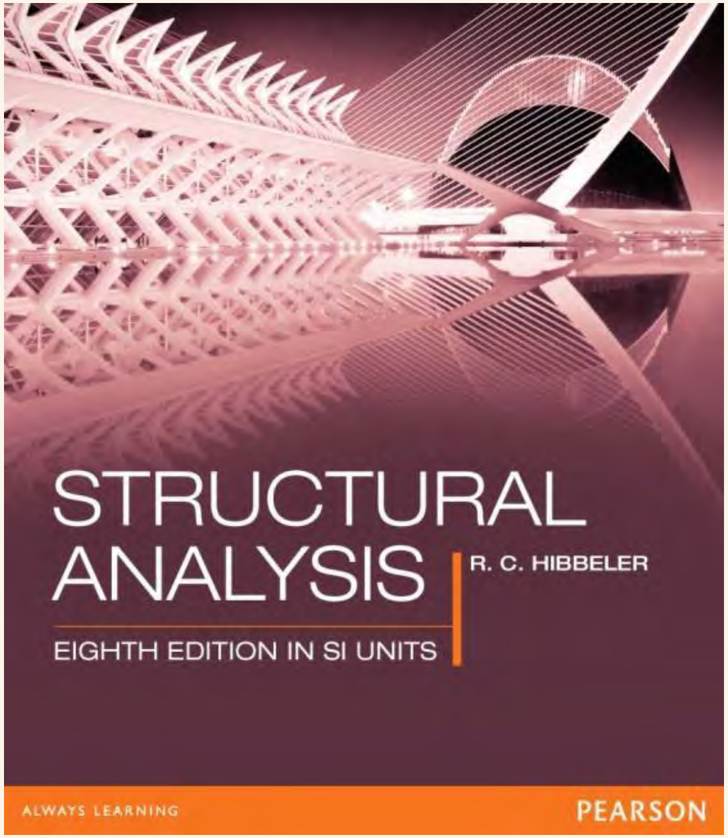
ALWAYS
LEARNING

STRUCTURAL ANALYSIS

EIGHTH EDITION IN SI UNITS

R. C. HIBBELER



CHAPTER 11:
DISPLACEMENT METHOD OF ANALYSIS: SLOPE-DEFLECTION EQUATIONS

11

Chapter Outline

11.1

DISPLACEMENT METHOD OF ANALYSIS: GENERAL PROCEDURES

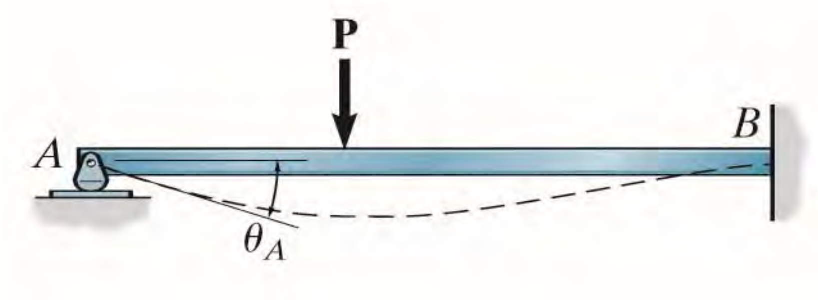
11.1

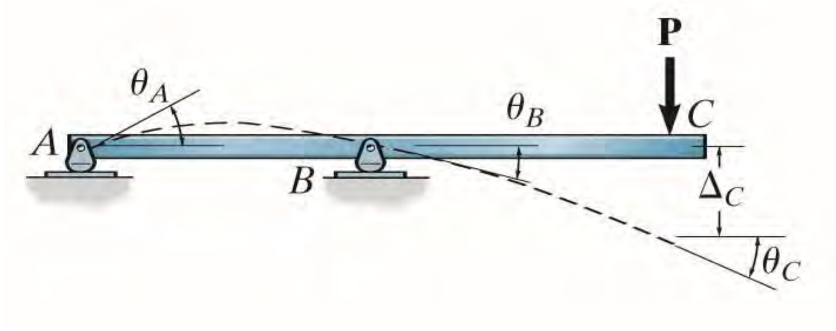
Displacement Method of Analysis: General Procedures

- Displacement (disp) method requires satisfying equilibrium eqn for the structures
- The unknown disp are written in terms of the loads by using the load-disp relations
- These eqn are solved for the disp
- Once the disp are obtained, the unknown loads are determined from the compatibility eqn using the load-disp relations
- When a structure is loaded, specified points on it called nodes, will undergo unknown disp
- These disp are referred to as the degree of freedom
- The no. of these unknowns is referred to as the degree in which the structure is kinematically indeterminate
- We will consider some examples

Displacement Method of Analysis: General Procedures

- Any load applied to the beam will cause node *A* to rotate
- Node *B* is completely restricted from moving
- Hence, the beam has only one unknown degree of freedom





- The beam has nodes at *A*, *B* & *C*
- There are 4 degrees of freedom θ_A, θ_B, θ_C and Δ_C

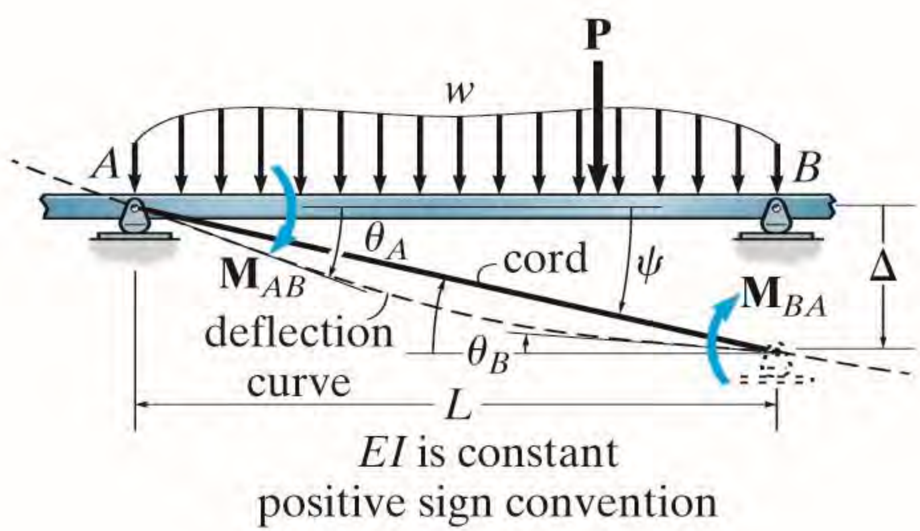
11.2

SLOPE –DEFLECTION EQUATIONS

11.2

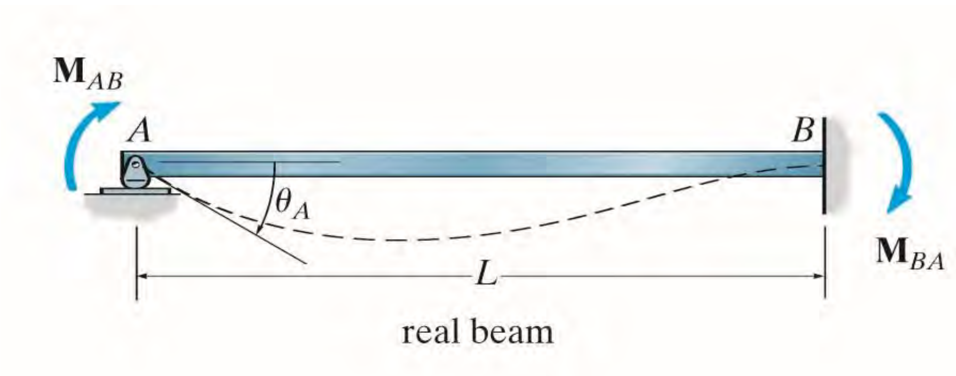
Slope-Deflection Equations

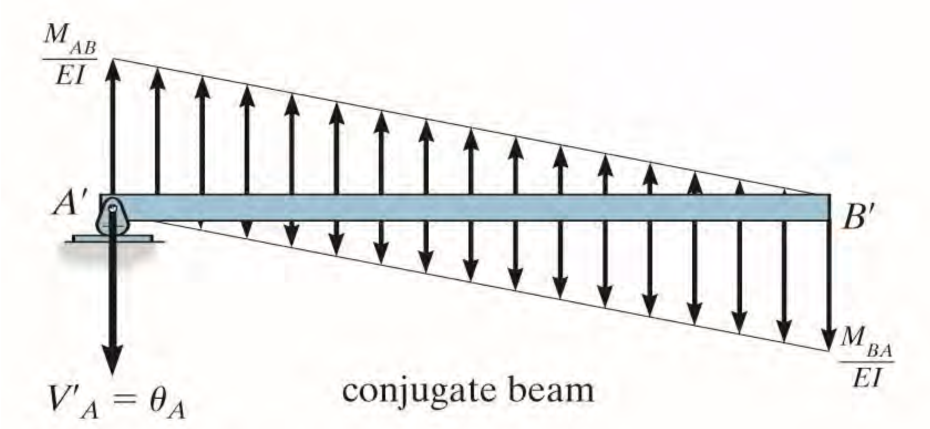
- Slope deflection method requires less work both to write the necessary eqn for the solution of a problem & to solve these eqn for the unknown disp & associated internal loads
- General Case
  - To develop the general form of the slope-deflection eqn, we will consider the typical span *AB* of the continuous beam when subjected to arbitrary loading



Slope-Deflection Equations

- Angular Displacement
  - Consider node *A* of the member to rotate θ_A while its end node *B* is held fixed
  - To determine the moment M_{AB} needed to cause this disp, we will use the conjugate beam method





Slope-Deflection Equations

- Angular Displacement

$$\sum M_{A'} = 0$$

$$\left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\sum M_{B'} = 0$$

$$\left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

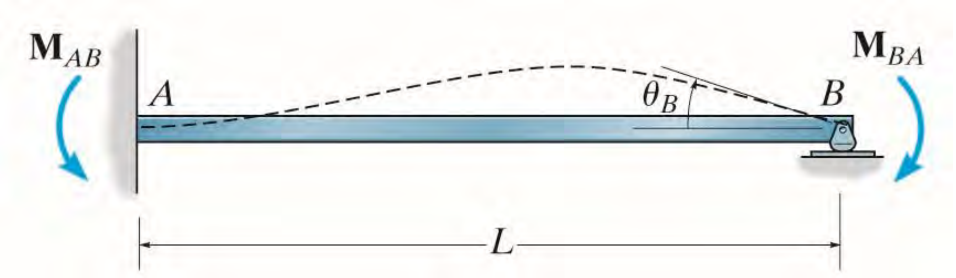
Slope-Deflection Equations

- Angular Displacement
  - From which we obtain the following:

$$M_{AB} = \frac{4EI}{L}\theta_A \qquad M_{BA} = \frac{2EI}{L}\theta_A$$

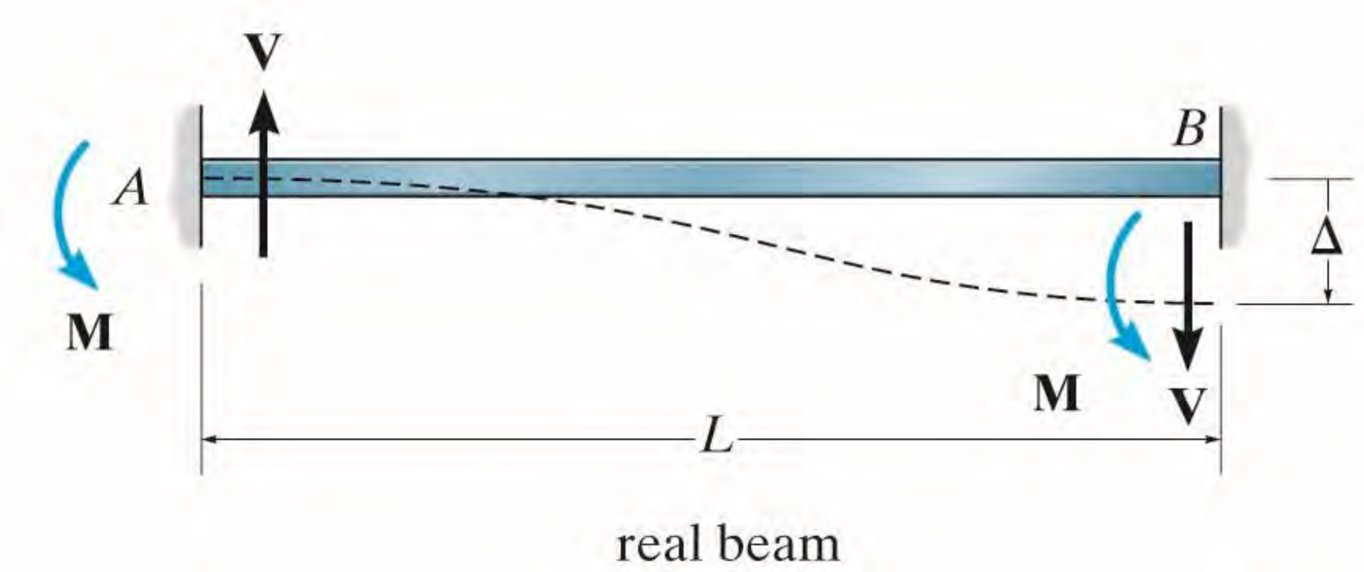
  - Similarly, when end B of the beam rotates to its final position while end A is held fixed, we can relate the applied moment M_{BA} to the angular disp θ_B & the reaction moment M_{AB} at the wall

$$M_{BA} = \frac{4EI}{L}\theta_B \qquad M_{AB} = \frac{2EI}{L}\theta_B$$



Slope-Deflection Equations

- Relative Linear Displacement
  - If the far node *B* of the member is displaced relative to *A*, so that the cord of the member rotates clockwise & yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member
  - Moment M can be related to the disp using conjugate beam method



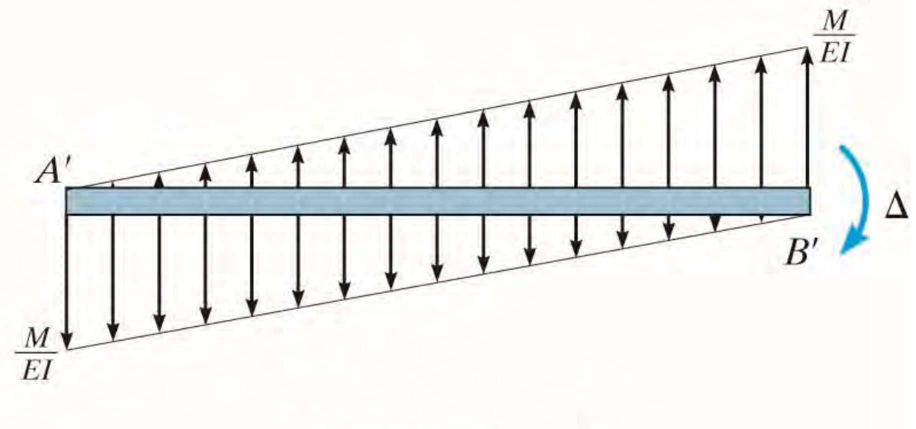
Slope-Deflection Equations

- Relative Linear Displacement
  - The conjugate beam is free at both ends since the real member is fixed supported
  - The disp of the real beam at *B*, the moment at end *B'* of the conjugate beam must have a magnitude of Δ as indicated

$$\sum M_{B'} = 0$$

$$\left[\frac{1}{2}\frac{M}{EI}(L)\frac{2L}{3}\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\frac{L}{3}\right] - \Delta = 0$$

$$M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta$$



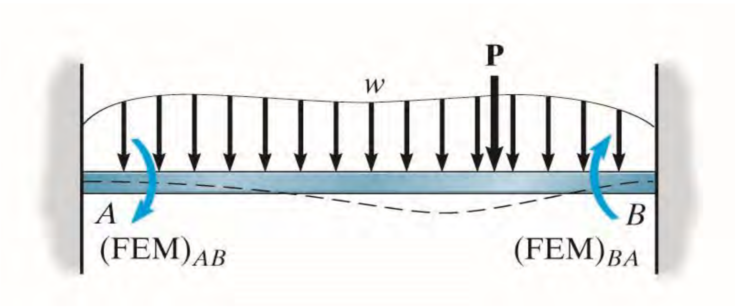
conjugate beam

Slope-Deflection Equations

- Fixed end moments
  - In general, linear & angular disp of the nodes are caused by loadings acting on the span of the member
  - To develop the slope-deflection eqn, we must transform these span loadings into equivalent moment acting at the nodes & then use the load-disp relationships just derived
  - If the end moments due to each disp & loading are added together, the resultant moments at the ends can be written as:

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + FEM_{AB} = 0$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + FEM_{BA} = 0$$



Slope-Deflection Equations

- Fixed end moments
  - The results can be expressed as a single eqn

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + FEM_N$$

M_N = internal moment at the near end of the span

E, k = modulus of elasticity & span stiffness

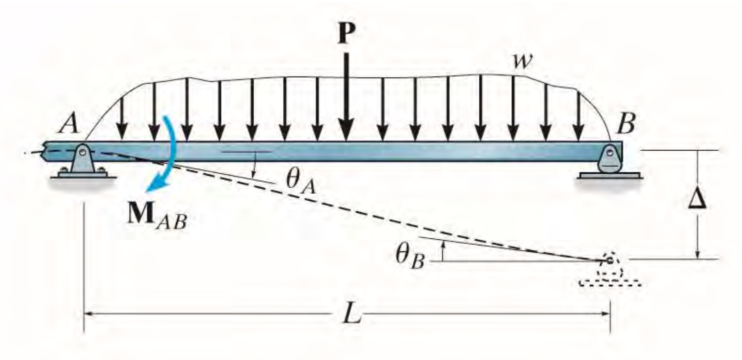
θ_N, θ_F = near and far end slopes or angular disp of the span at the supports

ψ = span rotation of its cord due to a linear disp

FEM_N = fixed end moment at the near end support

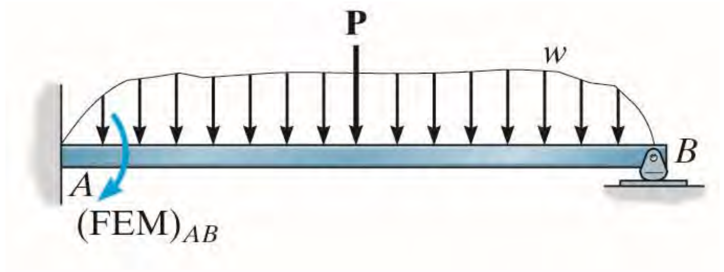
Slope-Deflection Equations

- Pin supported end span
  - Sometimes an end span of a beam or frame is supported by a pin or roller at its far end
  - The moment at the roller or pin is zero provided the angular disp at this support does not have to be determined



$$M_N = 2Ek[2\theta_N + \theta_F - 3\psi] + FEM_N$$

$$0 = 2Ek[2\theta_N + \theta_F - 3\psi] + 0$$



Slope-Deflection Equations

- Pin supported end span
  - Simplifying, we get:

$$M_N = 3Ek\left[\theta_N - \psi\right] + FEM_N$$

  - This is only applicable for end span with far end pinned or roller supported

11.3

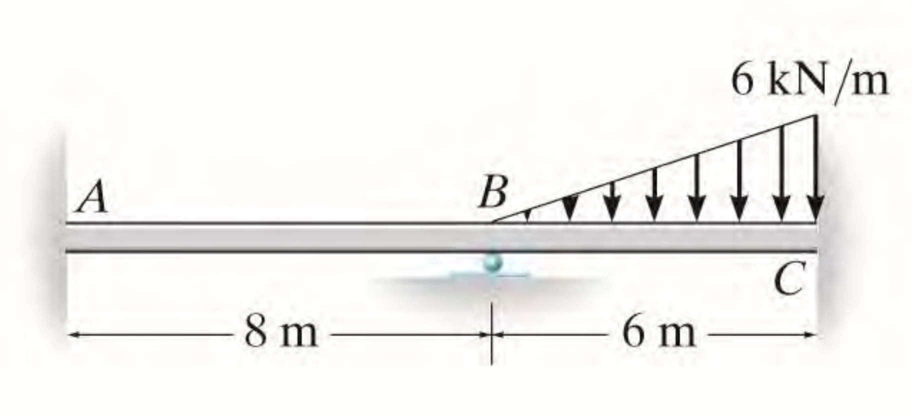
ANALYSIS OF BEAMES

11.3

Analysis of Beams

Example 11.1

Draw the shear and moment diagrams for the beam where *EI* is constant.



Analysis of Beams

Example 11.1 (Solution)

Slope-deflection equations

2 spans must be considered in this problem

Using the formulas for FEMs, we have:

$$(FEM)_{BC} = -\frac{wL^2}{30} = -\frac{6(6^2)}{30} = -7.2\,\text{kN}\bullet\text{m}$$

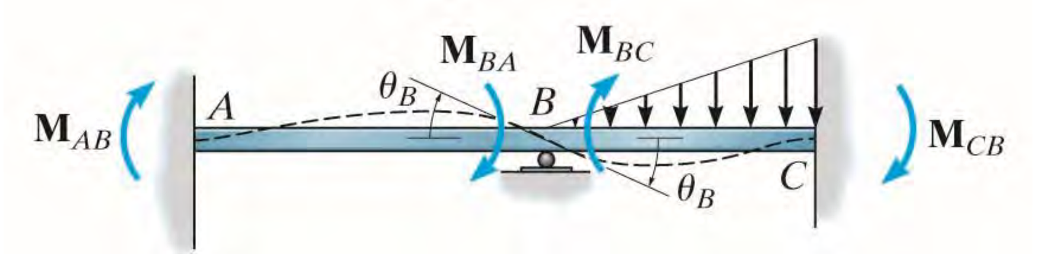
$$(FEM)_{CB} = \frac{wL^2}{20} = \frac{6(6^2)}{20} = 10.8\,\text{kN}\bullet\text{m}$$

Analysis of Beams

Example 11.1 (Solution)

Slope-deflection equations

- Note that $(FEM)_{BC}$ is –ve and
- $(FEM)_{AB}$ = $(FEM)_{BA}$ =0 since there is no load on span *AB*
- Since *A* & *C* are fixed support, $\theta_A = \theta_C = 0$
- Since the supports do not settle nor are they displaced up or down, $\psi_{AB} = \psi_{BC} = 0$



$$M_N = 2E\left(\frac{I}{L}\right)\left[2\theta_N + \theta_F - 3\psi\right] + FEM_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)\left[2(0) + \theta_B - 3(0)\right] + 0 = \frac{EI}{4}\theta_B \quad (1)$$

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Example 11.1 (Solution)

Slope-deflection equations

Similarly,

$$M_{BA} = \frac{EI}{2}\theta_B \qquad (2)$$

$$M_{BC} = \frac{2EI}{3}\theta_B - 7.2 \qquad (3)$$

$$M_{CB} = \frac{EI}{3}\theta_B + 10.8 \qquad (4)$$

Analysis of Beams

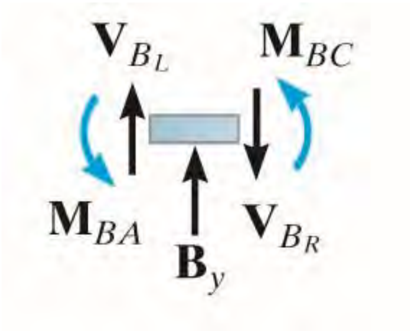
Example 11.1 (Solution)

Equilibrium equations

The necessary fifth eqn comes from the condition of moment equilibrium at support *B*

Here M_{BA} & M_{BC} are assumed to act in the +ve direction to be consistent with the slope-deflection eqn

$$M_{BA} + M_{BC} = 0 \qquad (5)$$



Analysis of Beams

Example 11.1 (Solution)

Equilibrium equations

$$\text{Sub eqn (2) and (3) into eqn (5) gives:}$$

$$\theta_B = \frac{6.17}{EI}$$

$$\text{Re-sub this value into eqn (1) to (4) gives:}$$

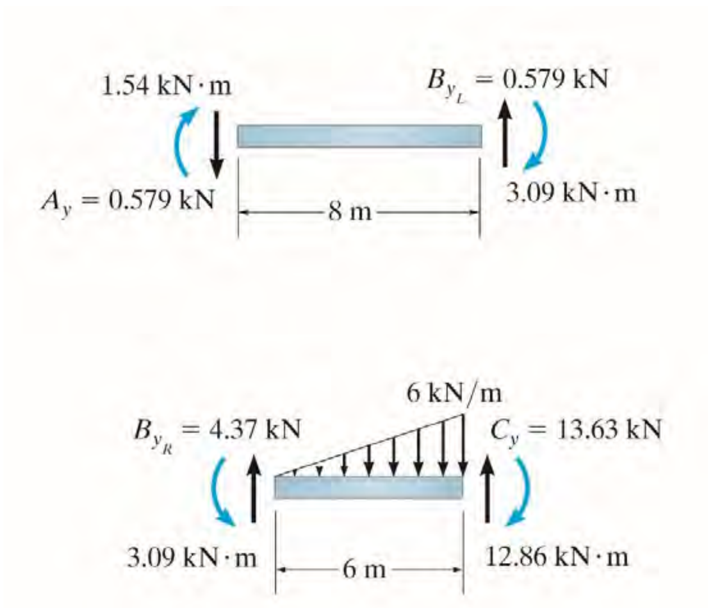
$$M_{AB} = 1.54\,\text{kN} \bullet \text{m}; \quad M_{BA} = 3.09\,\text{kN} \bullet \text{m};$$
$$M_{BC} = -3.09\,\text{kN} \bullet \text{m}; \; M_{CB} = 12.86\,\text{kN} \bullet \text{m}$$

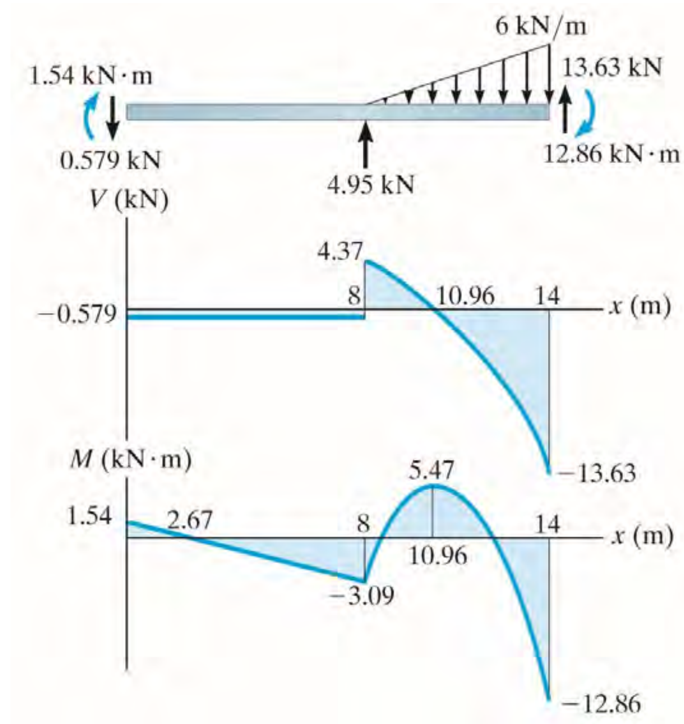
Analysis of Beams

Example 11.1 (Solution)

Equilibrium equations

- Using these results, the shears at the end spans are determined.
- The free-body diagram of the entire beam & the shear & moment diagrams are shown





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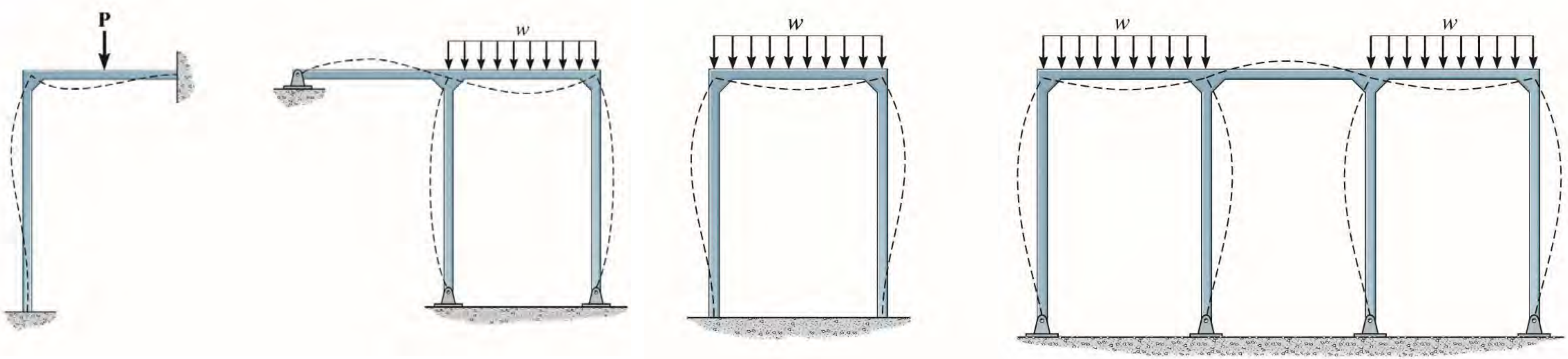
11.4

ANALYSIS OF FRAMES: NO SIDESWAY

11.4

Analysis of Frames: No Sidesway

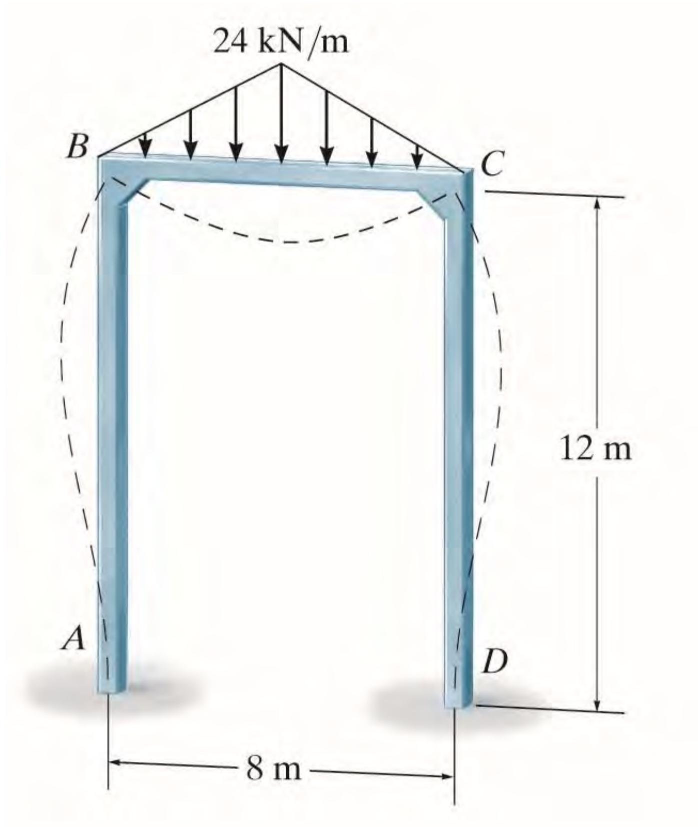
- A frame will not sidesway to the left or right provided it is properly restrained
- No sidesway will occur in an unrestrained frame provided it is symmetric wrt both loading and geometry



Analysis of Frames: No Sidesway

Example 11.5

Determine the moments at each joint of the frame. *EI* is constant.



Analysis of Frames: No Sidesway

Example 11.5 (Solution)

Slope-deflection equations

3 spans must be considered in this case: *AB*, *BC* & *CD*

$$(FEM)_{BC} = -\frac{5wL^2}{96} = -80\,\text{kN}\bullet\text{m}$$

$$(FEM)_{CB} = \frac{5wL^2}{96} = 80\,\text{kN}\bullet\text{m}$$

Note that $\theta_A = \theta_D = 0$ and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$

Analysis of Frames: No Sidesway

Example 11.5 (Solution)

Slope-deflection equations

We have

$$M_N = 2E\left(\frac{I}{L}\right)\left[2\theta_N + \theta_F - 3\psi\right] + FEM_N$$

$$\begin{aligned}
M_{AB} &= 0.1667EI\theta_B \\
M_{BA} &= 0.333EI\theta_B \\
M_{BC} &= 0.5EI\theta_B + 0.25EI\theta_C - 80 \\
M_{CB} &= 0.5EI\theta_C + 0.25EI\theta_B + 80 \\
M_{CD} &= 0.333EI\theta_C \\
M_{DC} &= 0.1667EI\theta_C
\end{aligned}$$

Analysis of Frames: No Sidesway

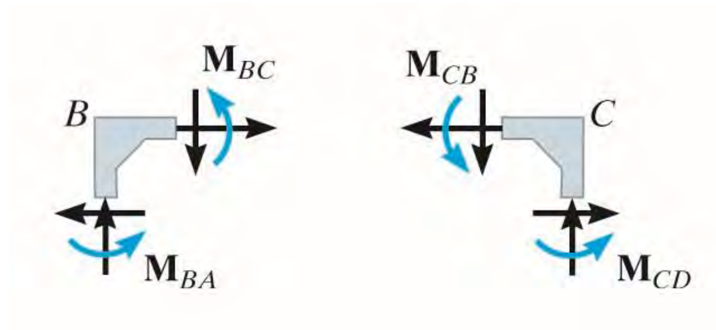
Example 11.5 (Solution)

Slope-deflection equations

The remaining 2 eqn come from moment equilibrium at joints *B* & *C*, we have:

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$



Solving for these 8 eqns, we get:

$$0.833EI\theta_B + 0.25EI\theta_C = 80$$

$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

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Analysis of Frames: No Sidesway

Example 11.5 (Solution)

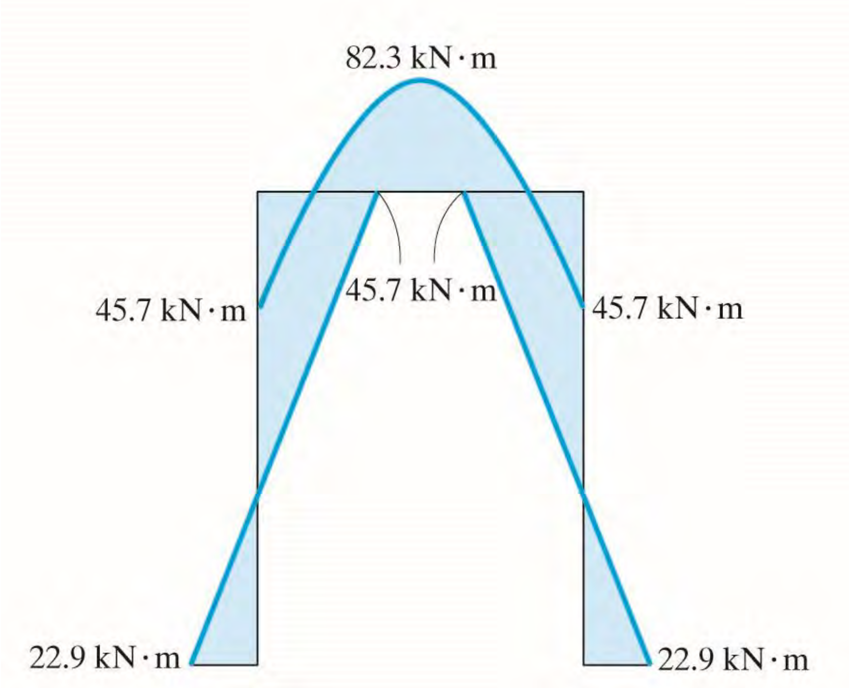
Slope-deflection equations
Solving simultaneously yields:

$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

$$M_{AB} = 22.9\,\text{kN}\bullet\text{m}; \quad M_{BA} = 45.7\,\text{kN}\bullet\text{m}$$

$$M_{BC} = -45.7\,\text{kN}\bullet\text{m}; \quad M_{CB} = 45.7\,\text{kN}\bullet\text{m}$$

$$M_{CD} = -45.7\,\text{kN}\bullet\text{m}; \quad M_{DC} = -22.9\,\text{kN}\bullet\text{m}$$



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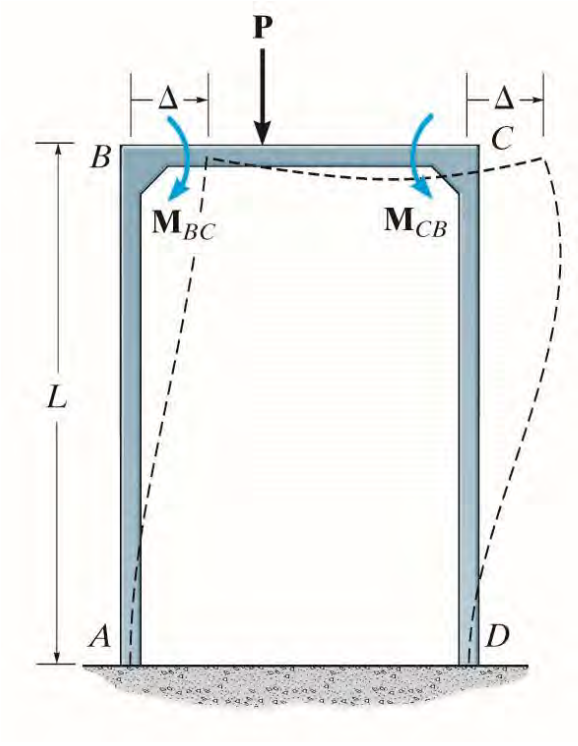
11.5
ANALYSIS OF FRAMES: SIDESWAY

11.5

Analysis of Frames: Sidesway

- A frame will sidesway when it or the loading acting on it is nonsymmetric
- The loading P causes an unequal moments at joint *B* & *C*
- M_{BC} tends to displace joint *B* to the right
- M_{CB} tends to displace joint *C* to the left
- Since $M_{BC} > M_{CB}$, the net result is a sidesway of both joint *B* & *C* to the right



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Analysis of Frames: Sidesway

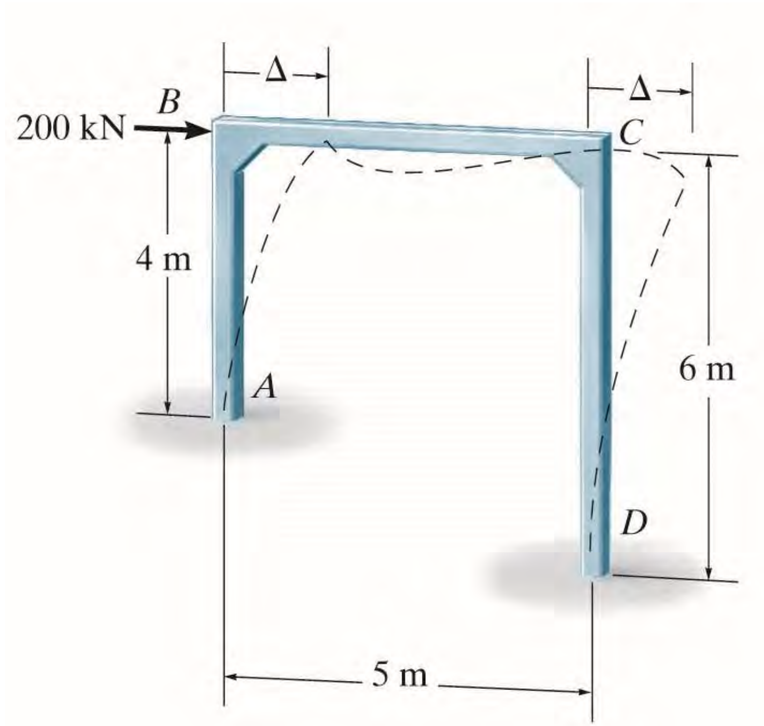
- When applying the slope-deflection eqn to each column, we must consider the column rotation, ψ as an unknown in the eqn
- As a result, an extra equilibrium eqn must be included in the solution
- The techniques for solving problems for frames with sidesway is best illustrated by e.g.

Analysis of Frames: Sidesway

Example 11.9

Explain how the moments in each joint of the two-story frame are determined.

EI is constant.



Analysis of Frames: Sidesway

Example 11.9 (Solution)

Slope-deflection equations

We have 12 equations that contain 18 unknowns

$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0 \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0 \qquad (4)$$

$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0 \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0 \qquad (6)$$

$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0 \qquad (7)$$

$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0 \qquad (8)$$

$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0 \qquad (9)$$

$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0 \qquad (10)$$

$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0 \qquad (11)$$

$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0 \qquad (12)$$

Analysis of Frames: Sidesway

Example 11.9 (Solution)

Equilibrium equations

No FEMs have to be calculated since the applied loading acts at the joints

Members *AB* & *FE* undergo rotations of $\psi_1 = \Delta_1/5$

Members *AB* & *FE* undergo rotations of $\psi_2 = \Delta_2/5$

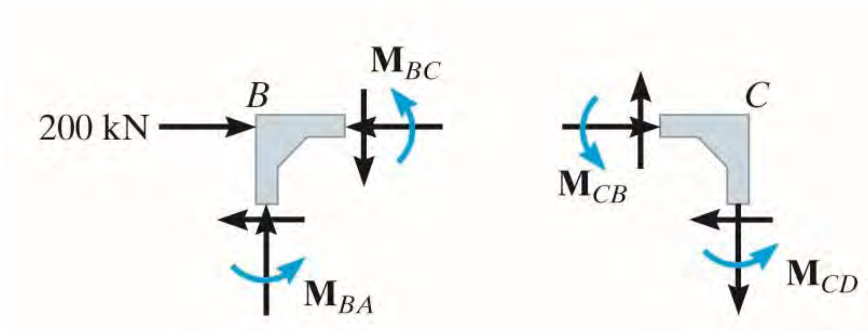
Moment equilibrium of joints *B*, *C*, *D* and *E*, requires

$$M_{BA} + M_{BE} + M_{BC} = 0 \qquad (13)$$

$$M_{CB} + M_{CD} = 0 \qquad (14)$$

$$M_{DC} + M_{DE} = 0 \qquad (15)$$

$$M_{EF} + M_{EB} + M_{ED} = 0 \qquad (16)$$



Analysis of Frames: Sidesway

Example 11.9 (Solution)

Equilibrium equations

Similarly, shear at the base of columns must balance the applied horizontal loads

$$\sum F_x = 0 \Rightarrow 40 - V_{BC} - V_{ED} = 0$$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0 \quad (17)$$

$$\sum F_x = 0 \Rightarrow 40 + 80 - V_{AB} - V_{FE} = 0$$

$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0 \quad (18)$$

Analysis of Frames: Sidesway

Example 11.9 (Solution)

Equilibrium equations

- Sub eqns (1) to (12) into eqns (13) to (18)
- These eqns can be solved simultaneously
- The results are resub into eqns (1) to (12) to obtain the moments at the joints

Thank you