Laplace Transform

The **Laplace Transform** is widely used in **engineering applications** (mechanical and electronic), especially where the driving force is discontinuous. It is also used in process control.

What Does the Laplace Transform Do?

The main idea behind the Laplace Transformation is that we can solve an equation (or system of equations) containing differential and integral terms by transforming the equation in ***t-space*** (or other spaces) to one in ***s-space***. This makes the problem much easier to solve. The kinds of problems where the Laplace Transform is invaluable occur in electronics.

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \qquad \text{..........1}$$

The parameter s is a complex number:

$s = \sigma + i\omega$, with real numbers σ and ω.

Laplace transforms of elementary functions

Using the definition of the Laplace transform in equation (1) a number of elementary functions may be transformed. For example:

(a) $\boldsymbol{f(t) = 1}$. From equation (1),

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}[e^{-s(\infty)} - e^0] = -\frac{1}{s}[0-1]$$

$$= \frac{\mathbf{1}}{\boldsymbol{s}} \text{ (provided } s > 0\text{)}$$

and $\mathcal{L}\{k\} = k\left(\frac{1}{s}\right) = \frac{\boldsymbol{k}}{\boldsymbol{s}}$

(c) $\boldsymbol{f(t) = e^{at}}$ (where a is a real constant $\neq 0$).

From equation (1),

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt,$$

from the laws of indices,

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{-(s-a)}(0-1)$$

$$= \frac{\mathbf{1}}{\boldsymbol{s-a}}$$

(provided $(s-a) > 0$, i.e. $s > a$)

No.	*f(t)*	*F(s)*
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^n (n=1, 2, 3, …)	$\frac{n!}{s^{n+1}}$
4	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
5	e^{at}	$\frac{1}{s-a}$
6	$t\,e^{at}$	$\frac{1}{(s-a)^2}$
7	$t^n\,e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
8	$\frac{1}{a-b}(e^{at}-e^{bt})$	$\frac{1}{(s-a)(s-b)}$
9	$\frac{1}{a-b}(ae^{at}-be^{bt})$	$\frac{s}{(s-a)(s-b)}$
10	$\frac{(c-b)e^{at}+(a-c)e^{bt}+(b-a)e^{ct}}{(a-b)(b-c)(c-a)}$	$\frac{1}{(s-a)(s-b)(s-c)}$
11	$\sin(at)$	$\frac{a}{s^2+a^2}$
12	$\cos(at)$	$\frac{s}{s^2+a^2}$
13	$1-\cos(at)$	$\frac{a^2}{s(s^2+a^2)}$
14	$at-\sin(at)$	$\frac{a^3}{s^2(s^2+a^2)}$

No.	*f(t)*	*F(s)*
15	$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
16	$\sin(at) + at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
17	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
18	$t\cos(at)$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
19	$\frac{\cos(at)-\cos(bt)}{(b-a)(b+a)}$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
20	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
21	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
22	$\sinh(at)$	$\frac{a}{s^2-a^2}$
23	$\cosh(at)$	$\frac{s}{s^2-a^2}$
24	$\sin(at)\cosh(at) - \cos(at)\sinh(at)$	$\frac{4a^3}{(s^4+4a^4)}$
25	$\sin(at)\sinh(at)$	$\frac{2a^2s}{(s^4+4a^4)}$
26	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
27	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$

Problems: Using a Laplace table, determine

- $\mathcal{L}\{6t - 3 + 9t^6\}$
- $\mathcal{L}\{e^{-t} + 3e^t\}$
- $sinht + cosh3t$
- $\mathcal{L}(2cos2\theta - sin3\theta)$
- ***3sin (ω t+α),* where ω and α are constants.**
- sin^2t
- $cosh^2 3t$ **if** $\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$
- $\mathcal{L}\{2t^4e^{3t}\}$
- $\mathcal{L}\{4e^{3t}cos5t\}$
- $\mathcal{L}\{e^{-2t}sin3t\}$
- $\mathcal{L}\{3e^{\theta}cosh4\theta\}$
- $e^{-t}(cosht - 3sinh2t)$
- $2e^{2t}(sin3t - 9cost)$

The Laplace transforms of derivatives

(a) First derivative

Let the first derivative of $f(t)$ be $f'(t)$ then, from equation (1),

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)\, dt$$

After solving the integration by the by part technique, we will get:

$$\left.\begin{aligned} &\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \\ \text{or} \quad &\mathcal{L}\left\{\frac{dy}{dx}\right\} = s\mathcal{L}\{y\} - y(0) \end{aligned}\right\}$$

………………. (2)

Where y(0) is the value of y at x =0.

(b) Second derivative

Let the second derivative of f (t) be f "(t), then from equation (1),

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

After solving the integration by the by part technique, we will get:

$$\left.\begin{aligned} &\mathcal{L}\{f''(t)\} \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \text{or}\quad &\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} \\ &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) \end{aligned}\right\} \qquad \text{..................... (3)}$$

Where y (0) is the value of y at x =0 and y'(0) is the value of *dy/dx* at *x=0*.

Equations (2) and (3) are used in the solution of differential equations and simultaneous differential equations

Inverse Laplace transforms

Definition of the inverse Laplace transform

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is written as $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

For example, since $\mathcal{L}\{1\} = \frac{1}{s}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$.

Similarly, since $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ then

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at, \text{ and so on.}$$

$$f(t) = \mathcal{L}^{-1}\{F\}(s) = \mathcal{L}_s^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\,F(s)\,ds$$

Ex: Find the following inverse Laplace transforms:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$ 2. $\mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\}$ 3. $\mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\}$ 4. $\mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\}$

5. $\mathcal{L}^{-1}\left\{\frac{4s}{s^2-16}\right\}$ 6. $\mathcal{L}^{-1}\left\{\frac{7s}{s^2+4}\right\}$ 7. $\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\}$ 8. $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\}$

9. $\mathcal{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\}$ 10. $\mathcal{L}^{-1}\frac{2s+5}{s^2+4s-5}$ 11. $\mathcal{L}^{-1}\frac{3s+2}{s^2-8s+25}$ 12. $\mathcal{L}^{-1}\frac{2(s-3)}{s^2-6s+13}$

Solving Inverse of Laplace transform using partial fractions

Sometimes the function whose inverse is not recognizable as a standard type, such as those listed in the Laplace table. In such cases it may be possible, by using partial fractions, to resolve the function into simpler fractions which may be inverted on sight.

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x-b)}+\frac{C}{(x+c)}$
2	Repeated linear factors	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
3	Quadratic factors	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$

Ex: Determine:

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s(s-3)}\right\},\quad \mathcal{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\},\ \text{and}\ \mathcal{L}^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$$

Exercise

Use partial fractions to find the inverse Laplace transforms of the following functions:

1. $\frac{11-3s}{s^2+2s-3}$ $[2e^{t}-5e^{-3t}]$
2. $\frac{2s^2-9s-35}{(s+1)(s-2)(s+3)}$ $[4e^{-t}-3e^{2t}+e^{-3t}]$
3. $\frac{5s^2-2s-19}{(s+3)(s-1)^2}$ $[2e^{-3t}+3e^{t}-4e^{t}t]$
4. $\frac{3s^2+16s+15}{(s+3)^3}$ $[e^{-3t}(3-2t-3t^2)]$
5. $\frac{7s^2+5s+13}{(s^2+2)(s+1)}$ $\left[2\cos\sqrt{2}t+\frac{3}{\sqrt{2}}\sin\sqrt{2}t+5e^{-t}\right]$
6. $\frac{3+6s+4s^2-2s^3}{s^2(s^2+3)}$ $[2+t+\sqrt{3}\sin\sqrt{3}t-4\cos\sqrt{3}t]$
7. $\frac{26-s^2}{s(s^2+4s+13)}$ $[2-3e^{-2t}\cos 3t-\frac{2}{3}e^{-2t}\sin 3t]$

The Solution of Differential Equations Using Laplace Transforms

Procedures to solve differential equations by using Laplace transform:

(i) Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives.

(ii) Put in the given initial conditions, i.e. y (0) and y' (0).

(iii) Rearrange the equation to make L {y} the subject.

(iv) Determine (y) by using, where necessary, partial fractions, and taking the inverse of each term by using the Table.

Ex: Use Laplace transforms to solve the differential equations:

(i) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$, given that when $x=0$, $y=3$ and $\frac{dy}{dx}=7$.

(ii) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}=9$, given that when $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

(iii) Using Laplace transforms, solve the following differential equations:

$$\frac{d^2y}{dt^2}-1\frac{dy}{dt}-2y=e^{-t}cosh2t$$

Given the initial conditions that when t = 0, y = 0 and $\left(\frac{dy}{dt}=0\right)$.

(iv) Using Laplace transform, solve the following second order D.E.

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+1y=\frac{1}{\operatorname{sech}(t)}$$

Given the initial conditions that when t = 0, y = 1 and $\left(\frac{dy}{dt}=0\right)$.

(v) Using Laplace transform, solve the following second order D.E.

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=3cosh3t-11sinh3t$$

Given the initial conditions that when t = 0, y = 0 and $\left(\frac{dy}{dt}=6\right)$.

(vi) Using Laplace transforms, solve the following differential equations:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 1 + 2e^x$$

Given the initial conditions that when x = 0, y = 1 and $\left(\frac{dy}{dx} = 2\right)$.

(vii) Using Laplace transforms, solve the following differential equations:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3x^2$$

Given the initial conditions that when x = 0, y = 0 and $\left(\frac{dy}{dx} = 0\right)$.

The solution of simultaneous differential equations using Laplace transforms

It is sometimes necessary to solve simultaneous differential equations. An example occurs when two electrical circuits are coupled magnetically where the equations relating the two currents i_1 and i_2 are typically:

$$L_1\frac{di_1}{dt}+M\frac{di_2}{dt}+R_1i_1=E_1$$

$$L_2\frac{di_2}{dt}+M\frac{di_1}{dt}+R_2i_2=0$$

Where L represents inductance, R resistance, M mutual inductance and E1 the p.d. applied to one of the circuits.

Procedure to solve simultaneous differential equations using Laplace transforms

i) Take the Laplace transform of both sides of each simultaneous equation by applying the formulae for the Laplace transforms of derivatives and using a list of standard Laplace transforms.

ii) Put in the initial conditions, i.e. x(0), y(0), x'(0), y'(0).

iii) Solve the simultaneous equations for L{y} and L{x} by the normal algebraic method.

iv) Determine y and x by using, where necessary, partial fractions, and taking the inverse of each term.

Ex: Solve the following pairs of simultaneous differential equations:

1. $2\frac{dx}{dt}+\frac{dy}{dt}=5e^t$

   $\frac{dy}{dt}-3\frac{dx}{dt}=5$

   given that when $t=0$, $x=0$ and $y=0$

2. $\frac{d^2x}{dt^2}-x=y$

   $\frac{d^2y}{dt^2}+y=-x$

   given that at $t=0$, $x=2$, $y=-1$, $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$.

3. $\frac{dy}{dt}+x=1$

   $\frac{dx}{dt}-y+4e^t=0$

   given that at $t=0$, $x=0$ and $y=0$.

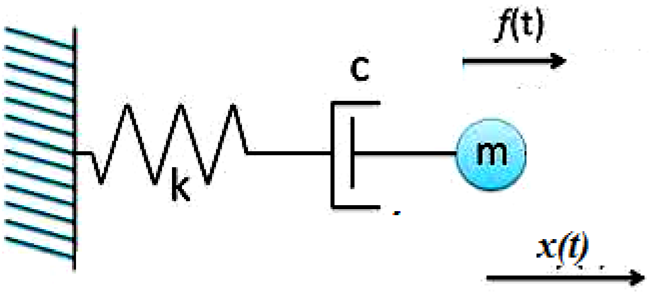
4. $3\frac{dx}{dt}-5\frac{dy}{dt}+2x=6$

   $2\frac{dy}{dt}-\frac{dx}{dt}-y=-1$

   given that at $t=0$, $x=8$ and $y=3$.

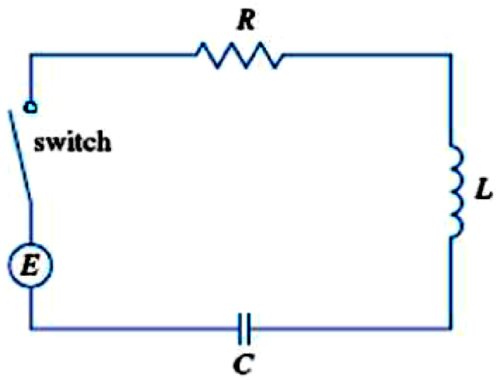
Applications on Second order Differential Equations

➢ Mechanical (Vibration) System



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

➢ Electrical System



$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{c}Q = E(t)$$

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

Ex: A spring-mass-dashpot system consists of a 1-kg mass attached to a spring with spring constant k=10 N/m; the dashpot has damping constant 7 kg/s: At time t = 0, the system is set into motion by pulling the mass 0.5 m from its equilibrium rest position while simultaneously giving it an initial velocity of 1 m/s. Using the Laplace transform, solve systems equation.

Ex: The displacement s of a body in a damped mechanical system, satisfies the following differential equation:

$$2\frac{d^2s}{dt^2} + 6\frac{ds}{dt} + 4.5s = 0$$

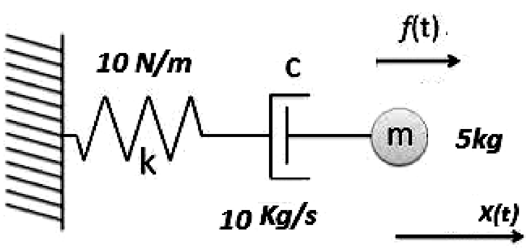
Where t represents time. If initially, when t =0, s=0 and ds/dt=4, solve the differential equation for s in terms of t.

Ex: Oscillations of a heavily damped pendulum satisfy the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = te^{2t}$$

Where *x* cm is the displacement of the bob at time *t* seconds. The initial *(when t=0*) displacement is equal to +4 cm and the initial velocity (dx/dt) is *8 cm/s*. Using the Laplace Transforms method, solve the equation for *x*.

Ex: For the system shown, drive the second order differential equation and find the general solution of it, if *f(t)=cos(t)*.



Ex: The charge q in an electric circuit at time t satisfies the equation:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E,$$

Where L, R, & C are constants. Solve the equation given L=2 H, R=200 Ω, C = 200×10^{-6} F , and E=250sinh(3t) V. Assume that when t=0, q=2 and dq/dt= 0. Using the Laplace Transforms method, solve the equation for q.

Ex: In a galvanometer the deflection θ satisfies the differential equation:

$$\frac{d^2\theta}{dt^2}+2\frac{d\theta}{dt}+\theta=0$$

Use Second Order Linear Homogeneous Differential Equations method to solve the equation for θ given that when t = 0, θ = 0 and dθ/dt =1.