Moment Method for Parameter of Geometric Distribution

$X \sim Goe(p)$, ***first case***: $f(x;p) = p\, q^{x}$ $E(X) = \frac{1-p}{p}$,

$m_k = \frac{\sum x_i^k}{n}$, $M_k = E(X^k)$,→ $m_k = M_k$

$m_1 = \frac{\sum x_i^1}{n} = \bar{X}$, $M_1 = E(X) = \frac{1-p}{p}$

$m_1 = M_1$

$\bar{X} = \frac{1-p}{p} \rightarrow \bar{X}\, p = 1 - p \rightarrow \hat{p}_{moment} = \frac{1}{1+\bar{X}}$

$X \sim Goe(p)$, **second case**: $f(x;p) = p\, q^{x-1}$ $E(X) = \frac{1}{p}$,

$m_k = \frac{\sum x_i^k}{n}$, $M_k = E(X^k)$,→ $m_k = M_k$

$m_1 = \frac{\sum x_i^1}{n} = \bar{X}$, $M_1 = E(X) = \frac{1}{p}$

$m_1 = M_1$

$\bar{X} = \frac{1}{p} \rightarrow \bar{X}\, p = 1 \rightarrow \hat{p}_{moment} = \frac{1}{\bar{X}}$

Note: In all distributions, it is not a requirement: Moment Method equal to the m.l.e Method.
For example, in Beta distribution, or in some cases we cannot use m.l.e. method.

Q// Is m.l.e $\hat{p} = \frac{1}{\bar{X}}$ unbiased estimator for p *in Geo*. distriution? $\hat{p} = \frac{n}{\sum x_i}$

H.W: Is m.l.e $\hat{p} = \frac{1}{1+\bar{X}} = \frac{n}{n+\sum x_i}$ unbiased estimator for *p in Geo*. distribution? $f(x;p) = p\,q^x$.

Exponential Family

Q// Is Geometric distribution Geo(p) belongs to exponential family?

1) $f(x;\ p) = \ p\,q^x \quad , \ x = 0, 1, \dots.$

$f(x;\ p) = \exp(\ln f(x;\ p))$

$= exp[\ln p + x\ln(1-p)]$

$A(\theta) = \ \ln(1-p) \quad , \ B(x) = x\ , C(\theta) = \ln p\ , \ \ D(x) = 0$

$\therefore\ f(x;\ p)\ of$ Geometric distribution belongs to exponential family.

In arssn;

$f(x_1, x_2, \dots, x_n;\ p) = exp(n\ln p + \ln(1-p)\sum x_i)$

$\therefore\ \sum B(X_i) = \ \sum x_i \ \ is\ suff.est.for\ p.$

2) $f(x;\ p) = \ p\,q^{x-1} \quad , \ x = \ 1, 2, \dots.$

$f(x;\ p) = \exp(\ln f(x;\ p))$

$= exp[\ln p + (x-1)\ln(1-p)]$

$= exp\left[ln\left(\frac{p}{1-p}\right) + x\ln(1-p)\right]$

$A(p) = \ \ln(1-p) \quad , \ B(x) = x\ , C(\theta) = ln\left(\frac{p}{1-p}\right) \quad , \ D(x) = 0$

$\therefore\ f(x;\ p)\ of$ Geometric distribution belongs to exponential family.

In arssn;

$f(x_1, x_2, \dots, x_n;\ p) = exp\left(nln\left(\frac{p}{1-p}\right) + \ln(1-p)\sum x_i\right)$

$\therefore\ \sum B(X_i) = \ \sum x_i \ \ is\ suff.est.for\ p$

5) Negative Binomial Distribution

Independent Bernoulli trials are performed until (r) successes appear, define the r.v. X is the number of failure trials before getting the *r–th* success trial, then a r.v. X defined to have (N.B.) dist. if the p.d.f. of X given by:

$$p(x) = \begin{cases} C_x^{x+r-1}\, p^r\, q^x & , \quad x = 0,1,2,\ldots.. \\ 0 & \quad o.w \end{cases}$$

Where the parameters (r) and (p) satisfy [r = 1, 2, …., $0 < p < 1$].

x: No. of failure trials before getting the r–th success.

r: No. of successes cases (fixed number).

Clearance: Let a coin tossed nine times, in ninth toss we get success trials, success trials (get 3 Heads). What is the probability of the following result?

$$p(TTT\underset{r=1}{H}T\underset{r=2}{H}TT\underset{r=3}{H}) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\overset{r=1}{\underline{\left(\tfrac{1}{2}\right)}}\left(\tfrac{1}{2}\right)\overset{r=2}{\underline{\left(\tfrac{1}{2}\right)}}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\overset{r=3}{\underline{\left(\tfrac{1}{2}\right)}}$$

$$= C_x^{x+r-1} = C_6^8 \quad p^r = p^3 \quad q^x = q^6$$

C_6^8 : No. of failure trials before getting the (3) success trials (H)

Properties of the Negative Binomial Distribution

Let X be a r.v. has Negative Binomial Distribution, X ~ N.B(r, p), then;

1) $mean = E(X) = \frac{r\,q}{p}$

$$\text{Proof}: E(X) = \sum_{x=0}^{\infty} x\ p(x) = \sum_{x=0}^{\infty} x\, C_x^{x+r-1}\, p^r\, q^x \quad , \quad But;\ C_x^{x+r-1} = (-1)^x\, C_x^{-r}$$

$$= p^r \sum_{x=0}^{\infty} x\,(-1)^x C_x^{-r}\ q^x = p^r \sum_{x=0}^{\infty} x\, C_x^{-r}\ (-q)^x = p^r \sum_{x=0}^{\infty} x\, \frac{-r!}{x!(-r-x)!}(-q)^x \quad = \frac{rq}{p}$$

$$\therefore E(X^2) = r^2\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + \frac{r\,q}{p}$$

$$\therefore v(X) = r^2\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + \frac{r\,q}{p} - \left(\frac{r\,q}{p}\right)^2 = r\,\frac{q^2}{p^2} + r\,\frac{q}{p} = r\,\frac{q}{p}\left(\frac{q}{p} + 1\right) = r\,\frac{q}{p}\left(\frac{q+p}{p}\right) = r\,\frac{q}{p}\left(\frac{1}{p}\right) = \frac{r\,q}{p^2}$$

3) The cumulative dist. function (c.d.f.) of X ~ N.B(r, p).

$$F(x) = p(X \le k) = \begin{cases} 0 & x < 0 \\ p^r \sum_{x=0}^{k} C_x^{x+r-1} q^x & , 0 \le x < \infty \\ 1 & x \to \infty \end{cases}$$

4) The moment generating function (m.g.f.) of X ~ N.B(r, p).

$$M_X(t) = \left(\frac{p}{1 - qe^t}\right)^r$$

Cumulates

The arithmetic mean, variance, etc. can be found through this important method.

$$m.g.f. \;\to\; M_X(t) = E(e^{tX}) = \left(\frac{p}{1-qe^t}\right)^r$$

$$g(t) = \ln E(e^{tX})$$

5) Additive Property

i- Let $X_1, X_2, \ldots, X_n$ be r.v.'s and independent, such that X_i ~ N.B(r_i, p) , $i = 1, 2, \ldots, n$,then;

$$\sum_{i=1}^{n} X_i \sim n.b\left(\sum_{i=1}^{n} r_i, p\right).$$

ii- Let $X_1, X_2, \ldots, X_r$ be r.v.'s and independent, such that X_i ~ Geo(p), then;

$$\sum_{i=1}^{r} X_i \sim n.b(r, p).$$

6) Mode of Negative Binomial Distribution

$\frac{f(m)}{f(m-1)} > 1$, $\frac{f(m)}{f(m+1)} > 1$

Y = [*x*] greatest integer less than or equal x. e.g: *m* = [1.3] = 1.3 – 0.3 = 1

Note: $\lim_{r \to \infty} N.Bin \quad \to \quad Poisson(\lambda)$

Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a rss*n* from Negative Binomial dist^n N.Bin(*r*, *p*), find the m.l.e for θ.

Sol: ∵ X ~ N.Bin(r, θ)

$f(x;\ p, r) = \binom{x+r-1}{x} p^r q^x \quad , \ x = 0, 1, \ldots.$

$L(p) = \prod_{i=1}^{n} \binom{x_i+r-1}{x_i} p^{n\,r} (1-p)^{\sum x_i}$

$\ln L(p) = \ln \prod_{i=1}^{n} \binom{x_i+r-1}{x_i} + n\,r \ln p + \sum x_i \ln(1-p)$

$\frac{\partial \ln L(p)}{\partial p} = \frac{n\,r}{p} - \frac{\sum x_i}{(1-p)} = \frac{nr - nrp - p\ \sum x_i}{p(1-p)} \qquad , \ \frac{\partial \ln L(p)}{\partial p} = 0$

$\to \ \frac{nr - nr\hat{p} - \hat{p}\ \sum x_i}{\hat{p}(1-\hat{p})} = 0$

$\to \ \frac{nr - \hat{p}(nr + \sum x_i\ )}{\hat{p}(1-\hat{p})} = 0$

$\to \ nr \ = p(nr + \sum x_i) \quad \to \ \hat{p} = \frac{nr}{nr + \sum x_i} = \frac{r}{r+\bar{x}}$ is m.l.e. estimator for probability of success

but this is a biased estimator. Its inverse (r + x)/r, is an unbiased estimator for 1/p. **H.W**

Ex: let (r -1) is number of success before the last success from ($x + r – 1$), then show that;

$\hat{p} = \frac{r-1}{x+r-1}$ unbiased *or* $\hat{p} = \frac{r}{x+r}$ biased?

Q// Is negative Binomial distribution N.Bin(r, p) belongs to exponential family?

$f(x;\ p, r) = \binom{x+r-1}{x} p^r q^x \quad , \ x = 0, 1, \ldots.$

$f(x;\ p, r) = \exp(\ln f(x;\ p, r))$

$= exp\left[\ln\binom{x_i+r-1}{x} + r\ln p + x\ln(1-p)\right]$

$A(p) = \ln(1-p) \quad , \ B(x) = x \ , C(p) = r\ln p \ , \quad D(x) = \ln\binom{x_i+r-1}{x}$

$\therefore f(x;\ p, r)$ *of* Negative Binomial distribution belongs to exponential family

Now in a rss(n) ; (for Sufficiency)

$L(\theta) = f(x_1, x_2, \ldots, x_n;\ p, r) = exp[A(\theta)\ \sum_{i=1}^{n} B(x_i) + n\ C(\theta) + \ \sum_{i=1}^{n} D(x_i)]$

$= exp[A(\theta)\ \sum_{i=1}^{n} B(x_i) + n\ C(\theta)].\ exp[\sum_{i=1}^{n} D(x_i)]$

$= g\left(\hat{\theta}, \theta\right).\ h(x_1, x_2, \ldots, x_n)$

$\therefore \sum_{i=1}^{n} B(x_i)\ is\ suff.est.\ \ for\ \ \ \theta\ and\ u(\theta)$

$\ln L(\theta) = A(\theta)\ \sum_{i=1}^{n} B(x_i) + n\ C(\theta) + \ \sum_{i=1}^{n} D(x_i)$

$\frac{\partial \ln L(\theta)}{\partial\ \theta} = A^{'}(\theta) \sum_{i=1}^{n} B(x_i) + n\ C'(\theta)$

$\frac{\partial^2 \ln L(\theta)}{\partial\ \theta^2} = A^{''}(\theta) \sum_{i=1}^{n} B(x_i) + n\ C^{''}(\theta)$

$-E\left(\frac{\partial^2 \ln L(\theta)}{\partial\ \theta^2}\right) = -[A^{''}(\theta) E \sum_{i=1}^{n} B(x_i) + n\ C^{''}(\theta)] \ \rightarrow\ Fisher\ information\ in\ arss(n)$

6) Poisson Distribution

Is the limiting form of the binomial distribution when ($n \to \infty$) and ($p \to 0$). So that (n p) is finite quantity such as (λ). In general if ($n \geq 50$) and ($\lambda \leq 5$), it can be taken to be a case of Poisson distribution such events are known as rare events.

Applications

1- The number of deaths from a disease such as heart attack.

2- The number of customers entering a service station per hour.

3- The number of defective material per packing manufactured.

4- The number of railroad accidents in same unit of time.

5- The number of telephone calls received at a particular switchboard per hour.

6- The number of cars passing a traffic intersection per minute.

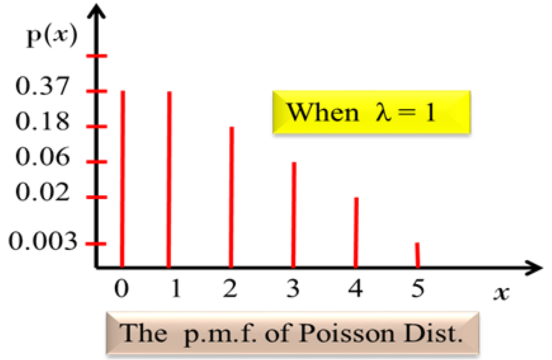
7- The number of insurance claims in same unit of time.

8- The number of errors a typist makes per page.

9- The number of customers to arrive in a bank per hour.

Definition: A r.v. X is defined to have a Poisson distribution if the p.m.f of given by:

$$p(x;\lambda)=p(X=x)=\begin{cases}\dfrac{e^{-\lambda}\lambda^{x}}{x!} & , \quad x=0,1,2,\dots\\ 0 & \quad o.w\end{cases}$$



Where the parameter (λ > 0) is a constant integer of function of any positive real number R^+.

[λ = *n*p], (λ: rate of failure)

Remark: Poisson: *n* large & p small

Binomial: *n* small & p large

Properties of a Poisson Distribution X ~ *poi*(λ)

1- The number of successes (events) that occur in a certain time interval is independent of the number of successes that occur in another time interval.

2- Mean and variance of Poisson Distribution X ~ poi(λ) $mean = variance = \lambda$

3- The moment generating function (m.g.f) of X ~ poi(λ)

$$M_X(t)=e^{\lambda(e^t-1)}$$

4- The c.d.f. of X ~ Poi(λ)

$$F(x) = p(X \le x) = \begin{cases} 0 & , \ x < 0 \\ \sum_{u=0}^{x} \frac{e^{\lambda} \cdot \lambda^{u}}{u!} & , \ 0 \le x < \infty \\ 1 & , \ x \to \infty \end{cases}$$

5- The additive property

Let $X_1, X_2, \dots, X_n$ be a Poisson distribution with $\lambda_1, \lambda_2, \dots, \lambda_n$ and X's are independent. Then $Y = X_1 + X_2 + \dots + X_n$ has a Poisson distribution $\sum_{i=1}^{n} X_i \sim Poi(\sum_{i=1}^{n} \lambda_i)$.

Maximum Likelihood Estimation

Ex: Let $X_1, X_2, \ldots, X_n$ denote a random sample from Poisson dist^n Poi(λ), find the m.l.e for λ.

Sol:

$\because$ X ~ Poi(λ)

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad , \qquad x = 0\,,1,\ldots\ldots$$

$\because X's$ *are indep.*

$$L(\lambda) = f(x_1, x_1, \ldots, x_1;\ \lambda) = \Pi\, f(x_i;\lambda)$$

$$= \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod(x_i)!}$$

$$\ln L(\lambda) = -n\,\lambda + \Sigma x_i \ln(\lambda) - \prod(x_i)!$$

$$\frac{\partial \ln L(\lambda)}{\partial\lambda} = -n + \frac{\Sigma x_i}{\lambda} \qquad , \qquad \frac{\partial \ln L(\lambda)}{\partial\lambda} = 0$$

$$-n + \frac{\Sigma x_i}{\hat{\lambda}} = 0$$

$$\frac{\Sigma x_i}{\hat{\lambda}} = n$$

$$\Sigma x_i = n\hat{\lambda}$$

$$\hat{\lambda}_{m.l.e} = \frac{\Sigma X_i}{n} = \overline{X}$$

$$\frac{\partial^2 \ln L(\lambda)}{\partial\lambda^2} = -\frac{\Sigma x_i}{\lambda^2} < 0$$

$\therefore\ \hat{\lambda} = \overline{X}$ is m.l.e for .

Ex: In a random sample of size (n). Is $T = \overline{X}$ unbiased estimator for λ of Poisson(λ).

$\because$ X ~ Poi(λ)

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , \ x = 0, 1, \dots \dots \quad , E(X) = \lambda$$

$$E(T = \bar{X}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{n\,E(X)}{n} = \lambda \quad , \rightarrow \bar{X} \textit{ is unbiased est. for } \lambda$$

H.W: 1) Is $\bar{X}$ is consistent estimator for λ.

2) Is $\sum X_i$ sufficient estimator for λ?

3) Let X be a random variable from Poisson dist^n poi(λ). Show that the family of X is complete.

Ex: Let $X_1, X_2, \dots, X_n$ be a rss$n$ from Poisson dist^n poi(θ). Show that $Y = \sum X_i$ is a complete sufficient estimator for θ. Find the unique continuous function of Y, which is the best estimator for θ (M.V.U.E).

Ex: Let $X_1, X_2, \dots, X_n$ be a rss$n$ from Poison dist^n Poi(θ), is $T = \overline{X}$ an efficient estimator for $\phi(\theta) = \theta$?

Rao-Blackwell Theorem

Let X has a p.d.f. $f(x;\theta)$, and u be an unbiased estimator for parameter θ, and T be a sufficient estimator for θ, then;

1) $E\left(E(U|T)\right) = E(U)$, $E(U) = \theta$

2) $Var\left(E(U|T)\right) \leq Var(U)$, Var(U) → M.V.L.B → M.V.U.E

For Poisson distribution $X \sim poisson(\lambda)$;

$E(\,X_i|\,\sum_{i=1}^{n} X_i) =$

$= E(\,X) = \lambda$

$Y = (\,X_i|\,\sum_{i=1}^{n} X_i) \sim Bin\left(\sum_{i=1}^{n} X_i\,, \frac{1}{n}\right)$, $\frac{1}{n} = \frac{\lambda_i}{\sum \lambda_i}$

$E(Y) = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$,, $Var(Y) = Var(\bar{X}) = \frac{\lambda}{n}$

Q// Let X_i ~Poi(θ), and $X_1, X_2, \ldots, X_n$ be a r.s. of size (n) generated from X, can we apply Rao Black well theorem to $E(\,X_i|\,\sum_{i=1}^{n} X_i)$, and then find $Var\,E(\,X_i|\,\sum_{i=1}^{n} X_i)$, $Var\,E(\,X_i|\,\sum_{i=1}^{n} X_i) < Var(X_i)$.

H.W: How can you prove that $Y = p(\,X_i|\,\sum_{i=1}^{n} X_i) \sim Bin\left(\sum_{i=1}^{n} X_i\,, \frac{1}{n}\right)$

Mode of a Poisson Distribution X ~ *poi*(λ)

If it is an integer; $m = [\lambda]$, $m = \lambda - 1$, $(\lambda, \lambda-1)$

$m = [\lambda]$

$m = \lambda - \varepsilon$

$$\frac{f(x)}{f(x-1)} > 1$$

$$\frac{\frac{e^{-\lambda}\lambda^{x}}{x!}}{\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x} > 1$$

$$\rightarrow m < \lambda \rightarrow I - \varepsilon$$

$$m = 2 < 2.1$$

$$2.1 - 0.1$$

$$\frac{f(x)}{f(x+1)} > 1$$

$$\frac{\frac{e^{-\lambda}\lambda^{x}}{x!}}{\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}} = \frac{x+1}{\lambda} > 1$$

$$\lambda < x + 1 \rightarrow \lambda - 1 < x = m$$

$$m = (\lambda - 1) + \varepsilon \quad , \; 0 < \varepsilon < 1$$

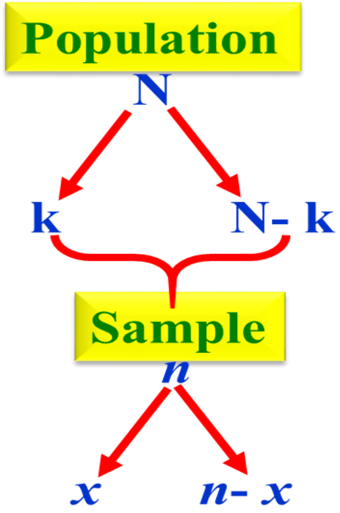
∴ Poisson dist. has two modes. $\frac{f(m)}{f(m-1)} = 1$, $\lambda > 0$

7) Hyper Geometric Distribution

Suppose that (n) objects are to be drawn at random, one at a time from a collection of (N) objects, (k) of one kind and ($N-k$) of another kind. The one kind of object will be thought of as "success" and coded (1); the other kind is coded (0), then a r.v. X is defined to have a hyper geometric distribution if the p.m.f. of X given by;

$$p(x) = p(x;N,k,n) = p(X=x) = \begin{cases} \dfrac{C_x^k \, C_{n-x}^{N-k}}{C_n^N} & , \quad x = 0,1,2,\ldots\ldots,n \\ 0 & o.w \end{cases}$$

Where; N, k, and n are parameters, such that $N \geq n$, $N \geq k$, and N, k, and n are all positive integer, X ~ H.G(N, k, n).



Remark: $x = \text{a}, \text{a}+1, \text{a}+2, \ldots\ldots, \text{b}$

Where;

$\text{a} = \text{Max}\,(0\,,\, n-(N-k))$

$\text{b} = \text{Min}\,(n\,,\, k)$

Where; N: size of population.

x: No. of defective items in the sample.

k : No. of defective items in the population.

$N-k$: No. of non-defective items in the population.

If $k < x$ then $C_x^k = 0$.

For $(k < x \leq n) \longrightarrow f(x) = 0$

This distribution would apply if X is the number of defective in a sample drawn ***without replacement*** from a batch of (n) objects there being (k) defective in the batch.

Properties of the Hyper Geometric Distribution

1) $\sum p(x) = 1 \Rightarrow \dfrac{\sum C_x^k \, C_{n-x}^{N-k}}{C_n^N} = \dfrac{C_{n+x-x}^{N+k-k}}{C_n^N} = \dfrac{C_n^N}{C_n^N} = 1$

2) Mean and variance of X ~ H.G(*N*, *k*, *n*).

$mean = n\,p = n\,\dfrac{k}{N}$, $where;$ $p = \dfrac{k}{N}$, $\dfrac{x}{n}$ $estimator\ of\ \dfrac{k}{N}$, $unbiased\ estimator\ \dfrac{k-1}{x+k-1}$

The mean of the H.G distribution is obtained from the representation of H.G variable as a sum of the Bernoulli trials.

3) The c.d.f. of X ~ H.G(*N*, *k*, *n*).

$$F(x) = p(X \le x) = \begin{cases} 0 & , \ x < 0 \\ \sum_{u=0}^{x} \dfrac{C_u^k \, C_{n-u}^{N-k}}{C_n^N} & , \ 0 \le x < n \\ 1 & , \ x \ge n \end{cases}$$

4) The m.g.f. of X ~ H.G(*N*, *k*, *n*).

The m.g.f. $M_X(t)$ of H.G distribution do not exist because is very complicated.

Note: 1) if $n \to \infty$ v(X) = $n\,p\,q$ (with replacement) (Binomial)

2) $\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$ $an\ estimator\ for\ p = \frac{k}{N}$.

To find (m.l.e) we need ($p = \frac{k}{N}$)

	Sample	Not- Sample	Total
Success	x	$k - x$	k
Failure	$n - x$	$N - n - k + x$	$N - k$
Total	n	$N - n$	N

H.W: Find (m.l.e) for Binomial, H.G, Uniform and Multinomial distribution.

Multivariate Hyper Geometric Distribution (Generalization)

$$f(X = x_1, X = x_2, \dots, X = x_r; N, n, k) = \frac{\binom{k_1}{x_1}\binom{k_2}{x_2}\dots\binom{k_r}{x_r}}{\binom{N}{n}}$$

$$N = k_1 + k_2 + \cdots + k_r = N_1 + N_2 + \cdots + N_r$$

$$n = x_1 + x_2 + \cdots + x_r \quad , (r\ types)$$

Ex: A box containing (N = 20) balls, (N_1 = 8 Red, N_2 = 5 Blue, N_3 = 7 Green). Draw a sample from this box, (n = 6). What is the probability that it is 1 red, 4 blue, 1 green?

Node: when $n \rightarrow \infty$ then the H.G. distribution approaches the Binomial distribution.

Note: Parameterization: It means we do something with the parameter.

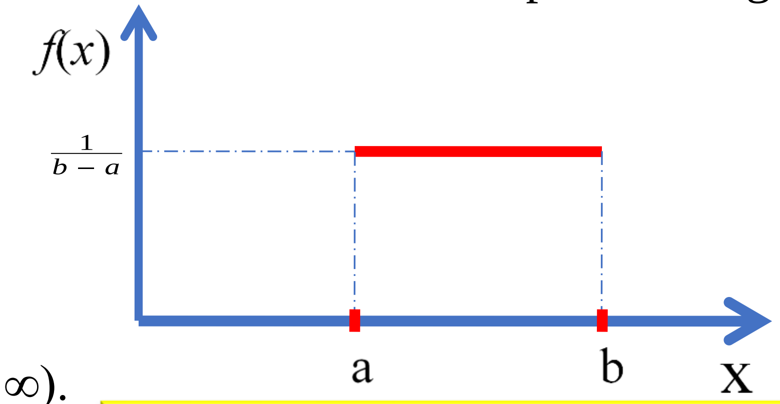
The Continuous Distribution

1. Continuous Uniform Distribution

- Used to model random variables that tend to occur "evenly" over a range of values.
- Probability of any interval of values proportional to its width.
- Used to generate (simulate) random variables from virtually any distribution.
- Used as "non-informative prior" in many Bayesian analyses.

Definition: A r.v. X is defined to have continuous uniform distribution iff the p.d.f. of X given by:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , \ a \le x \le b \\ 0 & o.w \end{cases}$$



The p.d.f. of C. uniform Distribution

Where (a and b) are two parameters ($-\infty < a < b < \infty$).

Symbol: X ~ C.U(a , b)

Example: The daily sale of gasoline is uniformly distributed between 2,000 and 5,000 gallons.
Remark: When (a = 0 and b = 1) is called Standard Uniform Distribution.
Special case: when a = 0, b = θ;

$$f(x;\theta) = \frac{1}{\theta} \quad , 0 < x < \theta$$

Properties of Continuous Uniform Distribution

1- $f(x)$ is a p.d.f. of X ~ C.U(a , b).

$$\int_a^b f(x)\,dx \overset{?}{=} 1 \quad \Rightarrow \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}\,x\Big|_a^b = \frac{1}{b-a}(b-a) = 1$$

2- The Cumulative distribution function (c.d.f.) of X ~ C.U(a , b).

$$F(x) = p(X \le x) = \int_a^x f(u)\,du = \int_a^x \frac{1}{b-a}\,du = \frac{1}{b-a}\int_a^x du = \frac{1}{b-a}\,u\Big|_a^x)$$

$$= \frac{1}{b-a}(x-a) \qquad \Rightarrow \qquad \therefore F(x) = \begin{cases} 0 & , \quad x \le a \\ \dfrac{x-a}{b-a} & , \quad a < x < b \\ 1 & , \quad b \ge b \end{cases}$$

3- The mean and the variance of X ~ C.U(a , b).

$$E(X) = \frac{a+b}{2} \quad , \qquad Var(X) = \frac{(b-a)^2}{12}$$

4- The moment generating function (m.g.f.) of $X \sim C.U(a , b)$.

$$M_X(t) = E\,e^{tX} = \int_a^b e^{tx} f(u)\,du = \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{t(b-a)}\, e^{tx}\Big|_a^b = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & , \; t > 0 \\ 1 & , \; t = 0 \end{cases}$$

$\because M_X(0) \neq 1$, *then we use the following method;*

$$M'_X(t) = \frac{g'_1(t)}{g'_2(t)} = \frac{b\,e^{tb} - a\,e^{ta}}{(b-a)}\Bigg|_{t=0} = \frac{b-a}{b-a} = 1$$

5) Mode and Median of Uniform Distribution

Mode in uniform dist^n are multi-mode. The median: [- median = mean]

Maximum Likelihood Estimation

To find m.l.e. for θ of uniform dist^n C.U(0, θ).

$$f(x;\ \theta) = \frac{1}{\theta} \quad ,0 < x < \theta$$

$$L(\theta) = \frac{1}{\theta^n}$$

$$\ln L(\theta) = -nln\,(\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial\ \theta} = -\frac{n}{\theta} \quad ,\frac{\partial \ln L(\theta)}{\partial\ \theta} = 0 \ \rightarrow \ \theta\ or\ n \neq 0\ (This\ is\ not\ logic\ )$$

The solution by the derivative is illogical.

$$L(\hat{\theta}) \geq L(\theta) \quad for\ \theta \in \Omega\ ,\Omega = [0, \infty]$$

Here we work with logic;

$Y_1, Y_2, \ldots, Y_n$ are an order statistics of the random sample of size (n).

$$0 < Y_1 \leq Y_2 \leq \cdots \leq Y_n < \theta$$

$$\theta \geq X_i \quad (Y_1 \leq Y_2 \leq \cdots \leq Y_n)$$

$$\theta \geq Max(X_i)$$

$$\therefore\ \hat{\theta} = Y_n\ is\ m.l.e\ for\ \theta \quad (For\ this\ reason\ (Y_n) is\ closer\ than\ the\ rest\ of the\ (Y's).$$

m.l.e. in general; $\hat{b} = y_n$

To find unbiased to (Y_n, Y_1, $Y_n - Y_1$) from uniform distribution?

H.W: In a rss(n) from uniform dist^n C.U(0, θ). Is an estimator Y_n unbiased in limit estimator for θ.

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from C.U(0, θ), and $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics, show that Y_n is sufficient estimator for the parameter (θ).

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from C.U($\theta_1 - \theta_2$, $\theta_1 + \theta_2$), and $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics, show that Y_1 and Y_n are the jointly sufficient estimators for the parameters (θ_1, θ_2) respectively.

2) Gamma, Exponential and Chi-Square Distribution

Gamma Distribution: Gamma function and Generated distribution for it:

Gamma distribution has important applications in waiting time and reliability analysis. This is used for the length of time it takes to do something or for the time between events. When we study lifetime of machines or devices or any other issue that involves time. That is, the time factor is one of the factors. [e.g.: 1) The length of time the machines have been operating in the factory. 2) Study of machinery stops in a particular factory.]. Gamma distribution: the time required (r) occurrences to occur that follow the Poisson distribution.

To define the family of gamma distributions, we first need to introduce a function that plays an important role in many branches of mathematics, i.e., the Gamma Function:

Definition: For ($x > 0$) , ($\alpha > 0$), the gamma function $\Gamma(\alpha)$ is defined by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

Properties of the Gamma Function

1. For any ($\alpha > 1$), $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$. ($\alpha$ = positive integer) , e.g: $\Gamma(6) = 5\,\Gamma(5)$

2. For any ($\alpha > 0$), $\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha)$. If $\alpha = 1 \rightarrow \Gamma(1) = 1$, $\therefore 0! = 1$

3. For any positive integer (α), $\Gamma(\alpha) = (\alpha - 1)!$

Gamma Distribution

The time required for (r^{α}) events to occur that follow the Poisson distribution. The Gamma distribution. The gamma distribution is derived from the gamma function.

$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$,

1) $let,\ \ y = \frac{x}{\beta} \ \rightarrow dy = \frac{1}{\beta}\, dx$, $or;$ **2**) $let, y = \beta y \rightarrow dy = \beta\, dx$

$\Gamma(\alpha) = \int_0^{\infty} \frac{x^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{x}{\beta}} \frac{1}{\beta} dx$, $\left(divided\ by\ \Gamma(\alpha)\right)$

$1 = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$

$\therefore$ **1**) $f(x;\ \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$, $x > 0$, $\alpha, \beta > 0$

2) $f(x;\ \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} = \frac{\beta}{\Gamma(\alpha)} (\beta x)^{\alpha-1} e^{-\beta x}$

are Gamma density function and has Gamma distribution.

Properties of the Gamma Distribution (First Form)

1. *f*(*x*) is a p.d.f. of X.

2. The mean and the variance of X ~ Γ(α, *β*).

The second form of Gamma dist.

$f(x;\ \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, $\mathrm{E(X)} = \frac{\alpha}{\beta}$, $V(X) = \frac{\alpha}{\beta^2}$

3. The m.g.f. of X ~ Γ(α, *β*)

The second form of Gamma dist.

$$f(x;\ \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad , \quad M_x(t) = \frac{1}{\left(1-\frac{t}{\beta}\right)^{\alpha}} = \left(\frac{\beta}{\beta-t}\right)^{\alpha}$$

Another method for m.g.f. $= (1-\ \beta t)^{-\alpha}$

4. The c.d.f. of X ~ Γ(α, *β*). = 1- c.d.f of Poisson distribution

5. Additive Property

If $X_1, X_2, \ldots, X_n$ be (n) independent r.v's , and has a distribution Γ(α, β), i.e., $X_i \sim \Gamma(\alpha_i, \beta)$ then $Y = \sum_{i=1}^{n} X_i$ have a distribution $\Gamma(\sum_{i=1}^{n} \alpha_i, \beta) \sim \Gamma(n\alpha, \beta)$, and $\bar{X} \sim \Gamma(n\ \alpha, \frac{\beta}{n})$.

6. Mode of Gamma Distribution

H.W: $f(x;\ \alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$

$$f'(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left[-\frac{1}{\beta}\ x^{\alpha-1} e^{-\frac{x}{\beta}} + (\alpha-1)x^{\alpha-2} e^{-\frac{x}{\beta}}\right] \quad , \quad f'(x) = 0$$

$$-\frac{1}{\beta}\ x^{\alpha-1} e^{-\frac{x}{\beta}} + (\alpha-1)x^{\alpha-2} e^{-\frac{x}{\beta}} = 0$$

$$(\alpha-1)x^{\alpha-2} = \frac{1}{\beta}\ x^{\alpha-1} \quad [\div\ x^{\alpha-1}]$$

$$\frac{(\alpha-1)}{x} = \frac{1}{\beta} \quad , \rightarrow x = mode = \beta\ (\alpha-1)\ , \quad \alpha \geq 1$$

Maximum Likelihood Estimation

Let X ~ Γ(α, β). Find m.l.e of both parameters (α, β).

Inverse Gamma Distribution

Theorem: Let; X ~ Γ(α, β), ; $Y = \frac{1}{X}$ ~ Γ^{-1}(α, β)

Proof:

$$f(x;\ \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta\, x}$$

$$y = \frac{1}{x} \ \rightarrow x = y^{-1} \ , dx = -y^{-2} \quad , \quad \left|\frac{dx}{dy}\right| = y^{-2}$$

$$g(y;\ \alpha,\beta) = f(x;\ \alpha,\beta)\ \left|\frac{dx}{dy}\right|$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\beta/y} . y^{-2}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, y^{-(\alpha+1)}\, e^{-\beta/y} \rightarrow p.d.f.of\ \Gamma^{-1}(\alpha,\beta)$$

Mean and Variance of Inverse Gamma Distribution

When X ~ Γ^{-1}(α, β)

$$E(X) = \frac{\beta}{\alpha - 1} \qquad \text{Var(X)} = \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2} = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$$

Three Parameters Gamma Distribution

$$f(x;\ \alpha, \beta, \gamma) = \frac{\gamma\ \beta^{\alpha/\gamma}}{\Gamma\left(\frac{\alpha}{\gamma}\right)}\ x^{\alpha-1} e^{-\beta\ x^{\alpha}}$$

Special Cases of Gamma Distribution

First: Exponential Distribution

An exponential r.v is a continuous r.v that measures the **lifetime** of some events.

In this distribution the random variable can only take on positive values, and it's Right-Skewed distribution with maximum at $x = 0$.

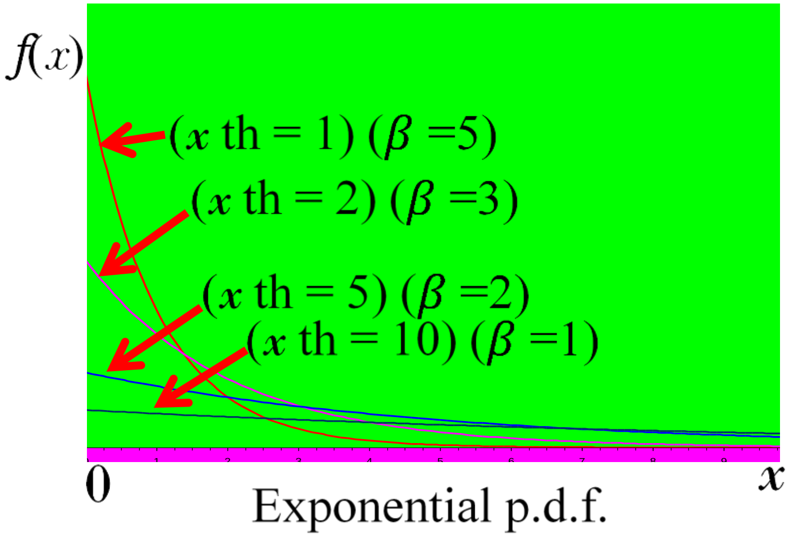
The exponential distribution can be used to model;

- The length of time between telephone calls.
- The mileage you get from one car of benzene.
- The length of time until a light bulb burns out.
- The length of time between arrivals at a service station (inter-arrival times).
- The lifetime of electronic components.

When the number of occurrences of an event follows the Poisson distribution, the time between occurrences follows the exponential distribution (distances for a Poisson process).

When (α = 1) in gamma distribution, then the gamma distribution reduces to the exponential distribution, then a random variable is exponentially distributed if its probability density function (p.d.f.) defined as;

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & , \ x > 0 \quad , \ \beta > 0 \\ 0 & o.w \end{cases}$$



Where β is a distribution parameter, and $\beta > 0$.

or we can get exponential distribution from Gamma function.

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy$$

1) The c.d.f. of X. X ~ Exp(1, β)

$$F(x) = p(X \le x) = \int_0^x \frac{1}{\beta} e^{-u/\beta} du = -e^{-u/\beta}\Big|_0^x$$

$$= -(e^{-x/\beta} - e^0) = \begin{cases} 0 & , \ x \le 0 \\ 1 - e^{-x/\beta} & , \ 0 < x < \infty \\ 1 & , \ x \to \infty \end{cases}$$

2) the mean and the variance of X. $mean = \beta$, $Var(X) = \beta^2$

3) The m.g.f. of X.

$$M_X(t) = (1 - \beta t)^{-1}$$

The Memory less Property: (forgets about its past)(Luck of Memory)

(Past history has no influence on the future)(The future is independent of the past): Let X is an Exponential random variable (as geometric r.v.) with parameter $\theta > 0$. Then X has the memory less property, which means that for any two real numbers (a, b > 0);

$$p(X > a + b \mid X > b) = p(X > a)$$

Consider the following statements

Relation between Exponential distribution and Poisson distribution

Q// How does the exponential distribution come from the Poisson distribution?

Maximum Likelihood Estimation

In a rss$n$ from exponential dist^n Exp($1/\theta$), find the m.l.e for θ.

Second: Chi-Square Distribution

When ($\alpha = r/2$), and ($\beta = 2$), where (r) is positive integer, then the gamma distribution reduces to the Chi-Square Distribution, with (r: positive integer) degrees of freedom; $X \sim \chi^2_{(r)}$ with p.d.f. and $\theta = \frac{1}{2}$ defined as;

$$f(x;r) = \begin{cases} \dfrac{1}{\Gamma(r/2)\, 2^{r/2}} \; x^{\frac{r}{2}-1} e^{-x/2} & , \; x > 0 \qquad , \; r > 0 \\ 0 & o.w \end{cases}$$

The mean and the variance of $X \sim \chi^2_{(r)}$ is; Mean = r Variance = 2r

The m.g.f. of $X \sim \chi^2_{(r)}$ is;

$$\because M_X(t) = (1-\beta t)^{-\alpha} = (1-2t)^{-r/2} = \left(\frac{1}{1-2t}\right)^{r/2} \quad , \; t < \frac{1}{2}$$

The second form of $X \sim \chi^2_{(r)}$ is; When ($\alpha = r/2$), and ($\beta = 1/2$),

$$E(X) = \frac{\alpha}{\beta} = \frac{r/2}{1/2} = r \;, var(X) = \frac{\alpha}{\beta^2} = \frac{r/2}{(\frac{1}{2})^2} = 2r \;, M_x(t) = \frac{1}{(1-2t)^{r/2}}$$

The c.d.f. of $X \sim \chi^2_{(r)}$ can't be found by direct integration, then;

$$p(0 < X < x_o) = \int_0^{x_o} \frac{1}{\Gamma(r/2)\, 2^{r/2}} \; x^{\frac{r}{2}-1} e^{-x/2} \, dx$$

Inverse Chi-Square Distribution

Theorem: Let; $X \sim \chi^2_{(r)}$, ; $Y = \frac{1}{X} \sim Inverse\ \chi^2_{(r)}$

$$= \frac{1}{\Gamma(r/2)2^{r/2}} y^{-(\frac{r}{2}+1)} e^{-1/2y} \rightarrow p.d.f.of\ Inverse\ \chi^2_{(r)}$$

Mean and Variance of Inverse Chi-Square Distribution

When $X \sim Inverse\ \chi^2_{(r)}$

$$f(x;\ r) = \frac{2^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} x^{-(\frac{r}{2}+1)} e^{-\frac{2}{x}} \rightarrow p.d.f.of\ Inverse\ \chi^2_{(r)}$$

$$mean = E(X) = \frac{\beta}{\alpha - 1} = \frac{2}{\frac{r}{2}-1}$$

$$\text{Var(X)} = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{2^2}{\left(\frac{r}{2}-1\right)^2\left(\frac{r}{2}-2\right)}$$

Mean and variance of Second form;

$$\text{Mean} = \frac{1}{(\alpha-1)\,\beta} = \frac{1}{\left(\frac{r}{2}-1\right)\times 2} = \frac{1}{r-2}$$

$$Var(Y) = \frac{1}{(\alpha-1)^2\,(\alpha-2)\beta^2} = \frac{1}{\left(\frac{r}{2}-1\right)^2\left(\frac{r}{2}-2\right)2^2}$$

3. Beta Distribution

Beta distribution, which is used to model percentages, proportions and in cases uncertainty, such as the proportion of lead in paint or the proportion of time that the FAX machine is under repair. The beta distribution is used as a prior distribution for Bernoulli, Binomial and Geometric proportions in Bayesian analysis. It is the special case of the Dirichlet distribution with only two parameters. Since the Dirichlet distribution is the conjugate prior of the multinomial distribution, the Beta distribution is the conjugate prior of the Binomial distribution.

Other examples of events that may be modeled by Beta distribution include:

1) The time it takes to complete a task.

2) The proportion of defective items in a shipment.

3) Batting averages in baseball.

4) Percentage of people with a disease in a country.

5) The distribution of activity times in project networks.

Definition: A r.v X is defined to have Beta Distribution. If the p.d.f. of X is given by:

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1\\ 0 & o.w\end{cases}$$

Where the shape parameters (α, β) are two positive integer, such that $\alpha, \beta > 0$:

$X \sim Beta\ (\alpha, \beta)$.

Beta function; $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

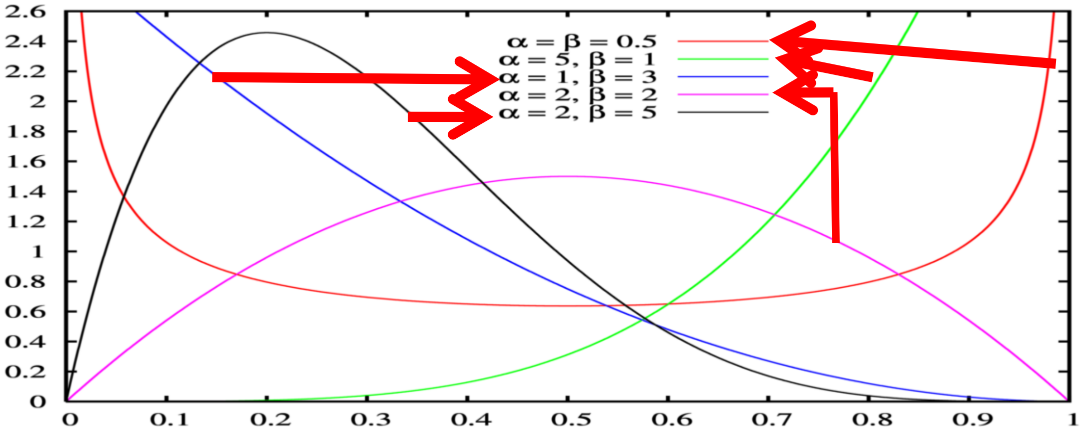
Then the p.d.f. of X is given by:

$$f(x;\alpha,\beta)=\begin{cases}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1\\ 0 & o.w\end{cases}$$

Special Values:

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}\quad,\quad B(1,1)=1\quad,\quad B(\tfrac{1}{2},\tfrac{1}{2})=\pi$$

Note: Special case: *Beta* (1, 1) → *U* (0, 1).



The p.d.f. of Beta Distribution

The Beta(1, 1) distribution is identical to the standard uniform distribution.

If X and Y are independently distributed Gamma(α, θ) and Gamma(β, θ) respectively, then $\frac{X}{(X+Y)}$ is distributed Beta (α, β).

Properties of Beta Distribution

1- $f(x)$ is a p.d.f. of X ~ Beta(α, β).

2- The Cumulative distribution function (c.d.f.) of X ~ *Beta* (α, β).

$$F(x) = p(X \le x) = \begin{cases} 0 & , x \le 0 \\ \int_0^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\, dx & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

3- The mean and the variance of X ~ Beta(α, β).

$$Mean = \frac{\alpha}{\alpha+\beta}, \quad \text{Var(X)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

3- The moment generating function of X ~ Beta(α, β).

$$\therefore M_X(t) = \sum_{k=0}^{\infty}\left[\frac{t^k}{k!}\ \frac{\beta(\alpha+k,\ \beta)}{\beta(\alpha,\ \beta)}\right] = 1 + \sum_{k=1}^{\infty}\left[\frac{t^k}{k!}\ \frac{\beta(\alpha+k,\ \beta)}{\beta(\alpha,\ \beta)}\right]$$

$$= 1 + \sum_{k=1}^{\infty}\left[\frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\right]$$

4- Mode of Beta Distribution

$$\rightarrow x = mode = \frac{(\alpha - 1)}{(\alpha + \beta - 2)}, \quad for\ \alpha > 1, \beta > 1$$

Dirichlet Distribution (Peter Gustav Lejeune Dirichlet)

The Dirichlet distribution is a generalization of the Beta distribution for multiple random variables. The Dirichlet distribution is over vectors whose values are all in the interval [0, 1] and the sum of values in the vector is (1). In other words, the vectors in the sample space of the Dirichlet distribution have the same properties as probability distributions. The Dirichlet distribution can be thought of as a "distribution over distributions".

The p.d.f. for a K-dimensional Dirichlet distribution (of order $K \geq 2$)(number of categories (integer)) has a vector of parameters denoted $\underline{\alpha}$ (α_1, ..., $\alpha_K > 0$) given by:

$$f(x_1, x_2, \dots, x_K;\ \alpha_1, \alpha_2, \dots, \alpha_K) = \frac{\Gamma\left(\sum_{i=1}^{K} \alpha_i\right)}{\prod_{i=1}^{K} \Gamma(\alpha_i)} \prod_{i=1}^{K} x_i^{\alpha_i - 1}$$

$\sum_{i=1}^{K} x_i = 1\ and\ x_i \in [0, 1]\ for\ all\ i \in \{1, \dots, K\}$.

The Dirichlet p.d.f. looks similar to the multinomial distribution.

- The Dirichlet density is proportional to: $\prod_{k=1}^{K} x_k^{\alpha_k - 1}$.
- The multinomial mass is proportional to: $\prod_{k=1}^{K} x_k^{\theta_k}$.

We conclude this analogy: Beta: Binomial :: Dirichlet: Multinomial.

4. Weibull Distribution

A random variable X is said to be distributed according to Weibull distribution if the p.d.f. is;

$$f(t;\alpha,\beta) = \alpha\,\beta\,t^{\beta-1}e^{-\alpha t^{\beta}} \quad ,t > 0 \quad ,\alpha,\beta > 0$$

α: scale parameter, is the characteristic life. β: Shape parameter [Determines the shape of the curve]. The (α, β) in Weibull dist. can represent decreasing, constant, or increasing failure rate.

If (β = 1) → $T \sim Exp(\alpha)$

Q// when use Weibull distribution in real-life?

Two main area this distribution be used are;

1) Lifetime Testing. **2)** Reliability.

T ⟶ Failure density $f(t)$

$R(t) = p(T > t) = 1 - F(t)$

Note: In the Weibull distribution: the distribution of failure resulting from a strong cause.

Reliability:

Probability of non-failure up to time (T = t) $\longrightarrow$ p(T > t)

$$f(t;\alpha,\beta) = \underbrace{\alpha\,\beta\,t^{\beta-1}}_{Z(t)}\ \underbrace{e^{-\alpha t^{\beta}}}_{R(t)}$$

Z(t) ← Hazard function

R(t)

$$R(t) = p(T > t) = \int_t^{\infty} \alpha\,\beta\,s^{\beta-1} e^{-\alpha s^{\beta}}\,ds = e^{-\alpha t^{\beta}}$$

Failure → Stochastically

Failure → Determinative

$$f(t;\ \alpha,\beta) = Z(t)\,.\,R(t)$$

$$F(t) = 1 - e^{-\alpha\,t}$$

$$R(t) = e^{-\alpha\,t}$$

$$f(t;\ \alpha) = \alpha \quad e^{-\alpha\,t}$$

Z(t) R(t)

Mean and Variance for Weibull Distribution

$$E(X^r) = \frac{\Gamma(\frac{r}{\beta}+1)}{\alpha^{r/\beta}} \quad ,\ r = 1,2,3,\ldots,n$$

$$E(X^1) = mean = \frac{\Gamma(\frac{1}{\beta}+1)}{\alpha^{1/\beta}} \quad , \qquad E(X^2) = \frac{\Gamma(\frac{2}{\beta}+1)}{\alpha^{2/\beta}}$$

$$Var(X) = \frac{\Gamma(\frac{2}{\beta}+1)}{\alpha^{2/\beta}} - \left(\frac{\Gamma\left(\frac{1}{\beta}+1\right)}{\alpha^{\frac{1}{\beta}}}\right)^2 = \frac{\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2}{\alpha^{2/\beta}}$$

How to extract the Weibull distribution from the gamma function

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$

$\Gamma(1) = \int_0^\infty e^{-x}\, dx$

$let; x = \alpha\, t^\beta \;\rightarrow\; dx = \alpha\beta t^{\beta-1} dt$

$\Gamma(1) = \int_0^\infty e^{-\alpha\, t^\beta}\, \alpha\beta t^{\beta-1} dt = 1$

$$f(t;\alpha,\beta) = \alpha\, \beta\, t^{\beta-1} e^{-\alpha t^\beta} \;\rightarrow\; Weibull\ Dist.$$

Maximum Likelihood Estimation (m.l.e) for Weibull Dist.

$f(t;\ \alpha,\beta) = Z(t)\,.\,R(t)$

$Z(t) = \alpha\beta t^{\beta-1}$

If;

1) $\beta = 1 \;\rightarrow\; Z(t) = \alpha$
2) $\beta = 2 \;\rightarrow\; Z(t) = 2\alpha\, t$
3) $\beta = 3 \;\rightarrow\; Z(t) = 3\alpha t^2$

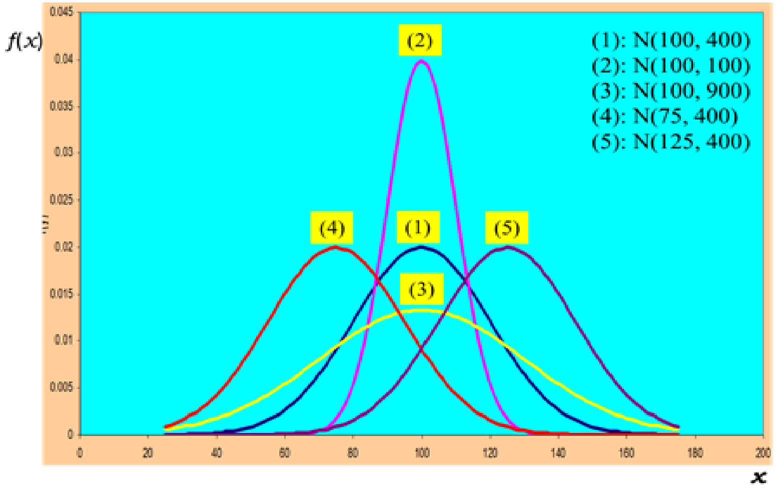
H.W: Find Mode and Median of Weibull Distribution

4. Normal (Gaussian) Distribution

The normal distribution is the cornerstone distribution of statistical inference. Many distributions can be approximated by a normal distribution. This distribution is considered the basis for the issue of monitoring the quality of production (quality control). The importance of this distribution is also highlighted by (the central limit theory), which proves that all probability distributions, whether discrete or continuous, approximate their distribution (according to certain conditions) to the normal distribution. Complex distributions may exist and therefore an approximation to a normal distribution can be used for them. In addition, all sampling distributions (Z, *t*, χ^2 and F) are derived primarily based on this distribution (Which assumes sampling from a normal distribution).

A r.v X is defined to be normal distribution, if the p.d.f. of X is given by:-

$$f(x;\mu,\sigma)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2} & ,-\infty<x<\infty\\ 0 & o.w\end{cases}$$



Where the parameter μ & σ must satisfy [$-\infty<\mu<\infty$, $\sigma>0$], and $\pi = 3.14285\ldots$, and e = 2.71828….., $X \sim N(\mu, \sigma^2)$.

Uses: 1) Biological statistics: Used to model many phenomena (spatially biological phenomena). If we calculate the ratio of male to female births in a specific area over a number of years, we will find that the distribution of this ratio follows a distribution similar to the natural distribution.
2) Organic measurements: Height and weight, for example, in a group of individuals with the same age, sex, and environment, are distributed in a distribution close to the normal distribution.
3) Social phenomena:

The special case of normal distribution, when ($\mu = 0$, and $\sigma^2 = 1$), then we called the resulting p.d.f. (Standard Normal Distribution), and denoted by; $Z \sim N(0, 1)$. Let;

$$z = \frac{x-\mu}{\sigma} \sim N(0,1) \quad \Rightarrow \therefore \quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad , \quad -\infty < z < \infty$$

Properties of Normal Distribution

1. $f(x)$ is a p.d.f. of X:

2. The mean and the variance of $X \sim N(\mu, \sigma^2)$. Mean $= \mu$, var(X) $= \sigma^2$.

3. The m.g.f. of $X \sim N(\mu, \sigma^2)$ is $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

4. The c.d.f. of $X \sim N(\mu, \sigma^2)$

$$F(x) = p(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(u-\mu)^2} du = N(x)$$

5. The c.d.f. of standard normal has been tabulated as follows; $X \sim N(0, 1)$.

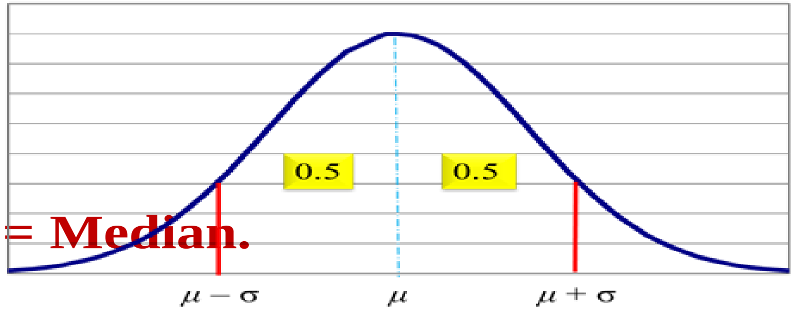
6. If X ~ N (μ , σ^2), then the r.v. $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ is called standard normal distribution.

7. The inflection points are ($\mu \pm \sigma$).

In the normal distribution, show the location of (σ) in an image.

8. The normal distribution curve is a bell shaped, and symmetrical around the mean μ.

$$f(x) = f(-x)$$



9. In normal distribution: Mean = Mode = Median.

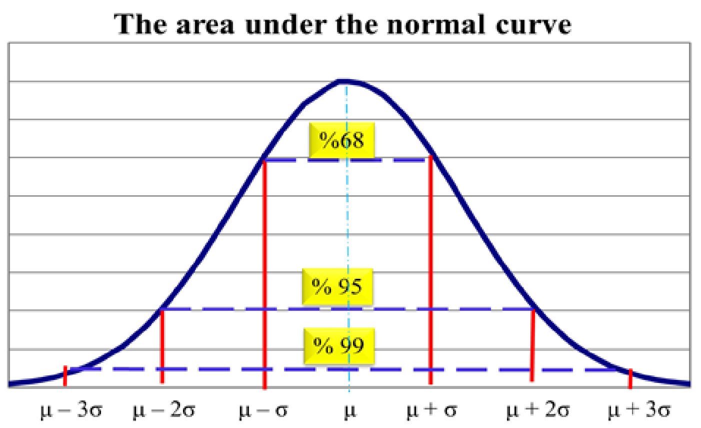
10. If X ~ N (μ, σ^2), then;

$$p(a < X < b) = \int_a^b f(x)\,dx = p\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$= p\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = N\left(\frac{b-\mu}{\sigma}\right) - N\left(\frac{a-\mu}{\sigma}\right)$$

11. The area under the normal distribution curve lies in the following;

$$p(\mu - \sigma < X < \mu + \sigma) = 0.68$$
$$p(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$$
$$p(\mu - 3\sigma < X < \mu + 3\sigma) = 0.99$$



H.W: Is necessary a consistent estimator to be unbiased estimator? Give an example.

In a random sample of size (n) from normal dist^n N(θ, σ^2), $S^2 = \frac{1}{n}\sum(X_i - \overline{X})^2$

Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from normal dist^n N(θ, σ^2), find m.l.e for parameters θ and σ^2.

Moments Estimation Method

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \quad , \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} \quad \Rightarrow \quad M_1 = E(X) = \theta$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \theta \quad \Rightarrow \therefore \hat{\theta} = \overline{X}$$

$$m_2 = \frac{\sum X_i^2}{n} \quad \Rightarrow \quad M_2 = E(X^2)$$

$$M_2 = E(X^2) = V(X) + \left(E(X)\right)^2 = \sigma^2 + \theta^2$$

$$m_2 = M_2$$

$$\frac{\sum X_i^2}{n} = \sigma^2 + \overline{X}^2$$

$$\therefore \hat{\sigma}^2 = \frac{\sum X_i^2}{n} - \overline{X}^2$$

Non-Informative prior probability

Find Bayes estimator for parameters of N(θ, σ^2) , using non informative prior probability.

1) $X \sim N(\theta, \sigma^2)$

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$L(\theta,\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

1) For θ

$$p(\theta \mid x_1, x_2, \ldots, x_n) \propto L(\theta)\ p(\theta)$$

$$L(\theta) \propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta+\bar{x}-\bar{x})^2} \quad \} \quad \mp \bar{x}$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\bar{x})^2}\ e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}$$

$$\therefore L(\theta) \propto e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}$$

$$\ln f(x;\theta,\sigma^2) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(x;\theta,\sigma^2)}{\partial\theta} = zero - \frac{2(x-\theta)(-1)}{2\sigma^2} = \frac{(x-\theta)}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial\theta^2} = -\frac{1}{\sigma^2}$$

$$-E\left(\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial\theta^2}\right)=\frac{1}{\sigma^2}=F.I.$$

$$p(\theta)\propto\left(I_s(\theta)\right)^{1/2}$$

$$\propto\left(\frac{1}{\sigma^2}\right)^{1/2}$$

$$\propto Cons\tan t$$

$$\propto 1$$

$$p(\theta\mid x_1,x_2,\ldots,x_n)\propto L(\theta)\,p(\theta)$$

$$\propto e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}\times(1)$$

$$\propto e^{-\frac{1}{2\sigma^2}n(\theta-\bar{x})^2}$$

$$p(\theta\mid x_1,x_2,\ldots,x_n)\sim N(\overline{X},\frac{\sigma^2}{n})$$

$$\because\ mean=\overline{X}\quad,\ \text{variance}=\frac{\sigma^2}{n}$$

$$\therefore\ p.d.f.\Rightarrow\ p(\theta\mid x_1,x_2,\ldots,x_n)=\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$

$$\therefore\hat{\theta}_{Bayes}=E\left(p(\theta\mid x_1,x_2,\ldots,x_n)\right)=\overline{X}$$

2) *For* σ^2

$$p(\sigma^2 \mid x_1, x_2, \ldots, x_n) \propto L(\sigma^2)\; p(\sigma^2)$$

$$L(\sigma^2) \propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

$$p(\sigma^2) \propto \left(I_s(\sigma^2)\right)^{1/2}$$

$$\ln f(x, \sigma^2) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(x; \sigma^2)}{\partial \sigma^2} = -\frac{1}{2}\,\frac{2\pi}{2\pi\sigma^2} + \frac{(x_i-\theta)^2\ (2)}{4\sigma^4} = -\ \frac{1}{2\sigma^2} + \frac{(x_i-\theta)^2}{2\sigma^4}$$

$$\frac{\partial^2 \ln f(x;\ \sigma^2)}{\partial (\sigma^2)^2} = \frac{2}{4(\sigma^2)^2} - \frac{(x_i-\theta)^2\ (4\sigma^2)}{4(\sigma^2)^4} = \frac{1}{2(\sigma^2)^2} - \frac{(x_i-\theta)^2}{(\sigma^2)^3}$$

$$-E\left(\frac{\partial^2 \ln f(x;\ \sigma^2)}{\partial(\sigma^2)^2}\right) = \frac{-1}{2(\sigma^2)^2} + \frac{E(x_i-\theta)^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{\sigma^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}$$

$$= \frac{-1+2}{2(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} = F.I.$$

$$p(\sigma^2) \propto (I_s(\theta))^{1/2}$$

$$\propto \left(\frac{1}{(\sigma^2)^2}\right)^{1/2}$$

$$\propto (\sigma^2)^{-1}$$

$$p(\sigma^2 \mid x_1, x_2, \ldots, x_n) \propto L(\sigma^2)\ p(\sigma^2)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2} \times (\sigma^2)^{-1}$$

$$\propto (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\sum (x_i-\theta)^2}$$

$$p(\sigma^2 \mid x_1, x_2, \ldots, x_n) \sim \Gamma^{-1}\left(\alpha = \frac{n}{2}\ ,\ \beta = \frac{\sum (x_i-\theta)^2}{2}\right)$$

$$\text{when}\ \ X \sim \Gamma^{-1}(\alpha, \beta)\ ,\quad f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}\quad ,\quad E(X) = \frac{\beta}{\alpha-1}$$

$$\therefore\ p.d.f. \Rightarrow\ p(\sigma^2 \mid x_1, x_2, \ldots, x_n) = \frac{\left(\frac{\sum (x_i-\theta)^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\left(\frac{\sum (x_i-\theta)^2}{2\sigma^2}\right)}$$

$$\therefore\ \hat{\sigma}^2_{Bayes} = E\left(\sigma^2 \mid x_1, x_2, \ldots, x_n\right) = \frac{\frac{\sum (X_i-\theta)^2}{2}}{\frac{n}{2}-1} = \frac{\sum (X_i-\theta)^2}{n-2}$$

Informative prior probability

Probability Distribution Normal $\sim N(\theta, \sigma^2)$ (θ known) → **Informative Prior Probability**
Inverse Gamma $\sim \Gamma^{-1}(\alpha_o/2, \beta_o/2)$

Probability Distribution Normal $\sim N(\theta, \sigma^2)$ (σ^2 known) → **Informative Prior Probability**
Normal $\sim N(\theta_o, \sigma_o^2)$

H.W: Find for $p(1/\sigma^2)$