زانكوّى سهلاحهددين - هـوليّر
Salahaddin University-Erbil

# Laplace Transform 

Research Project

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## By:

Soran Jalal Muhamadamin
Supervised by:
L.Mudhafar Hamed.H

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## Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:


Supervisor: M.Mudhafar Hamed.H
Scientific grade :Lecturer
Date: / 4 /2023

In view of the available recommendations, I forward this report for debate by the examining committee

Signature:


Name: Dr. Rashad Rashid Haji
Chairman of the Mathematics Department
Scientific grade: Assistant Professor
Date: / 4 / 2023

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#### Abstract

The Laplace transform is a powerful tool formulated to solve a wide variety of boundary value problems. The strategy is to transform the difficult differential equations into simple problems in the Laplace domain, where solutions can be easily obtained. One then applies the inverse Laplace transform to retrieve the solution of the original problems.

This project consists of three parts; the first part defines Laplace transform and the inverse Laplace transform of some elementary functions. The second part is concerned with the complex inversion formula and explains the modification of Bromwich contour in case of branch point. In the third part some applications are solved.


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## Introduction

The Laplace transform is a transformation-it changes a function into another function. This transformation is an integral transformation-the original function is multiplied by an exponential and integrated on an appropriate region. Such an integral transformation is the answer to very interesting questions: Is it possible to transform a differential equation into an algebraic equation? Is it possible to transform a derivative of a function into a multiplication? The answer to both questions is yes, for example with a Laplace transform.
This is how it works. You start with a derivative of a function, $y^{\prime}(t)$, then you multiply it by any function, we choose an exponential $e^{-s t}$, and then you integrate on $t$, so we get

$$
y^{\prime}(t) \rightarrow \int e^{-s t} y^{\prime}(t) d t
$$

which is a transformation, an integral transformation. And now, because we have an integration above, we can integrate by parts-this is the big idea,

$$
y^{\prime}(t) \rightarrow \int e^{-s t} y^{\prime}(t) d t=e^{-s t} y(t)+s \int e^{-s t} y(t) d t
$$

So we have transformed the derivative we started with into a multiplication by this constant $s$ from the exponential. The idea in this calculation actually works to solve differential equations and motivates us to define the integral transformation $y(t) \rightarrow \tilde{Y}(s)$ as follows,

$$
y(t) \rightarrow \tilde{Y}(s)=\int e^{-s t} y(t) d t .
$$

The Laplace transform is a transformation similar to the one above, where we choose some appropriate integration limits-which are very convenient to solve initial value problems.
We dedicate this section to introduce the precise definition of the Laplace transform and how is used to solve differential equations. In the following sections we will see that this method can be used to solve linear constant coefficients
differential equation with very general sources, including Dirac's delta generalized functions.

## CHAPTER ONE

### 1.1. Background

Definition 1.1: Equation (M, 2006)
An equation is a mathematical statement containing an equals sign. Numbers may be represented by unknown variables. To solve an equation, the value of these variables must be found

Definition 1.2: Differential Equation (Kishan, 2006)
A differential equation (DE) is an equation in solving a function and its derivatives.

Example 1.1: A few differential equation
1: $\frac{d y}{d x}=\sin \mathrm{x}$
2: $\frac{d y}{d x}=\frac{x+1}{y-2}$

## Definition 1.3: Partial Differential equation:

A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the indepen variables.

## Definition 1.4: Ordinary Differential equation

In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. He term ordinary is used in contrast with the term partial differentia equation which may be with respect to more then one independent variable.

## Definition1.5: Order of a differential equation

The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation in the simplest cases, equations may be solved by direct integration.

## Example 1.2:

1: $\frac{d y}{d x}=e^{x} \quad$ first order
2: $\frac{d^{4} y}{d x^{4}}+y=0 \quad$ fourth order
Definition 1.6 : Degree
Is the highest power of the highest derivative in which occurs in the D.E

## Definition 1.7: Linear O.D.E

A differential equation in any order is said to be linear if satisfies
1 The dep.v is exist and of the first degree.
2 The derivatives $y^{\prime} y^{\prime \prime} y^{\prime \prime \prime}$ exist and each of them of the first degree.
3 The dep.v and the derivatives not multiply each other.

## Definition 1.8: Non-Linear Differential Equation

When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

## CHAPTER TWO

### 2.1. Definition and Theorem

Definition 2.1: The Laplace transform of a function $f$ defined on $D_{f}=(0, \infty)$ is

$$
F(s)=\int_{0}^{\infty} e^{-s t} f(t) d t
$$

defined for all $s \in D_{F} \subset \mathbb{R}$ where the integral converges. In these note we use an alternative notation for the Laplace transform that emphasizes that the Laplace transform is a transformation: $\mathcal{L}[f]=F$, that is

$$
\mathcal{L}[]=\int_{0}^{\infty} e^{-s t}() d t
$$

So, the Laplace transform will be denoted as either $\mathcal{L}[f]$ or $F$, depending whether we want to emphasize the transformation itself or the result of the transformation. We will also use the notation $\mathcal{L}[f(t)]$, or $\mathcal{L}[f](s)$, or $\mathcal{L}[f(t)](s)$, whenever the independent variables $t$ and $s$ are relevant in any particular context.

The Laplace transform is an improper integral-an integral on an unbounded domain. Improper integrals are defined as a limit of definite integrals,

$$
\int_{t_{0}}^{\infty} g(t) d t=\lim _{N \rightarrow \infty} \int_{t_{0}}^{N} g(t) d t .
$$

An improper integral converges iff the limit exists, otherwise the integral diverges. Now we are ready to compute our first Laplace transform.

Example 2.1. Compute the Laplace transform of the function $f(t)=1$, that is, $\mathcal{L}[1]$.
Solution: Following the definition,

$$
\mathcal{L}[1]=\int_{0}^{\infty} e^{-s t} d t=\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-s t} d t
$$

The definite integral above is simple to compute, but it depends on the values of $s$. For $s=0$ we get

$$
\lim _{N \rightarrow \infty} \int_{0}^{N} d t=\lim _{n \rightarrow \infty} N=\infty
$$

So, the improper integral diverges for $s=0$. For $s \neq 0$ we get

$$
\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-s t} d t=\lim _{N \rightarrow \infty}-\left.\frac{1}{s} e^{-s t}\right|_{0} ^{N}=\lim _{N \rightarrow \infty}-\frac{1}{s}\left(e^{-s N}-1\right)
$$

For $s<0$ we have $s=-|s|$, hence

$$
\lim _{N \rightarrow \infty}-\frac{1}{S}\left(e^{-s N}-1\right)=\lim _{N \rightarrow \infty}-\frac{1}{S}\left(e^{|s| N}-1\right)=-\infty .
$$

So, the improper integral diverges for $s<0$. In the case that $s>0$ we get

$$
\lim _{N \rightarrow \infty}-\frac{1}{S}\left(e^{-s N}-1\right)=\frac{1}{S} .
$$

If we put all these result together we get

$$
\mathcal{L}[1]=\frac{1}{s}, s>0 .
$$

Example 2.2. Compute $\mathcal{L}\left[e^{a t}\right]$, where $a \in \mathbb{R}$.
Solution: We start with the definition of the Laplace transform,

$$
\mathcal{L}\left[e^{a t}\right]=\int_{0}^{\infty} e^{-s t}\left(e^{a t}\right) d t=\int_{0}^{\infty} e^{-(s-\alpha) t} d t .
$$

In the case $s=a$ we get

$$
\mathcal{L}\left[e^{a t}\right]=\int_{0}^{\infty} 1 d t=\infty
$$

so the improper integral diverges. In the case $s \neq a$ we get

$$
\begin{aligned}
\mathcal{L}\left[e^{a t}\right] & =\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-(s-a) t} d t, s \neq a \\
& =\lim _{N \rightarrow \infty}\left[\left.\frac{(-1)}{(s-a)} e^{-(s-a) t}\right|_{0} ^{N}\right] \\
& =\lim _{N \rightarrow \infty}\left[\frac{(-1)}{(s-a)}\left(e^{-(s-a) N}-1\right)\right] .
\end{aligned}
$$

Now we have to remaining cases. The first case is:

$$
s-a<0 \Rightarrow-(s-a)=|s-a|>0 \Rightarrow \lim _{N \rightarrow \infty} e^{-(s-a) N}=\infty
$$

so the integral diverges for $s<a$. The other case is:

$$
s-a>0 \Rightarrow-(s-a)=-|s-a|<0 \Rightarrow \lim _{N \rightarrow \infty} e^{-(s-a) N}=0
$$

so the integral converges only for $s>a$ and the Laplace transform is given by

$$
\mathcal{L}\left[e^{a t}\right]=\frac{1}{(s-a)}, s>a
$$

| $f(t)$ | $F(s)=c[f(t)]$ | $D p$ |
| :--- | :--- | :--- |
| $f(t)=1$ | $F(s)=\frac{1}{s}$ | $s>0$ |
| $f(t)=e^{a x}$ | $F(s)=\frac{1}{(s-a)}$ | $s>a$ |
| $f(t)=t^{n}$ | $F(s)=\frac{n!}{s^{(a n+1)}}$ | $s>0$ |
| $f(t)=\sin (a t)$ | $F(s)=\frac{a}{s^{2}+a^{2}}$ | $s>0$ |
| $f(t)=\cos (a t)$ | $F(s)=\frac{s}{s^{2}-a^{2}}$ | $s>0$ |
| $f(t)=\sinh (a t)$ | $F(s)=\frac{s}{s^{2}-a^{2}}$ | $s>\mid a \\|$ |
| $f(t)=\cosh (a t)$ | $F(s)=\frac{n!}{(s-a)^{(a+1)}}$ | $s>a$ |
| $f(t)=t^{n} e^{a t}$ | $F(s)=\frac{b}{(s-a)^{2}+b^{2}}$ | $s>a$ |
| $f(t)=e^{a x t} \sin (b t)$ | $F(s)=\frac{(s-a)}{(s-a)^{2}+b^{2}}$ | $s>a$ |
| $f(t)=e^{a x} \cos (b t)$ | $F(s)=\frac{b}{(s-a)^{2}-b^{2}}$ | $s-a>\\|b\\|$ |
| $f(t)=e^{a x} \sinh (b t)$ | $F(s)=\frac{(s-a)}{(s-a)^{2}-b^{2}}$ | $s-a>\\|b\\|$ |
| $f(t)=e^{a x} \cosh (b t)$ |  | $F l \mid$ |

Definition 2.2: A function $f$ defined on $[0, \infty)$ is of exponential order $s_{0}$, where $s_{0}$ is any real number, iff there exist positive constants $k, T$ such that

$$
|f(t)| \leqslant k e^{s_{0} t} \text { for all } t>T .
$$

Theorem (Convergence of LT) 2.1: If a function $f$ defined on $[0, \infty$ ) is piecewise continuous and of exponential order $s_{0}$, then the $\mathcal{L}[f]$ exists for all $s>$ $s_{0}$ and there exists a positive constant $k$ such that

$$
|\mathcal{L}[f]| \leqslant \frac{k}{s-s_{0}}, s>s_{0}
$$

Proof of Theorem : From the definition of the Laplace transform we know that

$$
\mathcal{L}[f]=\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-s t} f(t) d t
$$

The definite integral on the interval $[0, N]$ exists for every $N>0$ since $f$ is piecewise continuous on that interval, no matter how large $N$ is. We only need to check whether the integral converges as $N \rightarrow \infty$. This is the case for functions of exponential order, because

$$
\left|\int_{0}^{N} e^{-s t} f(t) d t\right| \leqslant \int_{0}^{N} e^{-s t}|f(t)| d t \leqslant \int_{0}^{N} e^{-s t} k e^{s_{0} t} d t=k \int_{0}^{N} e^{-\left(s-s_{0}\right) t} d t
$$

Therefore, for $s>s_{0}$ we can take the limit as $N \rightarrow \infty$,

$$
|\mathcal{L}[f]| \leqslant \lim _{N \rightarrow \infty}\left|\int_{0}^{N} e^{-s t} f(t) d t\right| \leqslant k \mathcal{L}\left[e^{s_{0} t}\right]=\frac{k}{\left(s-s_{0}\right)}
$$

Therefore, the comparison test for improper integrals implies that the Laplace transform $\mathcal{L}[f]$ exists at least for $s>s_{0}$, and it also holds that

$$
|\mathcal{L}[f]| \leqslant \frac{k}{s-s_{0}}, s>s_{0}
$$

Theorem (Linearity) 2.2: If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all $a, b \in \mathbb{R}$ holds

$$
\mathcal{L}[a f+b g]=a \mathcal{L}[f]+b \mathcal{L}[g] .
$$

Proof of Theorem: Since integration is a linear operation, so is the Laplace transform, as this calculation shows,

$$
\begin{aligned}
\mathcal{L}[a f+b g] & =\int_{0}^{\infty} e^{-s t}[a f(t)+b g(t)] d t \\
& =a \int_{0}^{\infty} e^{-s t} f(t) d t+b \int_{0}^{\infty} e^{-s t} g(t) d t \\
& =a \mathcal{L}[f]+b \mathcal{L}[g] .
\end{aligned}
$$

Example 2.3. Compute $\mathcal{L}\left[3 t^{2}+5 \cos (4 t)\right]$.
Solution: From the Theorem above and the Laplace

$$
\begin{aligned}
\mathcal{L}\left[3 t^{2}+5 \cos (4 t)\right] & =3 \mathcal{L}\left[t^{2}\right]+5 \mathcal{L}[\cos (4 t)] \\
& =3\left(\frac{2}{s^{3}}\right)+5\left(\frac{s}{s^{2}+4^{2}}\right), s>0 \\
& =\frac{6}{s^{3}}+\frac{5 s}{s^{2}+4^{2}} .
\end{aligned}
$$

Therefore,

$$
\mathcal{L}\left[3 t^{2}+5 \cos (4 t)\right]=\frac{5 s^{4}+6 s^{2}+96}{s^{3}\left(s^{2}+16\right)}, s>0 .
$$

The Laplace transform can be used to solve differential equations. The Laplace transform converts a differential equation into an algebraic equation. This is so because the Laplace transform converts derivatives into multiplications. Here is the precise result.

Theorem (Derivative into Multiplication) 2.3: If a function $f$ is continuously differentiable on $[0, \infty)$ and of exponential order $s_{0}$, then $\mathcal{L}\left[f^{\prime}\right]$ exists for $s>s_{0}$ and

$$
\mathcal{L}\left[f^{\prime}\right]=s \mathcal{L}[f]-f(0), s>s_{0} .
$$

Proof of Theorem: The main calculation in this proof is to compute

$$
\mathcal{L}\left[f^{\prime}\right]=\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-s t} f^{\prime}(t) d t
$$

We start computing the definite integral above. Since $f^{\prime}$ is continuous on $[0, \infty)$, that definite integral exists for all positive $N$, and we can integrate by parts,

$$
\begin{aligned}
\int_{0}^{N} e^{-s t} f^{\prime}(t) d t & =\left[\left.\left(e^{-s t} f(t)\right)\right|_{0} ^{N}-\int_{0}^{N}(-s) e^{-s t} f(t) d t\right] \\
& =e^{-s N} f(N)-f(0)+s \int_{0}^{N} e^{-s t} f(t) d t
\end{aligned}
$$

We now compute the limit of this expression above as $N \rightarrow \infty$. Since $f$ is continuous on $[0, \infty)$ of exponential order $s_{0}$, we know that

$$
\lim _{N \rightarrow \infty} \int_{0}^{N} e^{-s t} f(t) d t=\mathcal{L}[f], s>s_{0}
$$

Let us use one more time that $f$ is of exponential order $s_{0}$. This means that there exist positive constants $k$ and $T$ such that $|f(t)| \leqslant k e^{s_{0} t}$, for $t>T$. Therefore,

$$
\lim _{N \rightarrow \infty} e^{-s N} f(N) \leqslant \lim _{N \rightarrow \infty} k e^{-s N} e^{s_{0} N}=\lim _{N \rightarrow \infty} k e^{-\left(s-s_{0}\right) N}=0, s>s_{0} .
$$

These two results together imply that $\mathcal{L}\left[f^{\prime}\right]$ exists and holds

$$
\mathcal{L}\left[f^{\prime}\right]=s \mathcal{L}[f]-f(0), s>s_{0} .
$$

Example 2.4. Verify the result in Theorem 4.1.5 for the function $f(t)=\cos (b t)$. Solution: We need to compute the left hand side and the right hand side and verify that we get the same result. We start with the left hand side,

$$
\mathcal{L}\left[f^{\prime}\right]=\mathcal{L}[-b \sin (b t)]=-b \mathcal{L}[\sin (b t)]=-b \frac{b}{s^{2}+b^{2}} \Rightarrow \mathcal{L}\left[f^{\prime}\right]=-\frac{b^{2}}{s^{2}+b^{2}}
$$

We now compute the right hand side,

$$
s \mathcal{L}[f]-f(0)=s \mathcal{L}[\cos (b t)]-1=s \frac{s}{s^{2}+b^{2}}-1=\frac{s^{2}-s^{2}-b^{2}}{s^{2}+b^{2}},
$$

so we get

$$
s \mathcal{L}[f]-f(0)=-\frac{b^{2}}{s^{2}+b^{2}}
$$

We conclude that $\mathcal{L}\left[f^{\prime}\right]=s \mathcal{L}[f]-f(0)$.

Theorem (Higher Derivatives into Multiplication) 2.4. If a function $f$ is $n$-times continuously differentiable on $[0, \infty)$ and of exponential order $s_{0}$, then $\mathcal{L}\left[f^{\prime \prime}\right], \cdots, \mathcal{L}\left[f^{(n)}\right]$ exist for $s>s_{0}$ and

$$
\begin{aligned}
\mathcal{L}\left[f^{\prime \prime}\right]= & s^{2} \mathcal{L}[f]-s f(0)-f^{\prime}(0) \\
& \vdots \\
\mathcal{L}\left[f^{(n)}\right] & =s^{n} \mathcal{L}[f]-s^{(n-1)} f(0)-\cdots-f^{(n-1)}(0)
\end{aligned}
$$

Proof of Theorem: We need to use Eq. (4.1.4) $n$ times. We start with the Laplace transform of a second derivative,

$$
\begin{aligned}
\mathcal{L}\left[f^{\prime \prime}\right] & =\mathcal{L}\left[\left(f^{\prime}\right)^{\prime}\right] \\
& =s \mathcal{L}\left[f^{\prime}\right]-f^{\prime}(0) \\
& =s(s \mathcal{L}[f]-f(0))-f^{\prime}(0) \\
& =s^{2} \mathcal{L}[f]-s f(0)-f^{\prime}(0) .
\end{aligned}
$$

The formula for the Laplace transform of an $n$th derivative is computed by induction on $n$. We assume that the formula is true for $n-1$,

$$
\mathcal{L}\left[f^{(n-1)}\right]=s^{(n-1)} \mathcal{L}[f]-s^{(n-2)} f(0)-\cdots-f^{(n-2)}(0)
$$

Since $\mathcal{L}\left[f^{(n)}\right]=\mathcal{L}\left[\left(f^{\prime}\right)^{(n-1)}\right]$, the formula above on $f^{\prime}$ gives

$$
\begin{aligned}
\mathcal{L}\left[\left(f^{\prime}\right)^{(n-1)}\right] & =s^{(n-1)} \mathcal{L}\left[f^{\prime}\right]-s^{(n-2)} f^{\prime}(0)-\cdots-\left(f^{\prime}\right)^{(n-2)}(0) \\
& =s^{(n-1)}(s \mathcal{L}[f]-f(0))-s^{(n-2)} f^{\prime}(0)-\cdots-f^{(n-1)}(0) \\
& =s^{(n)} \mathcal{L}[f]-s^{(n-1)} f(0)-s^{(n-2)} f^{\prime}(0)-\cdots-f^{(n-1)}(0)
\end{aligned}
$$

Example 2.5. Verify Theorem for $f^{\prime \prime}$, where $f(t)=\cos (b t)$.
Solution: We need to compute the left hand side and the right hand side in the first equation in Theorem and verify that we get the same result. We start with the left hand side,

$$
\begin{aligned}
& \mathcal{L}\left[f^{\prime \prime}\right]=\mathcal{L}\left[-b^{2} \cos (b t)\right]=-b^{2} \mathcal{L}[\cos (b t)]=-b^{2} \frac{s}{s^{2}+b^{2}} \Rightarrow \mathcal{L}\left[f^{\prime \prime}\right] \\
&=-\frac{b^{2} s}{s^{2}+b^{2}}
\end{aligned}
$$

We now compute the right hand side,

$$
\begin{aligned}
& s^{2} \mathcal{L}[f]-s f(0)-f^{\prime}(0)=s^{2} \mathcal{L}[\cos (b t)]-s-0=s^{2} \frac{s}{s^{2}+b^{2}}-s \\
& =\frac{s^{3}-s^{3}-b^{2} s}{s^{2}+b^{2}},
\end{aligned}
$$

so we get

$$
s^{2} \mathcal{L}[f]-s f(0)-f^{\prime}(0)=-\frac{b^{2} s}{s^{2}+b^{2}}
$$

Theorem (Multiplication into Derivative) 2.5. If a function $f$ is of exponential order $s_{0}$ with a Laplace transform $F(s)=\mathcal{L}[f(t)]$, then $\mathcal{L}[t f(t)]$ exists for $s>s_{0}$ and

$$
\mathcal{L}[t f(t)]=-F^{\prime}(s), s>s_{0} .
$$

Proof of Theorem: From the definition of the Laplace Transform we see that

$$
\begin{aligned}
\mathcal{L}[t f(t)] & =\int_{0}^{\infty} e^{-s t} t f(t) d t \\
& =\int_{0}^{\infty} \frac{d}{d s}\left(-e^{-s t}\right) f(t) d t \\
& =-\frac{d}{d s} \int_{0}^{\infty} e^{-s t} f(t) d t \\
& =-\frac{d}{d s} \mathcal{L}[f(t)] \\
& =-F^{\prime}(s)
\end{aligned}
$$

Theorem (Higher Powers into Derivative) 2.6: If a function $f$ is of exponential order $s_{0}$ with a Laplace transform $F(s)=\mathcal{L}[f(t)]$, then $\mathcal{L}\left[t^{n} f(t)\right]$ exists for $s>$ $s_{0}$ and

$$
\mathcal{L}\left[t^{n} f(t)\right]=(-1)^{n} F^{(n)}(s), s>s_{0}
$$

where we denoted $F^{(n)}=\frac{d^{n}}{d s^{n}} F$.
Proof of Theorem: We use induction one more time. The case $n=1$ is done in Theorem 4.1.7. We now assume that

$$
\mathcal{L}\left[t^{n} f(t)\right]=(-1)^{n} \frac{d^{n}}{d s^{n}} \mathcal{L}[f(t)],
$$

and we try to show that a similar formula holds for $n+1$. But this is the case, since

$$
\begin{aligned}
\mathcal{L}\left[t^{(n+1)} f(t)\right] & =\mathcal{L}\left[t^{n}(t f(t))\right] \\
& =(-1)^{n} \frac{d^{n}}{d s^{n}} \mathcal{L}[t f(t)]
\end{aligned}
$$

Definition 2.4. The inverse Laplace transform, denoted $\mathcal{L}^{-1}$, of a function $F$ is

$$
\mathcal{L}^{-1}[F(s)]=f(t) \Leftrightarrow F(s)=\mathcal{L}[f(t)]
$$

## CHAPTER THREE

3.1. Solving Differential Equations: The Laplace transform can be used to solve differential equations. We Laplace transform the whole equation, which converts the differential equation for $y$ into an algebraic equation for $\mathcal{L}[y]$. We solve the Algebraic equation and we transform back.
$\mathcal{L}\left[\begin{array}{c}\text { differential } \\ \text { eq. for } y .\end{array}\right] \xrightarrow{(1)} \begin{gathered}\text { Algebraic } \\ \text { eq. for } \mathcal{L}[y] .\end{gathered} \xrightarrow{l} \begin{gathered}\text { (2) }\end{gathered} \begin{gathered}\text { Solve the } \\ \text { algebraic } \\ \text { eq. for } \mathcal{L}[y] .\end{gathered} \xrightarrow{(3)} \begin{gathered}\text { Transform back } \\ \text { to obtain } y .\end{gathered}$
Example 3.1. Use the Laplace transform to find $y$ solution of

$$
\begin{gathered}
y^{\prime \prime}+9 y=0, y(0)=y_{0}, y^{\prime}(0)=y_{1} . \\
p(r)=r^{2}+9 \Rightarrow r_{ \pm}= \pm 3 i,
\end{gathered}
$$

and then we get the general solution

$$
y(t)=c_{+} \cos (3 t)+c_{-} \sin (3 t) .
$$

Then the initial condition will say that

$$
y(t)=y_{0} \cos (3 t)+\frac{y_{1}}{3} \sin (3 t) .
$$

We now solve this problem using the Laplace transform method. Solution: We now use the Laplace transform method:

$$
\mathcal{L}\left[y^{\prime \prime}+9 y\right]=\mathcal{L}[0]=0 .
$$

The Laplace transform is a linear transformation,

$$
\mathcal{L}\left[y^{\prime \prime}\right]+9 \mathcal{L}[y]=\mathcal{L}[0]=0 .
$$

But the Laplace transform converts derivatives into multiplications,

$$
s^{2} \mathcal{L}[y]-s y(0)-y^{\prime}(0)+9 \mathcal{L}[y]=0 .
$$

This is an algebraic equation for $\mathcal{L}[y]$. It can be solved by rearranging terms and using the initial condition,

$$
\left(s^{2}+9\right) \mathcal{L}[y]=s y_{0}+y_{1} \Rightarrow \mathcal{L}[y]=y_{0} \frac{s}{\left(s^{2}+9\right)}+y_{1} \frac{1}{\left(s^{2}+9\right)} .
$$

But from the Laplace transform table we see that

$$
\mathcal{L}[\cos (3 t)]=\frac{s}{s^{2}+3^{2}}, \mathcal{L}[\sin (3 t)]=\frac{3}{s^{2}+3^{2}},
$$

therefore,

$$
\mathcal{L}[y]=y_{0} \mathcal{L}[\cos (3 t)]+y_{1} \frac{1}{3} \mathcal{L}[\sin (3 t)] .
$$

Once again, the Laplace transform is a linear transformation,

$$
\mathcal{L}[y]=\mathcal{L}\left[y_{0} \cos (3 t)+\frac{y_{1}}{3} \sin (3 t)\right]
$$

We obtain that

$$
y(t)=y_{0} \cos (3 t)+\frac{y_{1}}{3} \sin (3 t) .
$$

Definition 3.1. The step function at $t=0$ is denoted by $u$ and given by

$$
u(t)= \begin{cases}0 & t<0 \\ 1 & t \geqslant 0 .\end{cases}
$$

Example 3.2. Graph the step $u, u_{c}(t)=u(t-c)$, and $u_{-c}(t)=u(t+c)$, for $c>$ 0.

Solution: The step function $u$ and its right and left translations1.



Recall that given a function with values $f(t)$ and a positive constant $c$, then $f(t-$ $c)$ and $f(t+c)$ are the function values of the right translation and the left translation, respectively, of the original function $f$. In Fig. 2 we plot the graph of functions $f(t)=e^{a t}, g(t)=u(t) e^{a t}$ and their respective right translations by $c>$ 0.


Example 3.3. Graph the bump function $b(t)=u(t-a)-u(t-b)$, where $a<$ b.

Solution: The bump function we need to graph is

$$
b(t)=u(t-a)-u(t-b) b b(t)= \begin{cases}0 & t<a \\ 1 & a \leqslant t<b \\ 0 & t \geqslant b\end{cases}
$$

The graph of a bump function is given in Fig. 3, constructed from two step functions. Step and bump functions are useful to construct more general piecewise continuous functions.




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 خالّى لق. لـه بـه شى سيّيهه م چه ند داو اكاريیه ك چاره سـه ر كراون

ملخص: يعد تحويل لابلاس أداة فوية تمت صباغنها لحل مجمو عة متنو عة من مشاكل القيمة الحدية. تتمتل الاستر اتيجية في تحويل المعادلات اللتفاضلية الصعبة إلى مشاكل بسبطة في مجال لاباس ، حيث بيكن الحصول على لالى الحلول بسهولة. ثم بطبق المر ء تحويل لابلاس العكسي لاسترداد حل المشكلات الأصلية. يتكون هذا المشروع من ثلاثة أجزاء. يحدد الجزء الأول تحويل لابلاس وتحويل لابلاس العكسي لبعض الوظائف الأولية. الجزء الثاني معني بصبغة الانعكاس المعقدة ويشرح تعديل كفاف بروميتش في حالة نقطة الفرع. في الجزء الثالث يتم حل بعض النطبيقات

