زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Boundary value problems

Research Project

Submitted to the department of (mathematics) in partial fulfillment of the requirements for the degree of BSc in (mathematic)

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Apirl-2023

Acknowledgement

I would like to thank Allah for giving me the power to complete this work.

I would like to present and thanks to supervisor M.Mudhafar Hamed.H for her kind and valuable suggestion that head assisted me to a complete this work.

I would also to extend my gratitude to head of the Mathematics Department Dr.Rashad Rashid Haji.

Thank my family for all the love and support they provided, and then all my friend loves, and thank all Teacher Mathematic Department.

Abstract

In this project, we study boundary-value problem. We review Theory of linear differential equations to certain boundary-value .This problems for ordinary differential equations and partial differential equations.

Briefly introduces some notation and defines the two partial differential equations is a first exposure to solving partial differential equations, working with boundary-value problems and working with eign and initial value problem . In the end Comparison between IVP and BVP. I got some example from this boundary-value problem.

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Introduction

(Gladwell 2015):.A Boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

A two-point boundary value problem (BVP) of total order n on a finite interval $[a, b]$ may be written as an explicit first order system of ordinary differential equations (ODEs) with boundary values evaluated at two points as

$$y'(x) = f(x, y(x)), \qquad x \in (a, b), \qquad g(y(a), y(b)) = 0$$

Here, $y, f, g \in R^n$ and the system is called explicit because the derivative y' appears explicitly. The n boundary conditions defined by g must be independent; that is, they cannot be expressed in terms of each other (if g is linear the boundary conditions must be linearly independent). (Gladwell 2015)

In practice, most BPs do not arise directly but instead as a combination of equations defining various orders of derivatives of the variables which sum to n. In an explicit BVP system, the boundary conditions and the right hand sides of the ordinary differential equations (ODEs) can involve the derivatives of each solution variable up to an order one less than the highest derivative of that variable appearing on the left hand side of the ODE defining the variable. To write a general system of ODEs of different orders we can define y as a vector made up of all the solution variables and their derivatives up to one less than the highest derivative of each variable, then add trivial ODEs to define these derivatives. See the section on initial value problems for an example of how this is achieved. See also Ascher et al. (1995) who show techniques for rewriting boundary value problems of various orders as first order systems. Such rewritten systems may not be unique and do not necessarily provide the most efficient approach for computational solution.

The words two-point refer to the fact that the boundary condition function g is evaluated at the solution at the two interval endpoints $a \, and \, b$ unlike for initial value problems (IVPs) where the n initial conditions are all evaluated at a single point.

Occasionally, problems arise where the function g is also evaluated at the solution at other points in(a, b).

Chapter one

Review

Section one

1.1.1 Equation :.{ (M.Eather 2006)}

An equation is a mathematical statement containing an equal's sign. Numbers may be represented by unknown variables. To solve an equation, the value of these variables must be found.

Example: $5x^2+2x=16$

1.1.2. Differential equation: (Nagy 2021,18, January)A differential equation is an equation in solving a function and its derivatives.

$$\left(\frac{dy}{dx}\right)^3 + \frac{d'y}{dx^4} + y = 2\sin(x) + \ cos^3(x)$$

1.1.3. Partial Differential equation: (D.Polyanin 2008) A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the indepen variables.

1.1.4. Ordinary Differential equation: (John W. Cain 2010) In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. He term ordinary is used in contrast with the term partial differentia equation which may be with respect to more than one independent variable.

1.1.5. Order of a differential equation: (John W. Cain 2010)The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation in the simplest cases, equations may be solved by direct integration.

1.1.6. Semi-linear equation: (John W. Cain 2010) A first order p.d.e. is said to be a semi-linear equation if it is linear in p and g and the coefficients of p and g are functions of x and y only, i.e., if it is of the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

1.1.7. Quasi-linear equation: (D.Polyanin 2008)A first order p.d.e. Is said to be a quasi-linear equation if it is linear in p and q, i.e., if it is of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

1.1.8. Degree: (Nagy 2021,18,January)Is the highest power of the highest derivative in which occurs in the D.E

1.1.9. Linear O.D.E: (https) A differential equation in any order is said to be linear if satisfies:

1- The dep.v is exist and of the first degree.
2- The derivatives y', y'', y''' exist and each of them of the first degree.
3- The dep.v and the derivatives not multiply each other.

1.1.10. Non-Linear Differential Equation:. (https) When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

What is the difference between linear and nonlinear equations?

1. A linear equation will always exist for all values of x and y but nonlinear equations may or may not have solutions for all values of x and y.
2. A linear differential equation is defined by a linear equation in unknown variables and their derivatives. A nonlinear differential equation is not linear in unknown variables and their derivatives.

1.1.11. Independent: (https)The independent variable is the variable the experimenter manipulates or changes, and is assumed to have a direct effect on the dependent variable.

1.1.12. Dependent: (https) The dependent variable is the variable being tested and measured in an experiment, and is 'dependent' on the independent variable.

Chapter Two

Section one Boundary value problems

2.1.1. Boundary value problems: (L.POWERS 1972) More often than not, significant practical problems in partial-and even ordinary- differential equations cannot be solved by analytical methods .Difficulties may arise from variable coefficients, irregular regions, unsuitable boundary conditions, interfaces, or just overwhelming detail.

2.1.2. Definition (BVP): (Nagy 2021,18,January) Find all solutions of the differential equation

$y'' + a_1 y' + a_0 y = 0$ Is satisfying the boundary condition (BC)

Remarks: (Nagy 2021,18,January) In a boundary value problem we usually the following happen.

- The variable x represents position.
- The variable y may represents a physical quantity such us temperature.
- The BC is the temperature at two different positions.

Example: Find a solution to the BVP $\frac{d^2y}{dx^2} - y = 0$, $y(0) = 0$, $y(1) = 1$ if we know $y(x) = c_1 e^x + c_2 e^{-x}$ is a general soluation to the differential equation

Solution:

$$y(0) = 0 \rightarrow c_1 e^0 + c_2 e^0 = 0 \rightarrow c_1 + c_2 = 0 \rightarrow c_1 = -c_2$$

$$y(1) = 1 \rightarrow c_1 e^1 + c_2 e^{-1} = 1 \rightarrow -c_2 e^{-1} + c_2 e^1 = 1 \rightarrow c_2(e^{-1} - e^1) = 1$$

$$\rightarrow c_2 = \frac{1}{(e^{-1} - e^1)} \rightarrow c_1 = \frac{-1}{(e^{-1} - e^1)}$$

$$y(x) = \frac{-1}{(e^{-1} - e^1)} e^x + \frac{1}{(e^{-1} - e^1)} e^{-x} \rightarrow y(x) = \frac{e^{-x} - e^x}{e^{-1} - e^1}$$

2.1.3. Boundary conditions: (Nagy 2021,18,January) Instead of specifying requirements that y and its derivatives must satisfy at one particular value of the independent variable x, we could instead impose requirements on y and its derivatives at different x values. The result is called a boundary value problem (BVP).

Section two

2.2.1. Eigenvalue: (Richard Courant 2008)Eigenvalues are the special set of scalars associated with the system of linear equations. It is mostly used in matrix equations.

2.2.2. Eigenvector: (Richard Courant 2008)Eigenvector of a square matrix is defined as a non-vector in which when a given matrix is multiplied, it is equal to a scalar multiple of that vector. Let us suppose that A is an n x n square matrix, and if v be a non-zero vector, then the product of matrix A, and vector v is defined as the product of a scalar quantity λ and the given vector, such that:

$$Av = \lambda v$$

Where v = Eigenvector and λ be the scalar quantity that is termed as eigenvalue associated with given matrix A

2.2.3. Scalar Quantity: (Nagy 2021,18,January) A scalar quantity only has a magnitude and it can be represented by a number only a scalar does not have any direction the addition of scalars follows the generic rules of the addition of numbers.

2.2.4..Eigen function: (Richard Courant 2008)An Eigen function is a type of Eigen vector that is also a function for a given square matrix. A if we could find values of λ which we could find non zero solutions i.e. $x \neq 0$ to

$$Ax = \lambda x$$

That we called λ an eigenvalue of A and x was its corresponding eigenvector the Eigen value λ helps to find nonzero solutions to the equation

Section three Initial value problem

2.3.1. Initial value problem: (John W. Cain 2010)In multivariable calculus, an initial value problem a] (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain. Modeling a system in physics or other sciences frequently amounts to solving an initial value problem. .In that context, the differential initial value is an equation which specifies how the system evolves with time given the initial conditions of the problem.

2.3.2. Definition (IVP): (Nagy 2021,18,January)Find all solutions of the differential equation $y'' + a_1 y' + a_0 y = 0$ satisfying the initial condition (IC)

$y(t_0) = y_0$, $y'(t_0) = y_1$

Remarks: In an initial value problem we usually the following happen.

- The variable t represents time.
- The variable y represents position.
- The IC are position and velocity at the initial time.

2.3.3.Initial condition: (John W. Cain 2010)An initial condition is an extra bit of information about a differential equation that tells you the value of the function at a particular point. Differential equations with initial conditions are commonly called initial value problems.

Theorem*(IVP): (Nagy 2021,18,January)If the functionsa_1, a_0, b are continuous on a closed interval $I\epsilon R$ *the constant* $t_0 \epsilon I$ *and* $Y_0, Y_1 \epsilon R$ are arbitrary constants, then there is a unique solution$Y$, defined on I, of the initial value problem

$$y' + a_1(t)y' + a_0(t)y = b(t), \qquad y_0(t) = y_0, \qquad y'(t_0) = y_1$$

Section four Comparison and example

2.4.1. Comparison: IVP and BVP. We now review the initial boundary value:

(Nagy 2021,18,January) Problem for the equation above, where we showed in Theorem * that this initial value problem always has a unique solution.

Example: Find all solutions to the BVPs $y" + y = 0$ with the BCs:

(a) (a) $\begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases}$ (b) $\begin{cases} y\ (0) = 1, \\ y\left(\frac{\pi}{2}\right) = 1\ . \end{cases}$ (c) $\begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$

Solution: We first the roots of the characteristic polynomial$r^2 + 1 = 0$, that is, r±=±i. so the general solution of the differential equation is

$$y(x) = c_1 cos(x) + c_2 sin(x)$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$0 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 0$$

Therefore, there is no solution.

BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$1 = y(\frac{\pi}{2}) = c_2 \quad \Rightarrow \quad c_2 = 1$$

So there is a unique solution $y(x) = cos(x) + sin(x)$.

BC (c):

$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$

$-1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_2 = 1$

Therefore, c_2 is arbitrary, so we have infinitely many solutions.

$y(x) = cos(x) + c_2\, sin(x), \quad c_2 \epsilon R.$

Example: Find all solutions to the BVPs y" $+ 4y = 0$ with the BCs:

(A) $\begin{cases} y(0) = 1, \\ y\left(\frac{\pi}{4}\right) = -1. \end{cases}$ (B) $\begin{cases} y(0) = 1, \\ y\left(\frac{\pi}{2}\right) = -1. \end{cases}$ (C) $\begin{cases} y(0) = 1, \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

Solution: We first find the roots of the characteristic polynomial $r_2 + 4 = 0$, that is, $r_\pm = \pm 2i$. So the general solution of the differential equation is

$$y(x) = c_1 \, cos(2x) + c_2 \, sin(2x)$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$-1 = y(\frac{\pi}{4}) = c_2 \quad \Rightarrow \quad c_2 = -1$$

Therefore, there is a unique solution y(x) = cos (2x) $-$ sin(2x).

BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$-1 = y(\frac{\pi}{2}) = -c_1 \quad \Rightarrow \quad c_1 = 1$$

So, c_2 is arbitrary and we have infinitely many solutions

$$y(x) = cos(2x) + c_2 sinx(2x), \qquad c_2 \epsilon R$$

BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$$

$$1 = y(\frac{\pi}{2}) = -c_1 \quad \Rightarrow \quad c_2 = -1$$

Therefore, we have no solution.

Example: Find all numbers λ and nonzero functions y solutions of the BVP

$y'' + \lambda y = 0$ With $y(0) = 0,$ $y(L) = 0,$ $L > 0$

Solution: We divide the problem in three cases:

(a) $\lambda < 0,$ and (b) $\lambda = 0,$ and (c) $\lambda > 0.$

Case (a) $\lambda = -\mu^2 < 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$r^2 - \mu^2 = 0 \Rightarrow r_{\pm} = \pm\mu.$

The general solution is $y = c_+ e^{\mu x} + c_- e^{-\mu x}$. The BC imply

$0 = y(0) = c_+ + c_-,$ $0 = y(L) = c_+ e^{\mu L} + c_- e^{-\mu L}$

So from the first equation we get $c_+ = -c_-$, so

$0 = -c_- e^{\mu L} + c_- e^{-\mu L} \Rightarrow -c_-(e^{\mu L} - e^{-\mu L}) = 0 \Rightarrow c_- = 0 , c_+ = 0.$

So the only the solution is $y = 0$, then there are no Eigen functions with negative eigenvalues.

Case (b) : $\lambda = 0$, so the D.E is

$y'' = 0 \Rightarrow y = c_0 + c_1 x.$

The BC imply

$0 = y(0) = c_0,$ $0 = y(L) = c_1 L \Rightarrow c_1 = 0.$

So the only solution is $y = 0$, then there are no eigenfunctions with eigenvalue $\lambda = 0$.

Case (c): $\lambda = \mu^2 > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu i.$

The general solution is $y = c_+ \cos(\mu x) + c_- \sin(\mu x)$. The BC imply

$0 = y(0) = c_+, \quad 0 = y(L) = c_+ \cos(\mu L) + c_- \sin(\mu L).$

Since $c_+ = 0$, the second equation above is

$c_- \sin(\mu L) = 0, \quad c_- \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu n L = n\pi.$

So we get $\mu_n = n\pi / L$, hence the eigenvalue eigenfunction pairs are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$$

Since we need only one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n \geq 1$$

Example: Find the numbers λ and the nonzero functions y solutions of the BVP

$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0, \quad L > 0.$

Solution: We divide the problem in three cases:

(a) $\lambda < 0$, (b) $\lambda = 0$, and (c) $\lambda > 0$.

Case (a): Let $\lambda = \mu^2$, with $\mu > 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r\pm = \pm\mu$

The general solution is $y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$ The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 0 &= y'(L) = -\mu c_1 e^{-\mu L} + \mu c_2 e^{\mu L} \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{vmatrix} = \mu(e^{\mu L} + e^{-\mu L}) \neq 0$$

So, the linear system above for c_1, c_2 has a unique solution $c_1 = c_2 = 0$. Hence, we get theonly solution $y = 0$. This means there are no eigenfunctions with negative eigenvalues.

Case (b): let $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x, \qquad c_1, c_2 \epsilon R$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$0 = y(0) = c_1, \qquad 0 = y'(L) = c_2.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 0$.

Case (c): Let $\lambda = \mu^2$, with $\mu > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c1 \\ 0 &= y'(L) = -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L). \end{aligned}\right\} \Rightarrow \quad c_2 \cos(\mu L) = 0$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$cos(\mu L) = 0 \quad \Leftrightarrow \quad \mu_n L = (2n - 1)\frac{\pi}{2}, \quad n \geq 1.$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = c_n \sin(\frac{(2n-1)\,\pi\, x}{2L}), \qquad n \geq 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = \sin(\frac{(2n-1)\,\pi\, x}{2L}), \qquad n \geq 1.$$

Example: Find the numbers λ and the nonzero functions y solutions of the BVP

$$x^2\, y'' - x\, y' = -\lambda y, \qquad y(1) = 0, \qquad y(\ell) = 0, \quad \ell > 1$$

Solution: Let us rewrite the equation as

$$x^2\, y'' - x\, y' + \lambda y = 0$$

This is an Euler equidimensional equation

$$r(r-1) - r + \lambda = 0 \quad \Rightarrow \quad r^2 - 2r + \lambda = 0 \quad \Rightarrow \quad r_{\pm} = 1 \pm \sqrt{1-\lambda}$$

Case (a): Let$1 - \lambda = 0$, so we have a repeated root $r_+ = r_- = 1$. The general solution .To the differential equation is

$$y(x) = (c_1 + c_2 ln(x))x.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{array}{c} 0 = y(1) = c_1 \\ 0 = y(\ell) = (c_1 + c_2 \ln(\ell))\ell \end{array}\right\} \quad \Rightarrow \quad c_2 ln(\ell) = 0 \quad \Rightarrow \quad c_2 = 0.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 1$.

Case (b): Let $1 - \lambda > 0$, so we can rewrite it as $1 - \lambda = \mu^2$, with $\mu > 0$. Then , $r_{\pm} = 1 \pm \lambda$, and so the general solution to the differential equation is given by

$$y(x) = c_1 x^{1-\mu} + c_2 x^{1+\mu}$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{array}{r} 0 = y(1) = c_1 + c_2 \\ 0 = y(\ell) = c_1 \ell^{(1-\mu)} + c_2 \ell^{(1+\mu)} \end{array}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{vmatrix} = \ell(\ell^{\mu} - \ell^{-\mu}) = 0 \quad \Leftrightarrow \quad \ell \neq_{\pm} 1.$$

Since $\ell > 1$, the matrix above is invertible, and the linear system for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$. Hence we get the only solution $y = 0$. This means there are no eigenfunction with eigenvalues $\lambda < 1$.

Case (c): Let $1 - \lambda < 0$, so we can rewrite it as $1 - \lambda = -\mu^2$, with $\mu > 0$. Then $r_{\pm} = 1 i\mu$, and so the general solution to the differential equation is

$$y(x) = x[c_1 \, cos(\mu \, ln(x)) + c_2 \, sin(\mu \, ln(x))]$$

The boundary conditions imply the following conditions on $c_1 \; and \; c_2$,

$$\left.\begin{array}{r} 0 = y(0) = c_1, \\ 0 = y(\ell) = c_1 \ell \cos(\mu \ln(\ell)) + c_2 \ell \, sin(\mu \, ln(\ell)) \end{array}\right\} \quad \Rightarrow \quad c_2 \ell \, sin(\mu \, ln(\ell)) = 0$$

Since we are interested in nonzero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\sin(\mu \ln(\ell)) = 0 \quad \Leftrightarrow \quad \mu_n \, ln(\ell) = n\pi, \quad n \geq 0.$$

Recalling that $1 - \lambda_n = -\mu_n^2$ we get $\lambda_n = 1 + \mu_n^2$, hence

$$\lambda_n = 1 + \frac{n^2\pi^2}{ln^2(\ell)} \qquad y_n(x) = c_n x\, sin\left(\frac{n\pi\, ln(x)}{ln(\ell)}\right), \qquad n \geq 1$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = 1 + \frac{n^2\pi^2}{ln^2(\ell)} \qquad y_n(x) = x\, sin\left(\frac{n\pi\, ln(x)}{ln(\ell)}\right), \qquad n \geq 1$$

References

n.d. https://byjus.com/jee/eigenvalues-properties/.

D.Polyanin, Dr. Andrei. *scholarpedia.* 2008. http://www.scholarpedia.org/article/Partial_differential_equation (accessed 2 19, 2014).

Gladwell, Dr.Ian. *Boundary value problem.* Dallas,TX: Southern Methodist University, 2015.

John W. Cain, Angela Reynolds. *Ordinary and Partial Differential Equations.* Virginia Commonwealth University, Math Department, 2010.

L.POWERS, DAVID. *Boundary Value Problems.* New York and London: Eisevier, 1972.

M.Eather, jenny. *A Maths Dictionary for kids.* Aussie School House, 2006.

Nagy, Gabriel. *Ordinary Differential Equation.* East Lansing: Michigan State University, 2021,18,January.

Richard Courant, David Hilbert. *Richard Courant, David Hilbert.* Berlin: wiley, 2008.

پوختە

لەم پرۆژەیەدا، ئێمە لێکۆڵینەوە لە کێشەی بەهای سنوور دەکەین. ئێمە پێداچوونەوە بە بیردۆزی هاوکێشەی جیاوازی هێڵی دەکەین بۆ هەندێک بەهای سنووری دیاریکراو. ئەم کێشانە بۆ هاوکێشە جیاوازەکانی ئاسایی و هاوکێشەی جیاوازی بەشی

بە کورتی هەندێک تێبینی پێشکەش دەکات و پێناسەی دوو هاوکێشەی جیاوازی بەشی دەکات کە یەکەم بەرکەوتنە بۆ چارەسەرکردنی بەشێک لە هاوکێشە جیاوازەکان و کارکردن لەگەڵ کێشەکانی بەهای سنوور من نموونەیەکم لە .BVP و IVP و کارکردن لەگەڵ کێشەی بەهای سەرەتایی. لە کۆتاییدا بەراوردی نێوان کێشەی بەهای سنوورەوە وەرگرتووە