therefore \ (3.488 \le \sigma^2 \le 28)$$