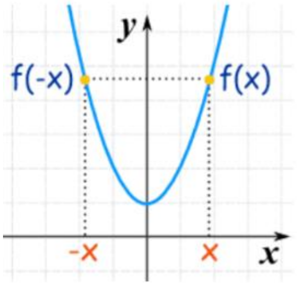
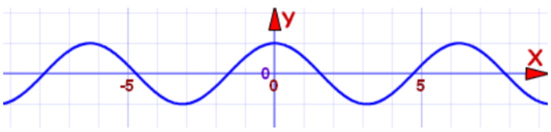
Fourier series

Introduction

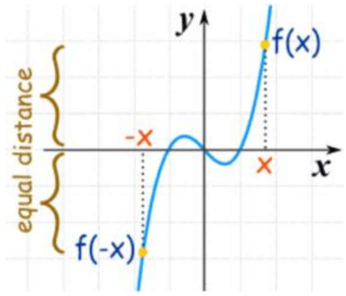
Even and odd functions

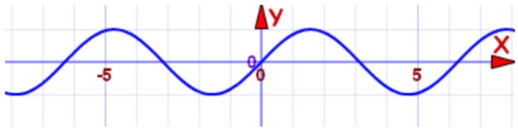
A function is "***even***" when is symmetry about y-axis and $\boxed{b_n=0}$. For eg. $f(x)= x^2 +1$ and $f(x)=\cos(x)$



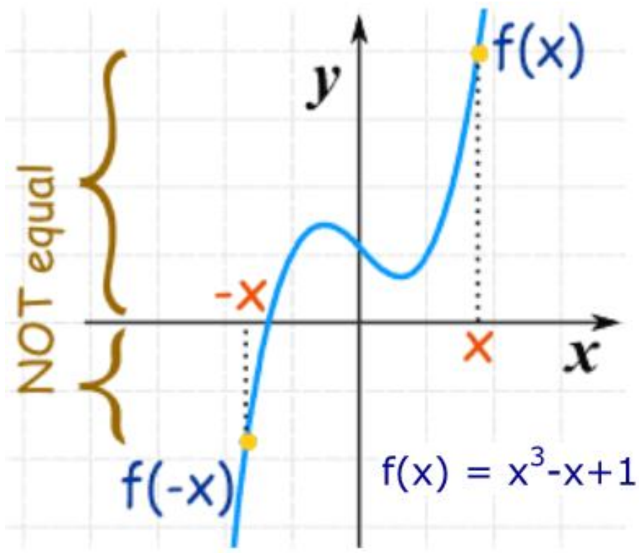


A fn. is "***odd***" when is symmetry about origin and $\boxed{a_n=0}$, and $\boxed{a_o=0}$. For eg. $f(x)= x^3 -x$ and $f(x)=\sin(x)$

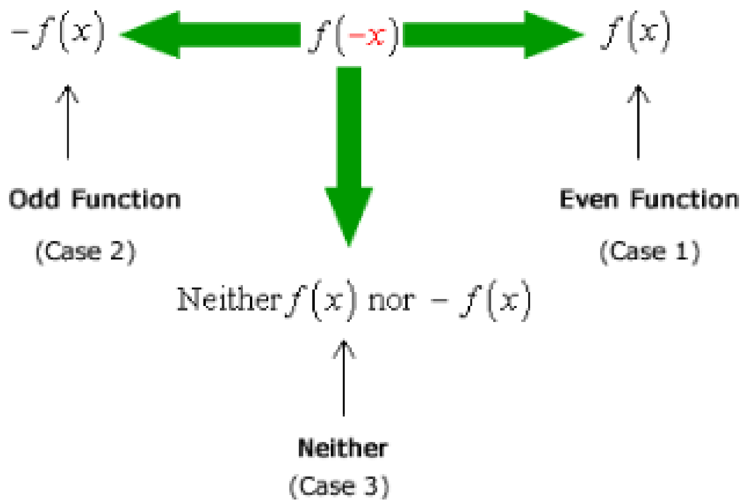




Don't be misled by the names "odd" and "even" ... **they are just names** ... and a function does **not have to be even or odd.** In fact most functions are neither odd nor even. For example:

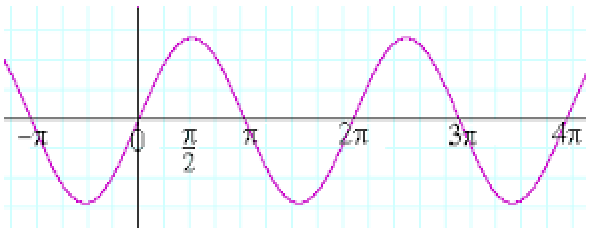


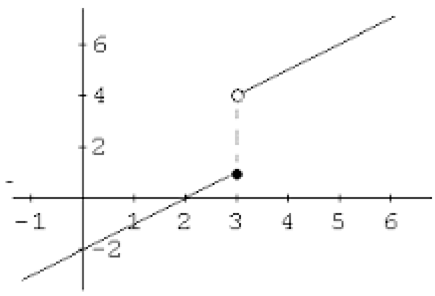
Or we can represent the three cases as below:

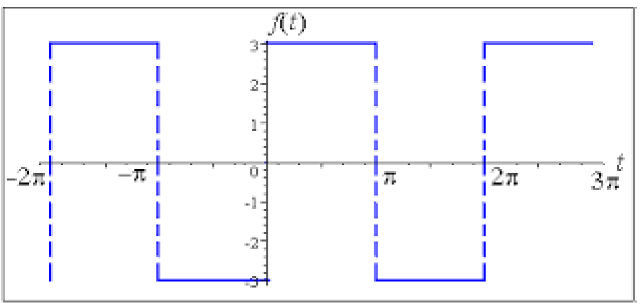


Periodic functions

A function $f(x)$ is said to be **periodic** if $f(x+T)=f(x)$ for all values of x, where T is some positive number. T is the interval between two successive repetitions and is called the **period** of the functions $f(x)$. For example, $y=\sin x$ is periodic in x with period 2π since $\sin x = \sin(x+2\pi) = \sin(x+4\pi)$, and so on.
If a graph of a function has no sudden jumps or breaks it is called a **continuous function**, examples being the graphs of sine and cosine functions. However, other graphs make finite jumps at a point or points in the interval. The square wave has finite discontinuities at $x=\pi, 2\pi, 3\pi$, and so on.







Fourier Series Definition

It is possible to form any function f(x) as a summation of a series of sine and cosine terms of increasing frequency.

In other words, any space or time varying data can be transformed into a different domain called the ***frequency space***. A fellow called Joseph Fourier first came up with the idea in the 19th century, and it was proven to be useful in various applications, mainly in signal processing. A great advantage of Fourier series over other series is that it can be applied to functions which are discontinuous as well as those which are continuous.

Introduction and Background Information

In the mid eighteenth century, physical problems such as the conduction of heat and the study of vibrations and oscillations led to the study of Fourier series. Of central interest was the problem of how arbitrary real valued functions could be represented by sums of simpler functions. As we shall see later, a Fourier series is an infinite sum of trigonometric functions that can be used to model real valued, periodic functions.

A Fourier series decomposes a periodic function or periodic signal into a sum of simple oscillating functions, namely sins and cosines (or complex exponentials). The original motivation was to solve the heat equation in a metal plate, which is a partial differential equation. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding Eigen solutions. This superposition or linear combination is called the Fourier series.

It later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems. The Fourier series has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, , displacement, velocity and acceleration of slider-crank mechanisms econometrics… etc.

Fourier series over the range of 2π

(i) The basis of a Fourier series is that all functions of practical significance which are defined in the interval $-\pi \leq x \leq \pi$ can be expressed in terms of a convergent trigonometric series of the form:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

When $a0, a1, a2, \ldots b1, b2, \ldots$ are real constants, i.e.

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)} \qquad (1)$$

Where for the range 2π (a↔b = 2π):

$$a_o = \frac{1}{2\pi}\int_a^b f(x)dx, \quad a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\, dx, \ and \ \ b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\, dx$$

When n=1,2,3,…….

(ii) a_0, *an* and *bn* are called the **Fourier coefficients** of the series and if these can be determined, the series of equation (1) is called the **Fourier series** corresponding to $f(x)$.

Fourier series over any range

A periodic function $f(x)$ of period $2L$ repeats itself when x increases by $2L$, $f(x+2L) = f(x)$. The change from functions dealt with previously having period 2π to functions having period $2L$ is not difficult since it may be achieved by a change of variable. Hence the Fourier series expressed in terms of x is given by:

$$\boldsymbol{F(x) = a_o + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}\right)x + b_n \sin\left(\frac{n\pi}{L}\right)x\right]}$$

Where for the range $2L$ (a↔b = **$2L$**) , eg. from ***–L to +L*** or from ***0 to 2L***

For the integration limits of interval of length 2L the equations will be:

$$a_o = \frac{1}{2L}\int_a^b f(x)\ dx,$$

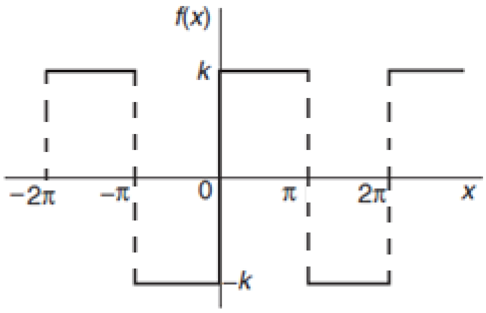
$$a_n = \frac{1}{L}\int_a^b f(x)\ \cos\left(\frac{n\pi}{L}\right)x\ dx,$$

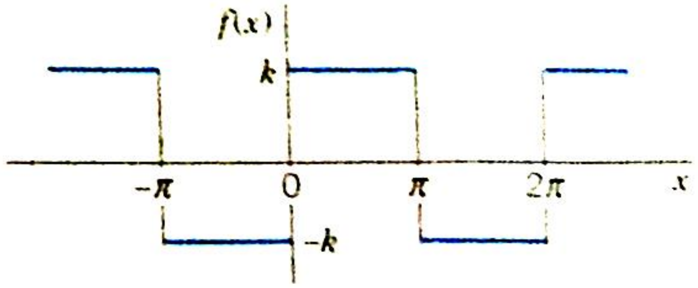
$$b_n = \frac{1}{L}\int_a^b f(x)\ \sin\left(\frac{n\pi}{L}\right)x\ dx,$$

a_0, *an* and *bn* are **Fourier coefficients**

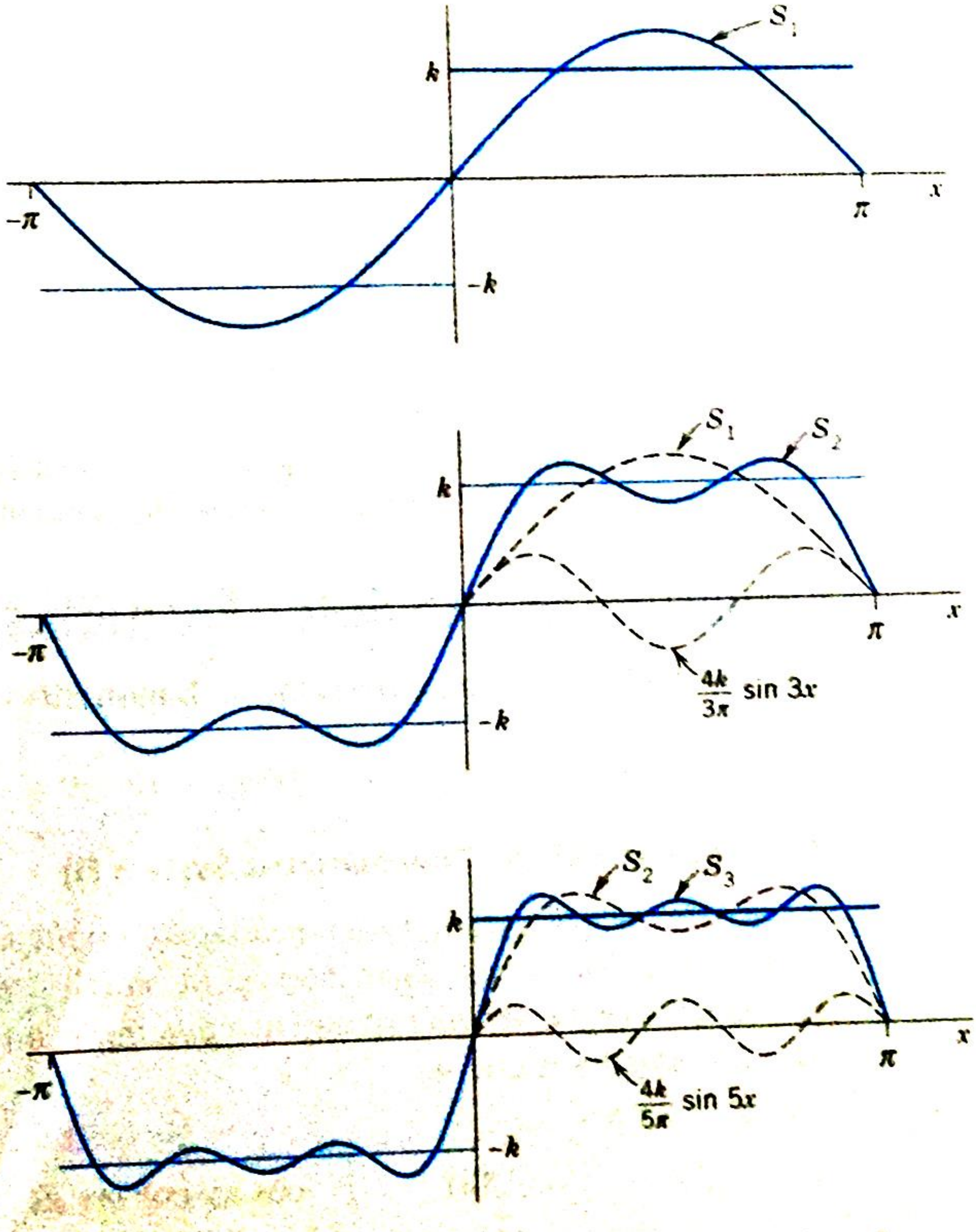
Ex1: Obtain a Fourier series for the periodic function $f(x)$ defined as:

$$f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ +k, & \text{when } \quad 0 < x < \pi \end{cases}$$

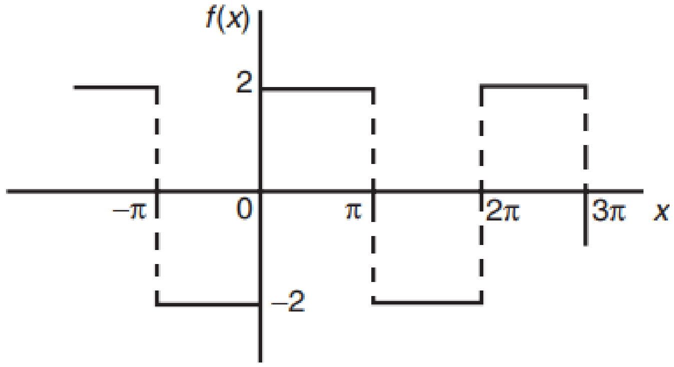




(a) The given function f(x) (Periodic rectangular wave)



Ex2: Obtain the Fourier series for the square wave shown in the figure below from (-π to π).



Ex3: Obtain the Fourier series expansion of the periodic function $f(t)$ of period 2π defined by $f(t) = t$ $(0 < t < 2\pi)$.

Ex4: Determine the Fourier series for the function $f(x)=x^2$ in the range $-\pi<x<\pi$. The function has a period of 2π.

Ex5: The voltage from a square wave generator is of the form:

$$v(t) = \begin{cases} 0, & -4 < t < 0 \\ 10, & 0 < t < 4 \end{cases}$$

Find the Fourier series for this periodic function.

Ex6: Determine the Fourier series for the function $f(x)=x+1$ in the range $0 < x < 2\pi$.

Ex7: Obtain the Fourier series expansion of the periodic function $f(x)$ defined by $f(x) = x$, $(0 < x < 3)$.

Ex8: Determine the Fourier series for the periodic function defined by:

$$f(x) = \begin{cases} -2, & \text{when } -\pi < x < -\frac{\pi}{2} \\ 2, & \text{when } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -2, & \text{when } \frac{\pi}{2} < x < \pi. \end{cases}$$

Ex9: Obtain the Fourier series for the square wave shown in the figure.