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# Autonomous Systems and Stability 

Research Project

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#### Abstract

In this project, we study Autonomous Systems and Stability. we concentrate on a particular type of separable equations, called autonomous, where the independent variable does not appear explicitly in the equation. For these systems we find a few qualitative properties of their solutions without actually computing the solution. And study two-dimensional nonlinear autonomous systems. We start reviewing the critical points of two-by-two linear systems and classifying them as attractors, repellers, centers, and saddle points. We then introduce a few examples of two-bytwo nonlinear systems. We define the critical points of nonlinear systems. We then compute the linearization of these systems and we study the linear stability of these two-dimensional nonlinear systems .


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## Introduction

By the end of the seventeenth century Newton had invented differential equations, discovered his laws of motion and the law of universal gravitation. He combined all of them to explain Kepler laws of planetary motion. Newton solved what now is called the two-body problem. Kepler laws correspond to the case of one planet orbiting the Sun. People then started to study the three-body problem. For example the movement of Earth, Moon, and Sun. This problem turned out to be far more difficult than the two-body problem and no solution was ever found. Around the end of the nineteenth century Henri Poincar'e proved a breakthrough result. The solutions of the three body problem could not be found explicitly in terms of elementary functions, such as combinations of polynomials, trigonometric functions, exponential, and logarithms. This led him to invent the so-called Qualitative Theory of Differential Equations. In this theory one studies the geometric properties of solutions-whether they show periodic behavior, tend to fixed points, tend to infinity, etc. This approach evolved into the modern field of Dynamics. In this chapter we introduce a few basic concepts and we use them to find qualitative information of a particular type of differential equations, called autonomous equations.

## CHAPTER ONE

## Background

Definition 1.1: Equation [T. Apostol, 1967]

An equation is a mathematical statement containing an equals sign. Numbers may be represented by unknown variables. To solve an equation, the value of these variables must be found

## Definition 1.2: Differential Equation [G. Simmons 1991]

A differential equation (DE) is an equation in solving a function and its derivatives.

Example : A few differential equation
1: $\frac{d y}{d x}=\sin \mathrm{x}$
2: $\frac{d y}{d x}=\frac{x+1}{y-2}$
Definition 1.3: Partial Differential equation: [S. Hassani 2006]
A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the in depended variables.

## Definition 1.4: Ordinary Differential equation [R. Churchill 1958]

In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. He term ordinary is used in contrast with the term partial differentia equation which may be with respect to more then one independent variable.

Definition1.5: Order of a differential equations [R. Churchill 1958]

The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation in the simplest cases, equations may be solved by direct integration.

## Example 1.5:

1: $\frac{d y}{d x}=e^{x} \quad$ first order
2: $\frac{d^{4} y}{d x^{4}}+y=0 \quad$ fourth order
Definition 1.6 : Degree [Nagy 2021,18,January]

Is the highest power of the highest derivative in which occurs in the D.E
Definition 1.7: Linear O.D.E [G. Simmons 1991]

A differential equation in any order is said to be linear if satisfies

1 The dep.v is exist and of the first degree.
2 The derivatives $y^{\prime} y^{\prime \prime} y^{\prime \prime \prime}$ exist and each of them of the first degree.

3 The dep.v and the derivatives not multiply each other.

## Definition 1.8: Non-Linear Differential Equation [Nagy 2021,18,January]

When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

## CHAPTER TWO

2.1. Flows on the Line: In this section we use these equations to present a new method to study qualitative properties of their solutions. Knowing the exact solution to the equation will help us understand how this new method works.

## 2.1: Autonomous Equations.

Definition 2.1.1: [Nagy 2021,18,January]
A first order autonomous differential equation is

$$
y^{\prime}=f(y)
$$

where $y^{\prime}=\frac{d y}{d t}$ and the function $f$ does not depend explictly on $t$.
Example 2.1.2: The following first order separable equations are autonomous:
(a) $y^{\prime}=2 y+3$.
(b) $y^{\prime}=\sin (y)$
(c) $y^{\prime}=r y\left(1-\frac{y}{K}\right)$.

The independent variable $t$ does not appear explicitly in these equations. The following equations are not autonomous.
(a) $y^{\prime}=2 y+3 t$.
(b) $y^{\prime}=t^{2} \sin (y)$.
(c) $y^{\prime}=t y\left(1-\frac{y}{K}\right)$

Example 2.1.3: Find all solutions of the first order autonomous system

$$
y^{\prime}=a y+b \quad, a, b>0
$$

## Solution:

This is a linear, constant coefficients equation, so it could be solved using the
integrating factor method. But this is also a separable equation, so we solve it as follows,

$$
\begin{gathered}
\int \frac{d y}{a y+b}=\int d t \Rightarrow \frac{1}{a} \ln (a y \\
\text { so we get, } \\
a y+b=e^{a t} e^{a c_{0}}
\end{gathered}
$$



And denoting $c=e^{a c o} / a$, we get the expression. $\quad y(t)=c e^{a t}-\frac{b}{a}$
However, sometimes it is not so simple to grasp the qualitative behavior of solutions of an autonomous equation. Even in the case that we can solve the differential equation.

Example 2.1.4: Sketch a qualitative graph of solutions to $y^{\prime}=\sin (y)$, for different initial data conditions $y(0)=y_{0}$.

Solution: We first find the exact solutions and then we see if we can graph them. The equation is separable, then

$$
\frac{y^{\prime}(t)}{\sin (y(t))}=1 \Rightarrow \int_{0}^{t} \frac{y^{\prime}(t)}{\sin (y(t))} d t=t
$$

Use the usual substitution $u=y(t)$, so $d u=y^{\prime}(t) d t$, so we get

$$
\int_{y_{0}}^{y(t)} \frac{d u}{\sin (u)}=t
$$

In an integration table we can find that

$$
\left.\ln \left[\frac{\sin (u)}{1+\cos (u)}\right]\right|_{30} ^{y(t)}=t \Rightarrow \ln \left[\frac{\sin (y)}{1+\cos (y)}\right]-\ln \left[\frac{\sin \left(y_{0}\right)}{1+\cos \left(y_{0}\right)}\right]=t .
$$

We can rewrite the expression above in terms of one single logarithm,

$$
\ln \left[\frac{\sin (y)}{(1+\cos (y))} \frac{\left(1+\cos \left(y_{0}\right)\right)}{\sin \left(y_{0}\right)}\right]=t
$$

If we compute the exponential on both sides of the equation above we get an implicit expression of the solution,

$$
\frac{\sin (y)}{(1+\cos (y))}=\frac{\sin \left(y_{0}\right)}{\left(1+\cos \left(y_{0}\right)\right)} e^{t}
$$

## 2.2: Critical Points and Linearization.

## Definition 2.2.1 [Richard courant 2008]

A point $y_{c}$ is a critical point of $y^{\prime}=f(y)$ iff $f\left(y_{c}\right)=0$. A critical points is:
(i) an Attractor (or sink), iff solutions flow toward the critical point;
(ii) a Repeller (or source), iff solutions flow away from the critical point;
(iii) Neutral, iff solution flow towards the critical point from one side and flow away from the other side.

In this section we keep the convention used in the Example 2.1.3, filled dots denote attractors, and white dots denote repellers. We will use a a half-filled point for neutral points. We recall that attractors have arrows directed to them on both sides, while repellers have arrows directed away from them on both sides. A neutral point would have an arrow pointing towards the critical point on one side and the an arrow pointing away from the critical point on the other side. We will usually mention critical points as stationary solutions when we describe them in a yt-plane, and we reserve the name critical point when we describe them in the phase line, the $y$-line.
We also talked about stable and unstable solutions. Here is a precise definition.
Definition 2.2.2. [S. Hassani 2006]
Let $y_{0}$ be a a constant solution of $y^{\prime}=f(y)$, and let $y$ be a solution with initial data $y(0)=y_{1}$. The solution given by $y_{0}$ is stable iff given any $\epsilon>0$ there is a
$\delta>0$ such that if the initial data $y_{1}$ satisfies $\left|y_{1}-y_{0}\right|<\delta$, then the solution values $y(t)$ satisfy $\left|y(t)-y_{0}\right|<\epsilon$ for all $t>0$. Furthermore, if $\lim _{t \rightarrow \infty} y(t)=$ $y_{0}$, then $y_{0}$ is asymptotically stable. If $y_{0}$ is not stable, we call it unstable.

$$
f(y)=f\left(y_{0}\right)+f^{\prime}\left(y_{0}\right)\left(y-y_{0}\right)+o\left(\left(y-y_{0}\right)^{2}\right) .
$$

Denote $f_{0}=f\left(y_{0}\right)$, then $f_{0}^{\prime}=f^{\prime}\left(y_{0}\right)$, and introduce the variable $u=y-y_{0}$. Then we get

$$
f(y)=f_{0}+f_{0}^{\prime} u+o\left(u^{2}\right)
$$

Let us use this Taylor expansion on the right hand side of the equation $y^{\prime}=f(y)$, and recalling that $y^{\prime}=\left(y_{0}+u\right)^{\prime}=u^{\prime}$, we get

$$
y^{\prime}=f(y) \Leftrightarrow u^{\prime}=f_{0}+f_{0}^{\prime} u+o\left(u^{2}\right)
$$

If $y_{0}$ is a critical point of $f$, then $f_{0}=0$, then

$$
y^{\prime}=f(y) \Leftrightarrow u^{\prime}=f_{0}^{\prime} u+o\left(u^{2}\right)
$$

From the equations above we see that for $y(t)$ close to a critical point $y_{0}$ the right hand side of the equation $y^{\prime}=f(y)$ is close to $f_{0}^{\prime} u$. Therefore, one can get information about a solution of a nonlinear equation near a critical point by studying an appropriate linear equation. We give this linear equation a name.

## Definition 2.2.3: [G. Simmons 1991]

The linearization of an autonomous system $y^{\prime}=f(y)$ at a critical point $y_{c}$ is the linear differential equation for the function u given by $u^{\prime}=f^{\prime}\left(y_{c}\right) u$.

Remark 2.2.4: [G. Simmons 1991] The prime notation above means, $u^{\prime}=\frac{d u}{d t}$, and $f^{\prime}=\frac{d f}{d y}$.
Example 2.2.5: Find the linearization of the equation $y^{\prime}=\sin (y)$ at the critical point $y_{n}=n \pi$. Write the particular cases for $n=0,1$ and solve the linear equations for arbitrary initial data.

Solution: If we write the nonlinear system as $y^{\prime}=f(y)$, then $f(y)=\sin (y)$. We then compute its $y$ derivative, $f^{\prime}(y)=\cos (y)$. We evaluate this expression at the
critical points, $f^{\prime}\left(y_{n}\right)=\cos (n \pi)=(-1)^{n}$. The linearization at $y_{n}$ of the nonlinear equation above is the linear equation for the unknown function $u_{n}$ given by

$$
u_{n}^{\prime}=(-1)^{n} u_{n}
$$

The particular cases $n=0$ and $n=1$ are given by

$$
u_{0}^{\prime}=u_{0}, u_{1}^{\prime}=-u_{1}
$$

It is simple to find solutions to first order linear homogeneous equations with constant coefficients. The result, for each equation above, is

$$
u_{0}(t)=u_{0}(0) e^{t}, u_{1}(t)=u_{1}(0) e^{-t}
$$

Theorem 2.2.6 [S. Hassani 2006]
(Stability of Linear Equations). The constant coefficent linear equation $u^{\prime}=a u$, with $a \neq 0$, has only one critical point $u_{0}=0$. And the constant solution defined by this critical point is unstable for $a>0$, and it is asymptotically stable for $a<0$.

Proof : of Theorem 2.2.1: The critical points of the linear equation $u^{\prime}=a u$ are the solutions of $a u=0$. Since $a \neq 0$, that means we have only one critical point, $u_{0}=0$. Since the linear equation is so simple to solve, we can study the stability of the constant solution $u_{0}=0$ from the formula for all the solutions of the equation,

$$
u(t)=u(0) e^{a t}
$$

The graph of all these solutions is sketch. in the case that $u(0) \neq 0$, we see that for $a>0$ the solutions diverge to $\pm \infty$ as $t \rightarrow \infty$, and for $a<0$ the solutions approach to zero as $t \rightarrow \infty$.



Theorem 2.2.7: [Nagy 2021,18,January]
(Stability of Nonlinear Equations). Let $y_{c}$ be a critical point of the autonomous system $y^{\prime}=f(y)$.
(a) The critical point $y_{c}$ is stable iff $f^{\prime}\left(y_{c}\right)<0$.
(b) The critical point $y_{c}$ is unstable iff $f^{\prime}\left(y_{c}\right)>0$.

Furthermore, If the initial data $y(0) \simeq y_{c}$, is close enough to the critial point $y_{c}$, then the solution with that initial data of the equation $y^{\prime}=f(y)$ are close enough to $y_{c}$ in the sense

$$
y(t) \simeq y_{c}+u(t)
$$

where $u$ is the solution to the linearized equation at the critical point $y_{c}$,

$$
u^{\prime}=f^{\prime}\left(y_{c}\right) u, u(0)=y(0)-y_{c}
$$

## 2.3: Population Growth Models.

Definition 2.3.1: [R. Churchill 1958]
The logistic equation describes the organism's population function $N$ in time as the solution of the autonomous differential equation

$$
N^{\prime}=r N\left(1-\frac{N}{K}\right)
$$

where the initial growth rate constant $r$ and the carrying capacity constant $K$ are positive..

Example 2.3.2: Find the exact expression for the solution to the logistic equation for population growth

$$
y^{\prime}=r y\left(1-\frac{y}{K}\right), y(0)=y_{0}, 0<y_{0}<K
$$

Solution: This is a separable equation,

$$
\frac{K}{r} \int \frac{y^{\prime} d t}{(K-y) y}=t+c_{0}
$$

The usual substitution $u=y(t)$, so $d u=y^{\prime} d t$, implies

$$
\frac{K}{r} \int \frac{d u}{(K-u) u}=t+c_{0} . \Rightarrow \frac{K}{r} \int \frac{1}{K}\left[\frac{1}{(K-u)}+\frac{1}{u}\right] d u=t+c_{0} .
$$

where we used partial fractions decomposition to get the second equation. Now, each term can be integrated,

$$
[-\ln (|K-y|)+\ln (|y|)]=r t+r c_{0}
$$

We reorder the terms on the right-hand side,

$$
\ln \left(\frac{|y|}{|K-y|}\right)=r t+r c_{0} \Rightarrow\left|\frac{y}{K-y}\right|=c e^{r t}, c=e^{r c 0}
$$

$$
\frac{y_{0}}{K-y_{0}}=c \Rightarrow y(t)=\frac{K y_{0}}{y_{0}+\left(K-y_{0}\right) e^{-r t}}
$$

## CHAPTER THREE

## Flows on the Plane

We now turn to study two-dimensional nonlinear autonomous systems. We start reviewing the critical points of two-by-two linear systems and classifying them as attractors, repellers, centers, and saddle points. We then introduce a few examples of two-by-two nonlinear systems. We define the critical points of nonlinear systems. We then compute the linearization of these systems and we study the linear stability of these two-dimensional nonlinear systems
3.1. Two-Dimensional Nonlinear Systems. We start with the definition of autonomous systems on the plane.

Definition 3.1.1: [Nagy 2021,18,January]
A first onder two-dimensional autonomous differential equation is

$$
x^{\prime}=f(x)
$$

where $\boldsymbol{x}^{\prime}=\frac{d \boldsymbol{x}}{d t}$, and the vector field $\boldsymbol{f}$ does not depend explicitly on $t$.
Remark 3.1.2: [Nagy 2021,18,January]
If we introduce the vector components $\boldsymbol{x}(t)=\left[\begin{array}{l}x_{1}(t) \\ x_{2}(t)\end{array}\right]$ and $\boldsymbol{f}(\boldsymbol{x})=\left[\begin{array}{l}f_{1}\left(x_{1}, x_{2}\right) \\ f_{2}\left(x_{1}, x_{2}\right)\end{array}\right]$, then the autonomous equation above can be written in components,

$$
\begin{aligned}
& x_{1}^{\prime}=f_{1}\left(x_{1}, x_{2}\right), \\
& x_{2}^{\prime}=f_{2}\left(x_{1}, x_{2}\right),
\end{aligned}
$$

where $x_{i}^{\prime}=\frac{d x_{i}}{d t}$, for $i=1,2$.
Example 3.1.3 (The Nonlinear Pendulum). A pendulum of mass $m$, length $\ell$, oscillating under the gravity acceleration $g$, moves according to Newton's second law of motion

$$
m(\ell \theta)^{\prime \prime}=-m g \sin (\theta)
$$

where the angle $\theta$ depends on time $t$. If we rearrange terms we get a second order scalar equation

$$
\theta^{\prime \prime}+\frac{g}{\ell} \sin (\theta)=0
$$

This scalar equation can be written as a nonlinear system. If we introduce $x_{1}=\theta$ and $x_{2}=\theta^{\prime}$, then

$$
\begin{aligned}
& x_{1}^{\prime}=x_{2} \\
& x_{2}^{\prime}=-\frac{g}{\ell} \sin \left(x_{1}\right) .
\end{aligned}
$$


3.2 Review: The Stability of Linear Systems. we used phase portraits to display vector functions

$$
\boldsymbol{x}(t)=\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]
$$

Solutions: of $2 \times 2$ linear differential systems. In a phase portrait we plot the vector $\boldsymbol{x}(t)$
on the plane $x 1 x 2$ for different values of the independent variable $t$. We then plot a curve
representing all the end points of the vectors $\boldsymbol{x}(t)$, for $t$ on some interval. The arrows in the curve show the direction of increasing $t$.


Figure A curve in a phase portrait represents all the end points of the vectors $x(t)$, for $t$ on some interval. The arrows in the curve show the direction of increasing $t$ solutions to 2-dimensional linear systems depend on the eigenvalues of the coefficient matrix. If we denote a general $2 \times 2$ matrix by

$$
A=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]
$$

then the eigenvalues are the roots of the characteristic polynomial,

$$
\operatorname{det}(A-\lambda I)=\lambda^{2}-p \lambda+q=0,
$$

where we denoted $p=a_{11}+a_{22}$ and $q=a_{11} a_{22}-a_{12} a_{21}$. Then the eigenvalues are

$$
\lambda_{ \pm}=\frac{p \pm \sqrt{p^{2}-4 q}}{2}=\frac{p}{2} \pm \frac{\sqrt{\Delta}}{2}
$$

where $\Delta=p^{2}-4 q$. We can classify the eigenvalues according to the sign of $\Delta$. In Fig 13 we plot on the $p q$-plane the curve $\Delta=0$, that is, the parabola $q=p^{2} / 4$. The region above this parabola is $\Delta<0$, therefore the matrix eigenvalues are complex, which corresponds to spirals in the phase portrait. The spirals are stable for $p<0$ and unstable for $p>0$. The region below the parabola corresponds to real disctinct eigenvalues. The parabola itself corresponds to the repeated eigenvalue case.


Figure The stability of the solution $\mathrm{x} 0=0$.
The trivial solution $x_{0}=O$ is called a critical point of the linear system $x^{\prime}=A \boldsymbol{x}$. Here is a more detailed classification of this critical point.
Definition 3.2.1. [Richard courant 2008]
The critical point $\boldsymbol{x}_{0}=0$ of a $2 \times 2$ linear system $\boldsymbol{x}^{\prime}=A \boldsymbol{x}$ is:
(a) an attractor (or sink ), iff both eigenvalues of $A$ have negative real part;
(b) a repeller (or source), iff both eigenvalues of $A$ have positive real part;
(c) a saddle, iff one eigenvalue of $A$ is positive and the other is negative;
(d) a center, iff both eigenvalues of $A$ are pure imaginary:
(e) higher order critical point iff at least one eigenvalue of $A$ is zero. The critical point $x_{0}=0$ is called hyperbolic iff it belongs to cases (a-c), that is, the real part of all eigenvalues of $A$ are nonzero..
3.3. Critical Points and Linearization. We now extended to two-dimensional systems the concept of linearization we introduced for one-dimensional systems. The hope is that solutions to nonlinear systems close to critical points behave in a similar way to solutions to the linearized system. We will see that this is the case if the linearized system has distinct eigenvalues. Se start with the definition of critical points.

Definition 3.3.1: [S. Hassani 2006]
A critical point of a two-dimensional system $x^{\prime}=f(x)$ is a vector $x_{0}$ where the field $f$ vanishes,

$$
f\left(x_{0}\right)=0
$$

Remark 3.3.2: [S. Hassani 2006]
A critical point defines a constant vector function $x(t)=x_{0}$ for all $t$, solution of the differential equation,

$$
x_{0}^{\prime}=0=f\left(x_{0}\right)
$$

In components, the field is $f=\left[\begin{array}{l}f_{1} \\ f_{2}\end{array}\right]$, and the critical point $x_{0}=\left[\begin{array}{l}x_{1}^{0} \\ x_{2}^{0}\end{array}\right]$ is solution of

$$
\begin{aligned}
& f_{1}\left(x_{1}^{\circ}, x_{2}^{0}\right)=0, \\
& f_{2}\left(x_{1}^{0}, x_{2}^{0}\right)=0 .
\end{aligned}
$$

When there is more than one critical point we will use the notation $x_{i}$, with $i=$ $0,1,2, \cdots$, to denote the critical points.
Example 3.3.3: Find all the critical points of the two-dimensional (decoupled) system

$$
\begin{aligned}
& x_{1}^{\prime}=-x_{1}+\left(x_{1}\right)^{3} \\
& x_{2}^{\prime}=-2 x_{2} .
\end{aligned}
$$

Solution: We need to find all constant vectors $\boldsymbol{x}=\left[\begin{array}{l}x_{1} \\ x_{2}\end{array}\right]$ solutions of

$$
\begin{aligned}
-x_{1}+\left(x_{1}\right)^{3} & =0 \\
-2 x_{2} & =0
\end{aligned}
$$

From the second equation we get $x_{2}=0$. From the first equation we get

$$
x_{1}\left(\left(x_{1}\right)^{2}-1\right)=0 \Rightarrow x_{1}=0, \text { or } x_{1}= \pm 1
$$

Therefore, we got three critical points, $x_{0}=\left[\begin{array}{l}0 \\ 0\end{array}\right], x_{1}=\left[\begin{array}{l}1 \\ 0\end{array}\right], x_{2}=\left[\begin{array}{c}-1 \\ 0\end{array}\right]$.

$$
\begin{aligned}
& x_{1}^{\prime}=f_{1}\left(x_{1}, x_{2}\right), \\
& x_{2}^{\prime}=f_{2}\left(x_{1}, x_{2}\right),
\end{aligned}
$$

Assume that $f_{1}, f_{2}$ have Taylor expansions at $x_{0}=\left[\begin{array}{l}x_{2}^{0} \\ x_{2}^{0}\end{array}\right]$. We denote $u_{1}=\left(x_{1}-x_{1}^{0}\right.$ $u_{2}=\left(x_{2}-x_{2}^{0}\right)$, and $f_{1}^{\circ}=f_{1}\left(x_{1}^{0}, x_{2}^{0}\right), f_{2}^{\circ}=f_{2}\left(x_{1}^{0}, x_{2}^{0}\right)$. Then, by the Taylor expansion the

$$
\begin{aligned}
& f_{1}\left(x_{1}, x_{2}\right)=f_{1}^{\circ}+\left.\frac{\partial f_{1}}{\partial x_{1}}\right|_{x_{0}} u_{1}+\left.\frac{\partial f_{1}}{\partial x_{2}}\right|_{x_{0}} u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right), \\
& f_{2}\left(x_{1}, x_{2}\right)=f_{2}^{\circ}+\left.\frac{\partial f_{2}}{\partial x_{1}}\right|_{x_{0}} u_{1}+\left.\frac{\partial f_{2}}{\partial x_{2}}\right|_{x_{0}} u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right) .
\end{aligned}
$$

Let us simplify the notation a bit further. Let us denote

$$
\begin{array}{ll}
\partial_{1} f_{1}=\left.\frac{\partial f_{1}}{\partial x_{1}}\right|_{x_{0}}, & \partial_{2} f_{1}=\left.\frac{\partial f_{1}}{\partial x_{2}}\right|_{x_{0}}, \\
\partial_{1} f_{2}=\left.\frac{\partial f_{2}}{\partial x_{1}}\right|_{x_{0}}, & \partial_{2} f_{2}=\left.\frac{\partial f_{2}}{\partial x_{2}}\right|_{x_{0}}
\end{array}
$$

then the Taylor expansion of $f$ has the form

$$
\begin{aligned}
& f_{1}\left(x_{1}, x_{2}\right)=f_{1}^{0}+\left(\partial_{1} f_{1}\right) u_{1}+\left(\partial_{2} f_{1}\right) u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right), \\
& f_{2}\left(x_{1}, x_{2}\right)=f_{2}^{0}+\left(\partial_{1} f_{2}\right) u_{1}+\left(\partial_{2} f_{2}\right) u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right)
\end{aligned}
$$

We now use this Taylor expansion of the field $f$ into the differential equation $x^{\prime}=$ $f$. Recall that $x_{1}=x_{1}^{0}+u_{1}$, and $x_{2}=x_{2}^{0}+u_{2}$, and that $x_{1}^{0}$ and $x_{2}^{0}$ are constants, then

$$
\begin{aligned}
u_{1}^{\prime} & =f_{1}^{0}+\left(\partial_{1} f_{1}\right) u_{1}+\left(\partial_{2} f_{1}\right) u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right), \\
u_{2}^{\prime} & =f_{2}^{0}+\left(\partial_{1} f_{2}\right) u_{1}+\left(\partial_{2} f_{2}\right) u_{2}+o\left(u_{1}^{2}, u_{2}^{2}\right) .
\end{aligned}
$$

Let us write this differential equation using vector notation. If we introduce the vectors and the matrix

$$
\boldsymbol{u}=\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right], \boldsymbol{f}_{0}=\left[\begin{array}{l}
f_{1}^{0} \\
f_{2}^{0}
\end{array}\right], D f_{0}=\left[\begin{array}{ll}
\partial_{1} f_{1} & \partial_{2} f_{1} \\
\partial_{1} f_{2} & \partial_{2} f_{2}
\end{array}\right],
$$

then, we have that

$$
x^{\prime}=f(x) \Leftrightarrow u^{\prime}=f_{0}+\left(D f_{0}\right) u+o\left(|u|^{2}\right)
$$

In the case that $x_{0}$ is a critical point, then $f_{0}=0$. In this case we have that

$$
x^{\prime}=f(x) \Leftrightarrow u^{\prime}=\left(D f_{0}\right) u+o\left(|u|^{2}\right) .
$$

The relation above says that the equation coefficients of $x^{\prime}=f(x)$ are close, order $o\left(|u|^{2}\right)$, to the coefficients of the linear differential equation $u^{\prime}=\left(D f_{0}\right) u$. For this reason, we give this linear differential equation a name.

Definition 3.3.4: [Nagy 2021,18,January]
The linearization of a two-dimensional system $x^{\prime}=f(x)$ at a critical point $x_{0}$ is the $2 \times 2$ linear system

$$
\boldsymbol{u}^{\prime}=\left(D f_{0}\right) \boldsymbol{u}
$$

where $u=x-x_{0}$, and we have introduced the Jacobian matrix at $x_{0}$,

$$
D f_{0}=\left[\begin{array}{ll}
\left.\frac{\partial f_{1}}{\partial x_{1}}\right|_{z_{0}} & \left.\frac{\partial f_{1}}{\partial x_{2}}\right|_{z_{0}} \\
\left.\frac{\partial f_{2}}{\partial x_{1}}\right|_{z_{0}} & \left.\frac{\partial f_{2}}{\partial x_{2}}\right|_{z_{0}}
\end{array}\right]=\left[\begin{array}{ll}
\partial_{1} f_{1} & \partial_{2} f_{1} \\
\partial_{1} f_{2} & \partial_{2} f_{2}
\end{array}\right] .
$$

Remark 3.3.5: [Nagy 2021,18,January]
In components, the nonlinear system is

$$
\begin{aligned}
& x_{1}^{\prime}=f_{1}\left(x_{1}, x_{2}\right), \\
& x_{2}^{\prime}=f_{2}\left(x_{1}, x_{2}\right),
\end{aligned}
$$

and the linearization at $x_{0}$ is

$$
\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{ll}
\partial_{1} f_{1} & \partial_{2} f_{1} \\
\partial_{1} f_{2} & \partial_{2} f_{2}
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]
$$

Example 3.3.6: Find the linearization at every critical point of the nonlinear system

$$
\begin{aligned}
& x_{1}^{\prime}=-x_{1}+\left(x_{1}\right)^{3} \\
& x_{2}^{\prime}=-2 x_{2} .
\end{aligned}
$$

Solution: We found earlier that this system has three critial points,

$$
x_{0}=\left[\begin{array}{l}
0 \\
0
\end{array}\right], x_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], x_{2}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right] .
$$

This means we need to compute three linearizations, one for each critical point. We start computing the derivative matrix at an arbitrary point $\boldsymbol{x}$,

$$
D f(x)=\left[\begin{array}{ll}
\frac{\partial f_{1}}{\partial x_{1}} & \frac{\partial f_{1}}{\partial x_{2}} \\
\frac{\partial f_{2}}{\partial x_{1}} & \frac{\partial f_{2}}{\partial x_{2}}
\end{array}\right]=\left[\begin{array}{cc}
\frac{\partial}{\partial x_{1}}\left(-x_{1}+x_{1}^{3}\right) & \frac{\partial}{\partial x_{2}}\left(-x_{1}+x_{1}^{3}\right) \\
\frac{\partial}{\partial x_{1}}\left(-2 x_{2}\right) & \frac{\partial}{\partial x_{2}}\left(-2 x_{2}\right)
\end{array}\right]
$$

so we get that

$$
D f(x)=\left[\begin{array}{cc}
-1+3 x_{1}^{2} & 0 \\
0 & -2
\end{array}\right]
$$

We only need to evaluate this matrix $D f$ at the critical points. We start with $x_{0}$,

$$
x_{0}=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \Rightarrow D f_{0}=\left[\begin{array}{cc}
-1 & 0 \\
0 & -2
\end{array}\right] \Rightarrow\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
-1 & 0 \\
0 & -2
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]
$$

The Jacobian at $x_{1}$ and $x_{2}$ is the same, so we get the same linearization at these points,

$$
\begin{aligned}
& x_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \Rightarrow D f_{1}=\left[\begin{array}{cc}
2 & 0 \\
0 & -2
\end{array}\right] \Rightarrow\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
2 & 0 \\
0 & -2
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right] \\
& x_{2}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right] \Rightarrow D f_{2}=\left[\begin{array}{cc}
2 & 0 \\
0 & -2
\end{array}\right] \Rightarrow\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
2 & 0 \\
0 & -2
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]
\end{aligned}
$$

Critical points of nonlinear systems are classified according to the eigenvalues of their corresponding linearization.
Definition 3.3.7: [G. Simmons 1991]
A critical point $x_{0}$ of a two-dimensional system $\boldsymbol{x}^{\prime}=f(x)$ is:
(a) an attractor (or $\sin k$ ), iff both eigenvalues of $D f_{0}$ have negative real part,
(b) a repeller (or source), iff both eigenvalues of $D f_{0}$ have positive real part;
(c) a saddle, iff one eigenvalue of $D f_{0}$ is positive and the other is negative;
(d) a center, iff both eigenvalues of $D f_{0}$ are pure imaginary;
(e) higher order critical point iff at least one eigenvalue of $D f_{0}$ is zero. A critical point $x_{0}$ is called hyperbolic iff it belongs to cases (a-c), that is, the real part of all eigenvalues of $D f_{0}$ are nonzero.
Example 3.3.8: Classify all the critical points of the nonlinear system

$$
\begin{aligned}
& x_{1}^{\prime}=-x_{1}+\left(x_{1}\right)^{3} \\
& x_{2}^{\prime}=-2 x_{2}
\end{aligned}
$$

Solution: We already know that this system has three critical points,

$$
x_{0}=\left[\begin{array}{l}
0 \\
0
\end{array}\right], x_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], x_{2}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right] .
$$

We have already computed the linearizations at these critical points too.

$$
D f_{0}=\left[\begin{array}{cc}
-1 & 0 \\
0 & -2
\end{array}\right], D f_{1}=D f_{2}=\left[\begin{array}{cc}
2 & 0 \\
0 & -2
\end{array}\right]
$$

We now need to compute the eigenvalues of the Jacobian matrices above. For the critical point $x_{0}$ we have $\lambda_{+}=-1, \lambda_{-}=-2$, so $x_{0}$ is an attractor. For the critical points $x_{1}$ and $x_{2}$ we have $\lambda_{+}=2, \lambda_{-}=-2$, so $x_{1}$ and $x_{2}$ are saddle points.
3.4. The Stability of Nonlinear Systems. Sometimes the stability of twodimensional nonlinear systems at a critical point is determined by the stability of the linearization at that critical point. This happens when the critical point of the linearization is hyperbolic, that is, the Jacobian matrix has eigenvalues with nonzero real part. We summarize this result in the following statement.

Theorem 3.4.1: [Nagy 2021,18,January]
Consider a two-dimensional nonlinear autonomous system with a continuously differentiable field $\boldsymbol{f}$,

$$
x^{\prime}=f(x)
$$

and consider its linearization at a hyperbolic critical point $x_{0}$,

$$
u^{\prime}=\left(D f_{0}\right) u
$$

Then, there is a neighborhood of the hyperbolic critical point $x_{0}$ where all the solutions of the linear system can be transformed into solutions of the nonlinear system by a continuous, invertible, transformation.

Remark 3.4.2: [Nagy 2021,18,January]
The Hartman-Grobman theorem implies that the phase portrait of the linear system in a neighborhood of a hyperbolic critical point can be transformed into the phase portrait of the nonlinear system by a continuous, invertible, transformation. When that happens we say that the two phase portraits are topologically equivalent.

Remark 3.4.3: [Nagy 2021,18,January]
This theorem says that, for hyperbolic critical points, the phase portrait of the linearization at the critical point is enough to determine the phase portrait of the nonlinear system near that critical point. Example 3.4.4: Use the Hartman-Grobman theorem to sketch the phase portrait of

$$
\begin{aligned}
& x_{1}^{\prime}=-x_{1}+\left(x_{1}\right)^{3} \\
& x_{2}^{\prime}=-2 x_{2} .
\end{aligned}
$$

Solution: We have found before that the critical points are

$$
x_{0}=\left[\begin{array}{l}
0 \\
0
\end{array}\right], x_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], x_{2}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right],
$$

Where $x_{0}$ is an attractor and $x_{1}, x_{2}$ are saddle points.
The phase portrait of the linearized systems at the critical points is given in These critical points have all linearizations with eigenvalues having nonzero real parts. This means that the critical points are hyperbolic, so we can use the HartmanGrobman theorem. This theorem says that the phase portrait in is precisely the phase portrait of the nonlinear system in this ex-


Since we now know also the phase portrait of the nonlinear, we only need to fill in the gaps in that phase portrait. In this example, a decoupled system, we can complete the phase portrait from the symmetries of the solution. Indeed, in the $x_{2}$ direction all trajectories must decay to exponentially to the $x_{2}=0$ line. In the $x_{1}$ direction, all trajectories are attracted to $x_{1}=0$ and repelled from $x_{1}= \pm 1$. The vertical lines $x_{1}=0$ and $x_{1}= \pm 1$ are invariant, since $x_{1}^{\prime}=0$ on these lines; hence any trajectory that start on these lines stays on these lines. Similarly, $x_{2}=0$ is an invariant horizontal line. We also note that the phase portrait must be symmetric in both $x_{1}$ and $x_{2}$ axes, since the equations
are invariant under the transformations $\mathrm{x} 1 \rightarrow-\mathrm{x} 1$ and $\mathrm{x} 2 \rightarrow-\mathrm{x} 2$. Putting all this extra information together we arrive to the phase portrait.


Figure Phase portraits of the nonlinear systems in the Example 3.4.4

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