زانکۆی سەلاحەدین - هەولێر
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Second order linear differential equation

Research Project

Submitted to the department of (Mathematics) in partial fulfillum of the requirements for the degree of BSc. in (MATHEMATICS)

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CERTIFICATION OF THE SUPERVISORS

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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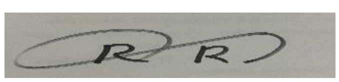
In view of the available recommendations, I forward this work for debate by the examining committee.

Signature: 

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ABSTRACT

We study linear second order ordinary differential equations. First we state some methods for solving these equations and find particular solution of homogenous linear second order ordinary differential equations. We give necessary conditions for linear second order ordinary differential equations to be exact equation and the integrating factor of linear second-order equations.

CONTENTS

INTRODUCTION

In this study consider linear homogeneous differential equation of second order with variable coefficient, A differential equation is an equation involving some function of interest along with a few of its derivatives. Typically, the function is unknown, and the challenge is to determine what that function could possibly be. Differential equations can be classified either as "ordinary" or as "partial". An ordinary differential equation is a differential equation in which the function in question is a function of only one variable. Hence, its derivatives are the "ordinary" derivatives encountered early in calculus. A partial differential equation is a differential equation in which the function of interest depends on two or more variables. Consequently, the derivatives of this function are the partial derivatives developed in the later part of most calculus courses. The order of a differential equation is simply the order of the highest order derivative explicitly appearing in the equation. In practice, higher-order differential equations are usually more difficult to solve than lower-order equations. Any function that satisfies a given differential equation is called a solution to that differential equation. "Satisfies the equation", means that, if you plug the function into the differential equation and compute the derivatives, then the result is an equation that is true no matter what real value we replace the variable with. And if that resulting equation is not true for some real values of the variable, then that function is not a solution to that differential equation. Our main interest will be second-order differential equations, both because it is natural to look at second-order equations after studying first-order equations, 2 and because second-order equations arise in applications much more often than do third-or fourth—order equations.

CHAPTER ONE

1.1.Background

Definition 1.1.1. (Kishan 2006) mathmattically an *equation* can be defined as astatment that supports the equality of two expressions which are connected by the equals sign " $=$ ".

Example : $5\mathrm{x} - 2 = 13$

Definition1.1.2. (H. 2006) An equation containting the dervatives of one or more dependent variable with respect to one or more independent variable is said to be *Differential equation* (D.E).

1-ordinary differential equation

2-partial differential equation

Example: $\frac{dy}{dx} + 10y = e(x)$

Definition 1.1.3. if a differential equation contains only ordinary derivatives of one or more dependent variables with respect it is said ordinary differential equation (O.D.E).

example : $\frac{dy}{dx} + 10y = e(x)$ one dependent variable and one independent variable

Definition 1.1.4. A differential equation involving partial differential of one or more dependent variable with respect to two pr more than two independent variable is said partial differential equation (P.D.E).

Definition 1.1.5. the *order of differential equation* is the highest derivative present in the equation.

Example : $\boldsymbol{y'} + \boldsymbol{y'''} + 4\mathrm{x} = \mathrm{sinx}$ third order (highest)

Definition 1.1.6. if a diffrentioal equation can be write as a polynomial in the dependent variable or its derivative than its degree is the exponent of the highest derivative is said *degree of differential equation*.

Example : $\boldsymbol{y' + 3y = 2}$ degree=1

Definition 1.1.7. A differential equation is said to be *linear ordinary differential equation* if

1- the dependent variable y and all its derivatives $y'.y''.y^n$ are of the first degree that is the power of each term in volving y is1.

2- the cofficients $a_0.a_1.a_2.\dots a_n$ of $y'.y''\dots y^n$ dependent on the independent variables x

3- no transce dental functions of the dependent variable y in volving like $(sin\, y. ey. log\, y\ .etc)$.

Definition 1.1.8. a differential is said to be *non linear ordinary equation* if it is not a linear.

Type of first order differential is:

1. separable equation .
2. homogeneous method.
3. integrating factor.
4. exact differential.

1-separable equation: $\frac{dy}{dx} = p(x)Q(x)$

$$ydy = xdx + c$$

2- Homogeneous method: A function $F(x.y)$ is called *homogeneos* function nth degree if satisfy the relation

tR $F(tx.ty) = tf_n(x.y)$

Example :. $f(x.y) = xny$ $t \in R$

$F(tx.ty) = (tx)^2(ty)$

$= (t^2x^2)(ty)$

$= t^{\,3}f(x.y) = F$

3- Integrating factor :when multiply both side of non-exact D.E by suitable factor in which changed in to exact D.E.

4- Exact diffrential eqe: A function $f(x\,.\,y)$ is called exact function if it can be written of the form

$$df(x.y) = (f/x)dx + (f/y)dy = 0$$

Definition 1.1.9. $M(x.y)dx + N(x.y)dy = 0$ is called Homogeneose .D.E

(H.D.E) if both functions M and N are H functions of the same degree .

Example: $(x^2 + y^2)dx + xy\,dy\ = 0$ (1)

Suppos $y/x = v$

$$y = xv \qquad (2)$$

$$dy = xdv + vdx$$

sub.equ (2) in equ (1) we gets D.E

$$(x^2 + x^2v^2)dx + x^2v(xdv + vdx) = 0$$

$$[x^2(1 + v^2) + x^2v^2]dx + x^3vdv = 0 \qquad * (1/x(1 + 2v)$$

$$\frac{dy}{x} + \frac{v}{1+2v^2}dv = 0$$

$$ln\,x\ + \frac{1}{4}\ ln(1 + 2v^2) = c$$

$$4ln\,x\ + ln(1 + 2v^2) = 4c$$

$$lnx^4 + ln(1 + 2v^2) = 4c$$

$$ln[x^4(1 + 2v^2)] = 4c$$

$x^4(1+2v^2)=c1 \quad where\ e^{4c}=c1$

$subst\ v=\frac{y}{x} \quad x^4(1+2(\frac{y^2}{x^2}))=c1$

$x^4+2y^2x^2=c1$ G.solu of D.E

Definition 1.1.10. (R.Diprima 2012)

second order linear equation Is an equation of the form $y''+p(x)y'+Q(x)y=R(x)$

Example: solve $y''+3y'+2y=e^x-3$

Solution: we have $\lambda^2+3\lambda+2=0$

Factoring gives $(\lambda+2)(\lambda+1)=0$ so $\lambda\in\{-1,-2\}$

and we have $y_1=e^{-x}$ and $y_2=e^{-2x}$

Let $y=Ae^x+B$ then

$y'=Ae^x$ and $y''=Ae^x$ giving as

$$y''+3y'+2y=Ae^x+3Ae^x+2Ae^x+2B$$

$$=6Ae^x+2B$$

$$6A=1 \implies A=\frac{1}{6} \quad and \quad 2B=-3 \Rightarrow B=\frac{-3}{2}$$

There fore $y_p=e^x/6-\frac{3}{2}$ we have

$$y=C_1e^{-x}+C_2e^{-2x}+\frac{e^x}{6}-\frac{3}{2}$$

1.2.Solutions to the Initial Value Problem (IVP) (chuchill n.d.)

Here is the first of the two main results in this section. Second order linear differential equations have solutions in the case that the equation coefficients are continuous functions. Since the solution is unique when we specify two initial conditions, the general solution must have two arbitrary integration constants.

Theorem1.2.1. (IVP). (chuchill n.d.) If the functions a_1, a_0,b are continuous on a closed interval $I \in R$, the constant $t_0 \in$ I, and $y_0 . y_1 \in R$ are arbitrary constants, then there is a unique solution (y), defined on I, of the initial value problem.

$$y'' + a_1(t)y' + a_0(t)y = b(t)$$

$$y(t_0) = y_0 \qquad\qquad y'(t_0) = y_0$$

Definition 1.2.1. Two functions $y_1 . y_2$ are called *linearly dependent* iff they are proportional. Otherwise, the functions are linearly independen.

Definition 1.2.2. An operator L is a *linear operator* iff for every pair of functions y_1, y_2 and constants $c_1 . c_2$ holds

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

Example:

(a) Show that y_1 (t) = sin(t), y_2 (t) = 2 sin(t) are linearly dependent.

(b) Show that y_1 (t) = sin(t), y_2 (t) = tsin(t) are linearly independen

Solution:

Part (a): This is trivial, since 2y_1 (t) − y_2 (t) = 0.

Part (b): Find constants c_1, c_2 such that for all t ∈ R holds

c_1 sin(t) + c_2 tsin(t) = 0.

Evaluating at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ we obtain

$c_1 + \frac{\pi}{2} \quad c_2$=0 if $t = \frac{\pi}{2}$

$c_1 + \frac{3\pi}{2} c_2$ = 0 ⇒ c_1 = 0, c_2 =0.

y_1, y_2 linearly independent

Definition 1.2.3. (R.Diprima 2012) any linearly independent set $y_1, y_2 .... y_n$ of n solution of the homogeneous nth-order differential equation on an interval is said to be *fundomental set of solution* on the interval .

Example : Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$are fundamental solutions to the equation

Solution: We first show that y_1 and y_2 are solutions to the differential equation, since

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & y^{-1} \\ \frac{1}{2} t^{-\frac{1}{2}} & -t^{-1} \end{vmatrix} = -t^{\frac{3}{2}} - \frac{1}{2} t^{-\frac{3}{2}} = \frac{3}{2} t^{-\frac{3}{2}} = \frac{-3}{\sqrt[2]{t^3}}$$

Since $w \neq 0$ for $t > 0$, $y_1 . y_2$ from a fandametsl equation.

$2t^2 y'' + 3ty' - y = 0 \quad t > 0$

It is not difficult to see that $y_1$and $y_2$are linearly independent. It is clear that they are not proportional to each other. A proof of that statement is the following: Find the constants c_1 and $c_2$such that

$0 = c_1 y_1 + c_2\, {}_{y_2} = c_1\, e^t + c_2\, e^{-2t}$

t ∈ R ⇒ 0 = $c_1\, e^t - 2c_2\, e^{-2t}$

The second equation is the derivative of the first one. Take t = 0 in both equations,

0 = c_1 + c_2, 0 = $c_1 - 2c_2$ ⇒ c_1 = c_2 = 0.

We conclude that y1 and y2 are fundamental solutions to the differential equation above.

1.3.Wronskian (R.Diprima 2012)

The Wronskian Function. We now introduce a function that provides important information about the linear dependency of two functions y1, y2. This function, W,

is called the Wronskian to honor the polish scientist Josef Wronski, who first introduced this function in 1821 while studying a different problem.

Definition:- The Wronskian of the differentiable functions y_1, y_2 is the function

Example :-Find the Wronskian of the functions:

(a) y_1 (t) = sin(t) and y_2 (t) = 2 sin(t). (ld)

(b) y_1 (t) = sin(t) and y_2 (t) = tsin(t). (li)

Solution:

Part (a): By the definition of the Wronskian:

$$w_{12}\text{ (t)} = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ cos(t) & 2\,cos(t) \end{vmatrix} = \sin(t)2\cos(t) - \cos(t)2\sin(t)$$

We conclude that w_{12} (t) = 0. Notice that y1 and y2 are linearly dependent.

Part (b): Again, by the definition of the Wronskian:

$$w_{12}\text{ (t)} = \begin{vmatrix} \sin(t) & t\sin(t) \\ cos(t) & t\,cos(t) \end{vmatrix} = \sin(t)[\sin(t) + t\cos(t) - \cos(t)t\sin(t)].$$

We conclude that w_{12} (t) = sin^2 (t).

Notice that y1 and y2 are linearly independent.

It is simple to prove the following relation between the Wronskian of two functions and the linear dependency of these two function

Theorem (IVP) 1.3.1. If the functions $a_1.\ a_2.b$ are continuous on a closed interval I ⊂ R,

the constant$t_0 \in I$, and $y_0.y_1 \in R$ are arbitrary constants, then there is a unique solution y, defined on I, of the initial value problem

$$y'' + a_1(t)y' + a_0(t)y = b(t).y(t_0) = y_0$$

$$y'(t_0) = y_1.$$

Theorem 1.3.2. (Linear Operator). (Agarwal 2008) The operator

L(y) = y'' + a_1 y'+ a_0 y, where a_1, a_0 are continuous functions and y is a twice differentiable function, is a linear operator.

Proof : This is a straightforward calculation:

$L(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)$'' + a_1 $(c_1y_1 + c_2y_2)$' + a_0 $(c_1y_1 + c_2y_2)$.

Recall that derivations is a linear operation and then reoorder terms in the following way,

L$(c_1y_1 + c_2y_2) = c_1y'' + a_1 c_1y'_1 + a_0 c_1y_1 + c_2 y''_2 + a_0 c_2$ Introduce the definition of L back on the right-hand side. We then conclude that

L$(c_1y_1 + c_2y_2)$ =c_1 L(y_1) + c_2L(y_2).

CHAPTER TWO

2.1. Reduction of Order Methods Sometimes a solution to a second order (S. &. Rao 1996)

differential equation can be obtained solving two first order equations, one after the other. When that happens we say we have reduced theorder of the equation. We use the ideas in Chapter 1 to solve each first order equation.

In this section we focus on three types of differential equations where such reduction oforder happens. The first two cases are usually called special second order equations and thethird case is called the conservation of the energy.

We end this section with a method that provides a second solution to a second order

equation if you already know one solution. The second solution can be chosen not proportional to the first one. This idea is called the reduction order method— although all four ideas we study in this section do reduce the order of the original equation.

2.1.1Theorem (Function y Missing) (G.simmons, Diffrentioal equations with applicationsand historical 1991)

If a second order differential equation has the form $.y'' = f(t.y')\ then\ v = y'$ satisfies the first order equation $v' = f(t\ .v)$ the proof is trivial s0 we go directly to an example .

Example . Find the general solution of the second order nonlinear equation

$y'' = -2t(y')^2$with initial conditions $y(0) = 2$. $y'(0) = -1$.

Solution: Introduce v = y' Then v'=y" and

$$V' = -2t\, v^2 \quad \Rightarrow \quad \frac{v'}{v^2} = -2t \quad \Rightarrow \quad -\frac{1}{v} = -t^2 + c.$$

So, $\frac{1}{y'} = t^2 - c$, that is, $y' = \frac{1}{t^2 - c}$. The initial condition implies

$$-1 = y'(0) = -\frac{1}{c} \Rightarrow c = 1 \Rightarrow y' = \frac{1}{(t^2 - 1)} .$$

Then. $y = \int \frac{dt}{t^2-1} + c$. We integrate using the method of partial fractions,

$\frac{1}{(t^2-1)} = \frac{1}{(t-1)(t+1)} = \frac{a}{(t-1)} + \frac{b}{(t+1)}$. Hence, $1 = a(t+1) + b(t-1)$. Evaluating

at t = 1 and t = −1 we get $a = \frac{1}{2}$ . . b= $-\frac{1}{2}$.

So $\frac{1}{t^2-1} = \frac{1}{2}$ [$1\,(t - 1) - \frac{1}{(t+1)i}$] . Therefore, the integral is simple to do.

$y = \frac{1}{2}(\ln|t - 1| - \ln|t + 1|) + c$. $2 = y(0) = \frac{1}{2}(0 - 0) + c$.

We conclude $y = \frac{1}{2}(ln\,|t - 1| - ln\,|t + 1|) + 2$.

The case (b) is way more complicated to solve.

Example :. Find a second solution y_2 linearly independent to the solution

$y_1(t) = t$ of the differential equation

$t^2 y'' + 2ty' - 2y = 0$.

Solution: We look for a solution of the form $y_2(t) = t\, v(t)$. This implies that

$$y'_2 = t\, v' + v \;\; .$$

$$y''_2 = t\, v'' + 2v'.$$

So, the equation for v is given by

$$0 = y''\, t^2 \; (t\, v'' + 2v') + 2t \; (t\, v' + v) - 2t\, v$$

$$= t3v'' + (2\, t^2 + 2t^2)v' + (2t - 2t)v$$

$$= t^3V'' + (4t^2)\, v' \Rightarrow v'' + \frac{4}{t}v' = 0.$$

Notice that this last equation is precisely Eq, since in our case we have

$$y_1 = t.\; p(t) = \frac{2}{t} \Rightarrow t\, v'' + [2 + \frac{2}{t}t]v' = 0.$$

The equation for v is a first order equation for w = v'

, given by

$$\frac{W'}{w} = -\frac{4}{t} \Rightarrow w(t) = c_1 t^{-4}$$

$$.\, c_1 \in R.$$

Therefore, integrating once again we obtain that v $= c_2 \; t^{-3} + c_3$, c_2, $c_3 \in R$.

and recalling that y2 = t v we then conclude that

$$y_2 = c_2 t^{-3} + c_3 t.$$

Choosing $c_2 = 1$ and $c_3 = 0$ we obtain that y_2 (t) = t^{-2}. Therefore, a fundamental solutionset to the original differential equation is given by

$$y_1(t) = t.\ y_2\ (t)\ = \frac{1}{t^2}$$

2.2. Charactaric polynomail (w.wright 2013) :

The characteristic polynomial and characteristic equation of the

second order linear homogeneous equation with constant coefficients

$$y'' + a_1 y' + a_0 y = 0$$

are given by

$$p(r) = r^2 + a_1 r + a_0\ \ .p(r) = 0.$$

Example:-. Find solutions to the equation

$$y'' + 5y' + 6y = 0.$$

Solution: We try to find solutions to this equation using simple test functions. For example,

it is clear that power functions y = t^n won't work, since the equation

$$n(n - 1)\, t^{(n-2)} + 5n\ t^{(n-1)} + 6\, t^n = 0$$

cannot be satisfied for all t ∈ R. We obtained, instead, a condition on t. This rules outmmpower functions. A key insight is to try with a test function having a derivative proportional to the original function,

$$y'(t) = r\, y(t).$$

Such function would be simplified from the equation. For example, we try now with the test function y(t) = e^{rt}.

If we introduce this function in the differential equation we get

$$(r^2 + 5r + 6)\, e^{rt} = 0 \Leftrightarrow r^2 + 5r + 6 = 0.$$

We have eliminated the exponential and any t dependence from the differential equation,

and now the equation is a condition on the constant r. So we look for the appropriate values

of r, which are the roots of a polynomial degree two,

$$r\pm = 1.2$$

$$-5 \pm \sqrt{25 - 24} - \frac{1}{2}(-5 \pm 1) \Rightarrow$$

$$r+ = -2.$$

$$r- = -3.$$

We have obtained two different roots, which implies we have two different solutions,

$$y_1(t) = e^{-2t}$$

$$.\, y_2(t) = e^{-3t}$$

These solutions are not proportional to each other, so the are fundamental solutions to the differential equation in . Therefore, Theorem in § 2.1 implies that we have found all possible solutions to the differential equation, and they are given by

$$y(t) = c_1 \; e^{-2t} + c_2 \; e^{-3t}. \qquad c_1 . c_2 \in R$$

Example :- Find the solution y of the initial value problem

$$Y'' + 5y' + 6y = 0 \qquad y(0) = 1 \qquad y'(0) = -1$$

Solution: We know that the general solution of the differential equation above $is \quad y_{gen}(t) = c_+ \ e^{-2t} + c_- \ e^{-3t}$

We now find the constants c+ and c- that satisfy the initial conditions above,

$$1 = y(0) = c_+ + c_-$$

$$-1 = y'(0) = -2\,c_+ - 3c_-$$

$$\Rightarrow c+ = 2 \ . \ c- = -1.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

Example:- Find the general solution ygen of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solutions of the form y(t) = e^{rt}, where r is solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \Rightarrow r = \frac{1}{4}(3 + \sqrt{9-8})$$

$$\Rightarrow r = \frac{1}{4}(3 - \sqrt{9-8})$$

$$r+ = 1 \ . \quad r- = -\frac{1}{2}.$$

Therefore, the general solution of the equation above $y_{gen}(t) = c_+ \ e^{t} + c_- \ e^{\frac{t}{2}}.$

CHAPTER THREE

3.1.Euler equidmential equation (E.coddington, An introduction to ordinary diffrentioal equations 1961)

The Euler equidimensional equation for the unknown y with singular

point at t_0 ∈ R is given by the equation below, where a1 and a0 are constants,

$$(t - t_0)2y'' + a1\,(t - t_0)\, y' + a_0 \; y = 0.$$

example:- Find the general solution of the Euler equation below for t > 0,

$$t^2 y'' + 4t\, y' + 2\, y = 0.$$

Solution: We look for solutions of the form $y(t) = t^r$

, which implies the $t\, y'(t) = r\, t^r \quad t^2\, y''(t) = r(r - 1)t^r$

therefore, introducing this function y into the differential equation we obtain

$$[r(r - 1) + 4r + 2]\, t^r = 0 \Leftrightarrow r(r - 1) + 4r + 2 = 0$$

The solutions are computed in the usual way,

$$r^2 + 3r + 2 = 0 \Rightarrow \quad r+= \frac{1}{2}\left[-3 + \sqrt{9-8}\right].r-= \frac{1}{2}[-3 - \sqrt{9-8}]$$

$$r- = 1 \quad .r+= 2$$

So the general solution of the differential equation above is given by

$$y_{gen}(t) = c_+\, t^{-1} + c_-\; t^{-2}$$

Example . Find the general solution of the Euler equation below for t > 0,

$$t^2y'' - 3t\,y' + 4\,y = 0.$$

Solution: We look for solutions of the form y(t) = t^r

, then the constant r must be solution of the Euler characteristic polynomial

r(r - 1) - 3r + 4 = 0 ⇔ r^2- 4r + 4 = 0 ⇒ r+ = r- = 2.

Therefore, the general solution of the Euler equation for t > 0 in this case is given by

$$ygen(t) = c + t^2 + c - t^2\ ln(t).$$

Example . Find the general solution of the Euler equation below for

t > 0,

$$t^2\,y'' - 3t\,y' + 13\,y = 0.$$

Solution: We look for solutions of the form y(t) = t^r , which implies that

$$ty'(t) = rt^r$$

$$t^2\,y''(t) = r(r - 1)t^r$$

therefore, introducing this function y into the differential equation we obtain

$$[r(r - 1) - 3r + 13] \qquad = 0 \Leftrightarrow r(r - 1) - 3r + 13 = 0.$$

The solutions are computed in the usual way,

$$r^2 - 4r + 13 = 0 \Rightarrow r+ = \frac{1}{2}\,[4 + \sqrt{-36}]$$

$$r- = \frac{1}{2}\,[4 - \sqrt{-36}]$$

$$r+ = 2 + 3i \;.\; r- = 2 - 3i.$$

So the general solution of the differential equation above is given by

$$ygen(t) = c + t^{2+3i} + c - t^{(2-3i)}$$

3.2. Real Solutions for Complex Roots (E.coddington, An introduction to ordinary diffrentioal equations 1999)

We study in more detail the solutions to the Euler equation in the case that the indicial polynomial has complex roots. Since these roots have the form

$$r+ = -\frac{a_1 - 1}{2} + \frac{1}{2}\sqrt{((a_1 - 1)2 - 4a_0)}$$

$$r\text{-}= -\frac{a1 - 1}{2} - \frac{1}{2}\sqrt{((a_1 - 1)2 - 4a_0)}$$

the roots are complex-valued in the case $(p_0 - 1)^2 - 4\,p_0 < 0$. We use the notation

r+- = α ± iβ, with $\alpha = -\frac{a1 - 1}{2}$ $\quad \beta = \sqrt{(a_0 - \frac{(a1 - 1)2}{4})}$.

The fundamental solutions in Theorem 2.4.2 are the complex-valued functions

$\tilde{y} + (t) = t^{(\alpha + i\beta)}$, $\tilde{y} - (t) = t^{(\alpha - i\beta)}$

The gebneral solution constructed from these solutions is

$ygen(t) \ = \ c\tilde{} + t^{(\alpha+i\beta})\quad + \quad c\tilde{} \ - t^{(\alpha+i\beta})\quad . c\tilde{}+. c\tilde{}- \in$ C.

This formula for the general solution includes real valued and complex valued solutions.

But it is not so simple to single out the real valued solutions. Knowing the real valued solutions could be important in physical applications. If a physical system is described by a differential equation with real coefficients, more often than not one is interested in finding real valued solutions. For that reason we now provide a new set of fundamental solutions

that are real valued. Using real valued fundamental solution is simple to separate all real valued solutions from the complex valued ones.

3.2.1. Theorem (Real valued fundamental solution) (E.coddington, An introduction to ordinary diffrentioal equations 1999)

If the differential equation

$$(t \ - \ t_0)^2 y'' + \ a_1(t \ - \ t_0)y' + \ a_0 \, y \ = \ 0 \quad t \ > t_0$$

where a1, a0, t0 are real constants, has indicial polynomial with complex roots

$r+- = \ \alpha \ \pm \ i\beta$ and complex valued fundamental solutions for t > t_0,

$y\tilde{} + (t) \ = \ (t \ - \ t_0)^{\, \alpha+i\beta)}$ $\qquad (y\tilde{} - (t) \ = \ (t \ - \ t_0)^{\, \alpha+i\beta)}$ then the equation also has real valued fundamental solutions for t > t0 given by

$$y + (t) = \ (t \ - \ t_0)\alpha \cos(\beta \ln(t \ - \ t_0))$$

$$y - (t) = \ (t \ - \ t_0) \, sin \ (\beta \ ln(t - \ t_0)$$

Proof : For simplicity consider the case to to = 0. Take the solutions

$$\tilde{y} + (t) = t^{(\alpha+i\beta).}$$

$$.\tilde{y} - (t) = t^{(\alpha-i\beta).}$$

Rewrite the power function as follows,

$$\tilde{y} + (t) = t^{(\alpha+i\beta).} = t^{\alpha} t^{i\beta} = t^{\alpha} e^{\ln(t\, i\beta)} \quad = t^{\alpha} \; e^{\beta i \ln(t)} \; \Rightarrow$$

$$\tilde{y} + (t) = t^{\alpha} \; e^{\beta i \ln(t)}$$

A similar calculation yields

$$\tilde{y} - (t) = t^{\alpha} \; e^{\beta i \ln(t)}$$

Recall now Euler formula for complex exponentials,

$e^{\theta i} = cos(\theta) + isin(\theta)$ then we get

$$\tilde{y} + (t) = t^{\alpha}[cos \; (\beta\, ln(t)) \quad \tilde{y} - (t) = e^{\alpha}[cos - \; (\beta\, ln(t))$$

Since ˜y+ and ˜y- are solutions to Eq, so are the function

$$y1(t) = \frac{1}{2}[\tilde{y}1(t) + \tilde{y}2(t)] \; . \; y2(t) = \frac{1}{2i}[\tilde{}1(t) - \tilde{y}2(t)]$$

It is not difficult to see that these functions are

$$y + (t) = t^{\alpha} cos\, \beta\, ln(t) \; . \; y - (t) = t^{\alpha} \; sin\, \beta\, ln(t)$$

To prove the case having t0 6= 0, just replace t by $(t - t0)$ on all steps above. This establishes the Theorem.

Example Find a real-valued general solution of the Euler equation below for t > 0,

$t^2 y'' - 3t\, y' + 13\, y = 0.$

Solution: The indicial equation is $r^{r-1} - 3r + 13 = 0$, with solutions

$$r^2 - 4r + 13 = 0 \Rightarrow r+ = 2 + 3i \qquad r- = 2 - 3i.$$

A complex-valued general solution for t > 0 is,

ygen(t) = ˜c+ $t^{(2+3\mathrm{i})}$+ ˜c- $t^{(2-3\mathrm{i})}$

c˜+, c˜-∈C

A real-valued general solution for t > 0 is

ygen(t) = c+ t^2cos(3 ln(t))+ c- t^2sin (3 ln(t))

c+, c- ∈C .

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پوختە:

ئێمە هاوکێشەی جیاکاریە ئاساییەکانی ڕیزبەندی دووەمی هێڵی دەخوێنین.سەرەتا هەندێك رادەگرین شێوازەکان بۆچارەسەرکردنی ئەم هاوکێشانە ودۆزینەوەی شیکاری تایبەتی بۆ هاوکێشە جیاکاریە ئاساییەکانی ڕیزبەندی دووەمی هێڵی.مەرجی پێویست دەبەخشین بۆ ڕیزبەندی دووەمی هێڵی هاوکێشەی جیاکاریە ئاسایی بۆ ئەوەی هاوکێشەی تەواو بێت ویەکخستنی فاکتەری هاوکێشەکانی ڕیزبەندی دووەمی هێڵی..