Applied Engineering Analysis - 0111

Exponential, Hyperbolic and Trigonometric Relations

Exponential function	Trigonometric functions	Hyperbolic functions
$e^{x} = \cosh x + \sinh x$	$\cos x = (e^{ix} + e^{-ix})/2$ $\sin x = (e^{ix} - e^{-ix})/2i$	$\cosh x = (e^{x} + e^{-x})/2$ $\sinh x = (e^{x} - e^{-x})/2$

$$\cos^2\theta + \sin^2\theta = 1.$$
$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin A\cos B = \tfrac{1}{2}[\sin(A + B) + \sin(A - B)]$$
$$\cos A\sin B = \tfrac{1}{2}[\sin(A + B) - \sin(A - B)]$$
$$\cos A\cos B = \tfrac{1}{2}[\cos(A + B) + \cos(A - B)]$$
$$\sin A\sin B = -\tfrac{1}{2}[\cos(A + B) - \cos(A - B)]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

Homogenous Linear Equations of Second Order

The standard form of the second order linear equation is:

$$\boxed{y'' + p(t)y' + q(t)y = g(t).}$$

Where *p(t)*, *q(t)* and ***g****(t)* are functions. Homogeneous Equations: If g(t) = 0, then the equation above becomes:

$$y'' + p(t)y' + q(t)y = 0$$

It is called a homogeneous equation. Otherwise, the equation is non-homogeneous (or inhomogeneous). A differential equation is called linear if no products of the function and/or its derivatives occur

Second Order Linear Homogeneous Differential Equations with Constant Coefficients

For the most part, we will only learn how to solve second order linear equation with constant coefficients (that is, when *p(t)* and *q(t)* are constants). Since a homogeneous equation is easier to solve compares to its non-homogeneous counterpart, we start with second order linear homogeneous equations that contain constant coefficients only:

$$ay'' + by' + cy = 0$$

Where *a, b,* and *c* are constants, *a≠0*

Solution of the second order linear homogeneous equations with Constant coefficients

Note that in the standard form below

$$ay'' + by' + cy = 0, \qquad a \neq 0 \quad \text{-----------(1}$$

Let **$y = e^{rt}$** be a solution for some as-yet-unknown constant(r). Substitute y, $y' = re^{rt}$, and $y'' = r^2 e^{rt}$, into (1), we get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0,$$
$$e^{rt}(ar^2 + br + c) = 0$$

Since e^{rt} is never zero, the above equation is satisfied (and therefore $y = e^{rt}$ is a solution of (1)) if and only if $ar^2 + br + c=0$. Notice that the expression (ar^2+br+c) is a quadratic polynomial with r as the unknown. It is always solvable, with roots given by the quadratic formula. Hence, we can always solve a second order linear homogeneous equation with constant coefficients.

This equation, (ar^2+br+c), is called the characteristic or quadratic polynomial of the differential equation. The roots r_1,r_2 will be found by:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will take a more detailed look of the 3 possible cases of the solutions thusly found:

1. (When $b^2-4ac > 0$) There are two distinct (dissimilar) real roots r_1,r_2.
2. (When $b^2-4ac=0$) There is one repeated real root r.
3. (When $b^2-4ac < 0$) There are two complex conjugate roots $r = \lambda \pm \mu i$.

Case 1 Two distinct (dissimilar) real roots

When $b^2-4ac > 0$, the characteristic polynomial have two distinct real roots r_1,r_2. They give two distinct solutions with a general solution of:

$$\boxed{y = C_1y_1 + C_2y_2 = C_1e^{r_1t} + C_2e^{r_2t}.}$$

Case 2 One repeated (Similar) real root

When $b^2 - 4ac = 0$, the characteristic polynomial has a single repeated real root, $r_1 = r_2$, the general solution in the case of a repeated real root is:

$$\boxed{y = C_1 e^{rt} + C_2 t e^{rt}}$$

Case 3 Two complex conjugate roots

When $b^2 - 4ac < 0$, the characteristic polynomial has two complex roots, which are conjugates, $r_1 = \lambda - \mu i$ and $r_2 = \lambda + \mu i$ (λ, μ are real numbers, $\mu > 0$), and its general solution is:

$$\boxed{y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t.}$$

When $r = \lambda \pm \mu i$, $\mu > 0$, are two complex roots. In mathematics, complex conjugates are a pair of complex numbers, both having the same real part, but with imaginary parts of equal magnitude and opposite signs. For example, $3 + 4i$ and $3 - 4i$ are complex conjugates

Examples on Homogeneous Linear Second Order Differential Equations

1) Find the general solution of the following second order liner homogeneous Differential equations:

I)) $x'' + 8x' + 7x = 0$ **II)) $y'' + 2.6y' + 1.69y = 0$**

2) Solve the following second order liner homogeneous Differential equations:

- ***$y'' - 4y' + 4y = 0$, when $y(0) = 4$, $y'(0) = 5$***
- ***$y'' + 2y' + 5y = 0$, when $y(0) = 4$, $y'(0) = 6$***
- ***$y'' + y' - 2y = 0$, when $y(0) = 7$, $y'(0) = 1$***

NON-HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

The general second-order linear non-homogeneous differential equation with constant coefficients is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Where a; b; and c are some constants, and ***f(x)*** is some function of x. The procedure of solving the preceding equation is:

<u>The first step</u> is to find the general solution of the homogeneous equation [$f(x) = 0$]. This gives us the "complementary function" y_{cf} .

<u>The second step</u> is to find a particular solution y_{ps} of the full equation

$$\boxed{\text{General solution} = y_{CF} + y_{PS}}$$

<u>Particular Solution Forms</u>

$f(x)$	Form of y_{PS}
k (a constant)	C
linear in x	$Cx + D$
quadratic in x	$Cx^2 + Dx + E$
$k \sin px$ or $k \cos px$	$C \cos px + D \sin px$
ke^{px}	Ce^{px}
sum of the above	sum of the above
product of the above	product of the above

Examples on Nonhomogeneous Linear Second Order Equations

1. **For the following exercises find the general solution of the Inhomogeneous Linear Second Order Equations:**

- $y'' - 3y' + 2y = -4$
- $y'' - 4y' + 3y = 2x$
- $y'' + 6y' + 9y = 9\cos(3x)$
- $y'' - y' - 6y = 8e^{2x}$
- $y'' - 4y' + 13y = 3e^{2x} - 5e^{3x}$

2. **Solve the following differential equations:**

- $y'' - 2y' + y = e^{t}$ $\quad y(0)=1,\ y'(0)=0$
- $y'' + 4y' = 8 + 34\cos(x)$ $\quad y(0)=3,\ y'(0)=2$
- $y'' + y' - 12y = -7e^{3x}$ $\quad y(0)=7,\ y'(0)=0$

Higher Order/n^{th} Order Differential Equations

Homogeneous, Linear, with constant coefficients

The solutions of linear differential equations with constant coefficients of the third order or higher can be found in similar ways as the solutions of second order linear equations. For an n-th order homogeneous linear equation with constant coefficients:

$$a_n y^{(n)'} + a_{n-1} y^{(n-1)'} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0, \; an \neq 0$$

It has a general solution of the form

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_{n-1} y_{n-1} + C_n y_n$$

Where $y_1, y_2, \ldots, y_{n-1}, y_n$ are any n linearly independent solutions of the equation. Such a set of linearly independent solutions, and therefore, a general solution of the equation, can be found by first solving the differential equation's characteristic equation:

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 = 0.}$$

This is a polynomial equation of degree n, therefore, it has n real and/or complex roots (not necessarily distinct). Those necessary n linearly independent solutions can then be found using the four rules below.

(i). If r is a distinct real root, then $y = e^{rt}$ is a solution.

(ii). If $r = \lambda \pm \mu i$ are distinct complex conjugate roots, then $y = e^{\lambda t}\cos \mu t$ and $y = e^{\lambda t}\sin \mu t$ are solutions.

(iii). If r is a real root appearing k times, then $y = e^{rt}$, $y = te^{rt}$, $y = t^2 e^{rt}$, … , and $y = t^{k-1} e^{rt}$ are all solutions.

(iv). If $r = \lambda \pm \mu i$ are complex conjugate roots each appears k times, then

$$\begin{array}{ll} y = e^{\lambda t}\cos \mu t, & y = e^{\lambda t}\sin \mu t, \\ y = te^{\lambda t}\cos \mu t, & y = te^{\lambda t}\sin \mu t, \\ y = t^2 e^{\lambda t}\cos \mu t, & y = t^2 e^{\lambda t}\sin \mu t, \\ \vdots & \vdots \\ y = t^{k-1} e^{\lambda t}\cos \mu t, \text{ and } & y = t^{k-1} e^{\lambda t}\sin \mu t, \end{array}$$

are all solutions.

Examples: Solve the following Differential Equations.

1. $\frac{d^3y}{dx^3} + 25\frac{dy}{dx} = 0$

2. $\frac{d^4y}{dx^4} - 18\frac{d^2y}{dx^2} + 81y = 0$

3. $\frac{d^5y}{dx^5} - 3\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 0$

4. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = 0$ $y(0) = 4, y'(0) = -7, y''(0) = 23$

5. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 10y = 0$ $y(0)=-3, y'(0)=12, y''(0)=-36$