زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Systems of Linear Deferential Equations

Research Project

Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of BSc. in (Mathematics)

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Abstract

In the work we Stade to system of Linear Differential Equation. We describe the main ideas to solve certain differential equation. We’re talking about a group of definition (equations, differential equation, ordinary differential equation, partial differential equation order, degree, Linear, Non linear,quasi-Linear differential equation). Also we talk about content system of Linear Differential equation, we’re talking about the ways in which the system of linear differential equation is implement. First by, first order linear differential system and homogeneous and diagonalizable ,we’re explain it on same example and Initial value problem and Theorem (Existence and uniqueuss) and Theorem (first order Reduction)and theorem (second order Reduction)we have another road and that’s it (homogeneous system). In this way we use some definition and explain them for example, we’re talk about homogenous diagonalizable system. In one of the other ways.

Acknowledgment

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My thanks appreciations go to Mathematical Department and all my valuable teachers.

Table of Content

Introduction

Newton's second law of motion for point particles is one of the first defferential equations ever written. Even this early example of a deferential equation consists not of a single equation but of a system of three equation on three unknowns. The unknown functions are the particle three coordinates in space as function of time. One important difficulty to solve a defferential system is that the equations in a system are usually coupled. One cannot solve for one unknown function without knowing the other unknowns. In this chapter we study how to solve the system in the particular case that the equations can be uncoupled. We call such systems diagonalizable. Explicit formulas for the solutions can be written in this case. Later we generalize this idea to systems that cannot be uncoupled.

CHAPTER ONE

Definition

Equation 1.1[(M, 2006)**]:** An equation is a mathematical statement containing an equals sign. Numbers may be represented by unknown variables. To solve an equation, the value of these variables must be found.

1.1 Example: 5x2 +2x=16

Differential Equation 1.2 [**(Kishan, 2006)**] Any relation involving the dependent variable, independent variable and the differential coefficient of the dependent variable with respect to the independent variable is known as a differential equation.

Example:

1.$\frac{dy}{dx} = cotx$

2.. $y' = 3x + y''$

Ordinary Differential Equation 1.3[(M, 2006)**] :** In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. He term ordinary is used in contrast with the term partial differentia equation which may be with respect to more then one independent variable

For Example .

$$\frac{dy}{dx} + 2xy = e^x \qquad y''' - 3y(y'') - 2x(y')^3 = 7$$

Partial Differential Equation 1.4[(M, 2006)**]:** A partial differential equation (or briefly a PDE) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the indepen variables.

For Example .

$$\frac{\partial^2 z}{\partial x \partial y} + 6xy\ \frac{\partial^2 z}{\partial x^2} = 2x$$

Order of a differential equation1.5[(M, 2006)**]**: The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation in the simplest cases, equations may be solved by direct integration

For Example

$\frac{dy}{dx} + 2xy = e^x$ order 1

$\frac{dy^2}{\partial d} + y\ cos\ x + \frac{dy}{dx} = \tan hx$ order 2

Degree of a differential equation 1.6: **[Kishan, 2006]** The degree of a differential equation is the degree of the highest differential coefficient, which occurs in it, after when the differential equation has been cleared of radical and fractional powers.

For Example

$\frac{dy^2}{dx^2} + (\frac{dy}{dx})^3 + y = o$ Degree 1

$(\frac{dy^2}{dx^2})^3 + \frac{dy}{dx} + y = o$ Degree 3

Liner Defferential Equation1.7[(M, 2006)**]**: A differential equation in any order is said to be linear if satisfies:

1- The dep.v is exist and of the first degree.
2- The derivatives y', y'', y''' exist and each of them of the first degree.
3- The dep.v and the derivatives not multiply each other.

For example :

$$y'' + xy'' + 2y = \tan x$$
$$y'' + y = o$$
$$y'' + y + x = o$$

Non-Linear Differential Equation1.8[(M, 2006)**]:.** When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

For example:

$$\frac{dy^2}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 + 5y = x^2$$
$$y'' + 4yy' + 2y = \cos x$$
$$y'' + 3xy' + 5y^6 = x^2$$

. Quasi-linear equation1.9[(M, 2006)**]:** A first order p.d.e. Is said to be a quasi-linear equation if it is linear in p and q,

For example :

$$(y''')^3 e^x + y'' + y = o$$

CHAPTER 2

Systems of Linear Defferential Equations

Definition 2.1[(Sons J. a., 1967)]: An $n \times n$ **first order linear differential system** is the equation

$$\acute{x}(t) = A(t)x(t) + b(t),$$

where the $n \times n$ coefficient matrix A, the source n-vector b, and the unknown n-vector x are given in components by

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad b(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

The system in is called **homogeneous** iff the source vector $b = 0$, of constant coefficients iff the matrix A is constant, and diagonalizable iff the matrix A is **diagonalizable.**

Example2.1: Use matrix notation to write down the 2×2 system given by

$$\begin{aligned} x_1' &= x_1 - x_2, \\ x_2' &= -x_1 + x_2. \end{aligned}$$

Solution: In this case, the matrix of coefficients and the unknown vector have the form

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This is an homogeneous system, so the source vector $\boldsymbol{b} = \boldsymbol{0}$. The differential equation can be written as follows,

$$\begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= -x_1 + x_2 \end{aligned} \Leftrightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow x' = Ax.$$

Example2.2 : Find the explicit expression for the linear system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},\ \boldsymbol{b}(t) = \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix},\ \Leftrightarrow \qquad \begin{aligned} x_1' &= x_1 + 3x_2 + e^t \\ x_2' &= 3x_1 + x_2 + 2e^{3t}. \end{aligned}$$

Example2.3 : Show that the vector valued functions $\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$ are solutions to the 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: We compute the left-hand side and the right-hand side of the differential equation above for the function $\boldsymbol{x}^{(1)}$ and we see that both side match, that is,

$$A\boldsymbol{x}^{(1)} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t};\ \boldsymbol{x}^{(1)\prime} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} (e^{2t})' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} 2e^{2t}$$

so we conclude that $x^{(1)\prime} = Ax^{(1)}$. Analogously,

$$\begin{aligned} A\boldsymbol{x}^{(2)} &= \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^{-t} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t};\ \boldsymbol{x}^{(2)\prime} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (e^{-t})' \\ &= -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \end{aligned}$$

so we conclude that $x^{(2)\prime} = Ax^{(2)}$.

Definition 2.2[(Sons J. a., 1967)] An **Initial Value Problem** for an $n \times n$ linear differential system is the following: Given an $n \times n$ matrix valued function A, and an n-vector valued function $\boldsymbol{b}$, a real constant t_0, and an n-vector $\boldsymbol{x}_0$, find an n-vector valued function $\boldsymbol{x}$ solution of

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{b}(t),\ x(t_0) = x_0.$$

Example2.4 : Write down explicitly the initial value problem for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given by

$$\boldsymbol{x}' = A\boldsymbol{x},\ \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Solution: This is a 2×2 system in the unknowns x_1, x_2, with two linear equations

$$\begin{aligned} x_1' &= x_1 + 3x_2 \\ x_2' &= 3x_1 + x_2, \end{aligned}$$

and the initial conditions $x_1(0) = 2$ and $x_2(0) = 3$.

The main result about existence and uniqueness of solutions to an initial value problem for a linear system is also analogous to Theorem 2.1.2

Theorem : (Existence and Uniqueness) 2.1 [(Sons J. a., 1967)]. If the functions A and $\boldsymbol{b}$ are continuous on an open interval $I \subset \mathbb{R}$, and if $\boldsymbol{x}_0$ is any constant vector and t_0 is any constant in I, then there exist only one function $\boldsymbol{x}$, defined an interval $\tilde{I} \subset I$ with $t_0 \in \tilde{I}$, solution of the initial value problem

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{b}(t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

Example2.5. Find the explicit expression of the most general 3×3 homogeneous linear differential system.

Solution: This is a system of the form $\boldsymbol{x}' = A(t)\boldsymbol{x}$, with A being a 3×3 matrix. Therefore, we need to find functions x_1, x_2, and x_3 solutions of

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{13}(t)x_3 \\ x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3. \end{aligned}$$

Remark: The initial condition vector $\boldsymbol{x}_0$ represents n conditions, one for each component of the unknown vector x.

Theorem . (First Order Reduction)2.2[(Sons J. a., 1967)]. A function y solves the second order equation

$$y'' + a_1(t)y' + a_0(t)y = b(t),$$

iff the functions $x_1 = y$ and $x_2 = y'$ are solutions to the 2×2 first order differential system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -a_0(t)x_1 - a_1(t)x_2 + b(t). \end{aligned}$$

Example2.6 . Express as a first order system the second order equation

$$y'' + 2y' + 2y = \sin{(at)}.$$

Solution: Introduce the new unknowns

$$x_1 = y,\ x_2 = y' \Rightarrow x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin{(at)}.$$

We conclude that

$$x_1' = x_2,\ x_2' = -2x_1 - 2x_2 + \sin{(at)}$$

Theorem. (Second Order Reduction). 2.3[(Sons J. a., 1967)] Any 2×2 constant coefficients linear system $\boldsymbol{x}' = A\boldsymbol{x}$, with $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, can be written as second order equations for x_1 and x_2,

$$\boldsymbol{x}'' - \operatorname{tr}{(A)}\boldsymbol{x}' + \det{(A)}\boldsymbol{x} = 0.$$

Furthermore, the solution to the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$, with $\boldsymbol{x}(0) = \boldsymbol{x}_0$, also solves the initial value problem given by Eq. (5.1.6) with initial condition

$$x(0) = x_0,\ x'(0) = Ax_0.$$

Example2.7 . Express as a single second order equation the 2×2 system and solve it,

$$\begin{aligned} x_1' &= -x_1 + 3x_2, \\ x_2' &= x_1 - x_2. \end{aligned}$$

Solution: Instead of using the result from Theorem 2.3, we solve this problem following the second proof of that theorem. But instead of working with x_1, we work with x_2. We start computing x_1 from the second equation: $x_1 = x_2' + x_2$. We then introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2 \ \Rightarrow\ x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

so we obtain the second order equation

$$x_2'' + 2x_2' - 2x_2 = 0$$

We solve this equation with the methods studied in Chapter 2, that is, we look for solutions of the form $x_2(t) = e^{rt}$, with r solution of the characteristic equation

$$r^2 + 2r - 2 = 0 \ \Rightarrow\ r_\pm = \frac{1}{2}[-2 \pm \sqrt{4+8}] \ \Rightarrow\ r_\pm = -1 \pm \sqrt{3}$$

Therefore, the general solution to the second order equation above is

$$x_2 = c_+ e^{(1+\sqrt{3})t} + c_- e^{(1-\sqrt{3})t},\ c_+, c_- \in \mathbb{R}.$$

Since x_1 satisfies the same equation as x_2, we obtain the same general solution

$$x_1 = \tilde{c}_+ e^{(1+\sqrt{3})t} + \tilde{c}_- e^{(1-\sqrt{3})t},\ \tilde{c}_+, \tilde{c}_- \in \mathbb{R}$$

Example2.8 . Write the first order initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x},\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \boldsymbol{x}(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

as a second order initial value problem for x_1. Repeat the calculations for x_2.

Solution: From Theorem 2.3 we know that both x_1 and x_2 satisfy the same differential equation. Since $\operatorname{tr}(A) = 1 + 4 = 5$ and $\det(A) = 4 - 6 = -2$, the differential equations are

$$x_1'' - 5x_1' - 2x_1 = 0, \; x_2'' - 5x_2' - 2x_2 = 0.$$

From the same Theorem we know that the initial conditions for the second order differential equations above are $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}'(0) = A\boldsymbol{x}_0$, that is

$$\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \; \boldsymbol{x}'(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

therefore, the initial conditions for x_1 and x_2 are

$$x_1(0) = 5, \; x_1'(0) = 17, \; \text{ and } \; x_2(0) = 6, \; x_2'(0) = 39.$$

Homogeneous Systems2.2:[**(Sons J. W., 1969)**]

Solutions to a linear homogeneous differential system satisfy the superposition property: Given two solutions of the homogeneous system, their linear combination is also a solution to that system.

Example2.9 . Verify that $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$ and $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ are solutions to the homogeneous linear system

$$\boldsymbol{x}' = A\boldsymbol{x}, \; A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

Solution: The function $\boldsymbol{x}^{(1)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(1)\prime} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \; A\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{-2t} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

We then conclude that $\boldsymbol{x}^{(1)\prime} = A\boldsymbol{x}^{(1)}$. Analogously, the function $\boldsymbol{x}^{(2)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(2)\prime} = 4\begin{bmatrix}-1\\1\end{bmatrix}e^{4t},\ A\boldsymbol{x}^{(2)} = \begin{bmatrix}1 & -3\\-3 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}e^{4t} = \begin{bmatrix}-4\\4\end{bmatrix}e^{4t} = 4\begin{bmatrix}-1\\1\end{bmatrix}e^{4t}$$

We then conclude that $\boldsymbol{x}^{(2)\prime} = A\boldsymbol{x}^{(2)}$. To show that $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ is also a solution we could use the linearity of the matrix-vector product, as we did in the proof of the Theorem . Here we choose the straightforward, although more obscure, calculation: On the one hand,

$$\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)} = \begin{bmatrix}e^{-2t} - e^{4t}\\e^{-2t} + e^{4t}\end{bmatrix} \Rightarrow \left(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\right)' = \begin{bmatrix}-2e^{-2t} - 4e^{4t}\\-2e^{-2t} + 4e^{4t}\end{bmatrix}$$

On the other hand,

$$A\left(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\right) = \begin{bmatrix}1 & -3\\-3 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} - e^{4t}\\e^{-2t} + e^{4t}\end{bmatrix} = \begin{bmatrix}e^{-2t} - e^{4t} - 3e^{-2t} - 3e^{4t}\\-3e^{-2t} + 3e^{4t} + e^{-2t} + e^{4t}\end{bmatrix}$$

that is

$$A\left(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\right) = \begin{bmatrix}-2e^{-2t} - 4e^{4t}\\-2e^{-2t} + 4e^{4t}\end{bmatrix}$$

We conclude that $\left(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\right)' = A\left(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\right)$.

We now introduce the notion of a linearly dependent and independent set of functions.

Definition2.3 [**(Sons J. a., 1967**)]. A set of n vector valued functions $\left\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\right\}$ is called **linearly dependent** on an interval $I \in \mathbb{R}$ iff for all $t \in I$ there exist constants $c_1, \cdots, c_n$, not all of them zero, such that it holds

$$c_1\boldsymbol{x}^{(1)}(t)+\cdots+c_n\boldsymbol{x}^{(n)}(t)=\mathbf{0}$$

A set of n vector valued functions is called **linearly independent** on I iff the set is not linearly dependent.

Definition2.4[(Sons J. W., 1969)] . (a) The set of functions $\{\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(n)}\}$ is **a fundamental set of solutions** of the equation $\boldsymbol{x}'=A\boldsymbol{x}$ iff the set $\{\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(n)}\}$ is linearly independent and $\boldsymbol{x}^{(i)\prime}=A\boldsymbol{x}^{(i)}$, for every $i=1,\cdots,n$.

(b) The **general solution** of the homogeneous equation $\boldsymbol{x}'=A\boldsymbol{x}$ denotes any vector valued function $\boldsymbol{x}_{\text{gen}}$ that can be written as a linear combination

$$\boldsymbol{x}_{\text{gen}}(t)=c_1\boldsymbol{x}^{(1)}(t)+\cdots+c_n\boldsymbol{x}^{(n)}(t)$$

where $\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(n)}$ are the functions in any fundamental set of solutions of $\boldsymbol{x}'=A\boldsymbol{x}$, while $c_1,\cdots,c_n$ are arbitrary constants.

Example2.10 . Show that the set of functions $\left\{x^{(1)}=\begin{bmatrix}1\\1\end{bmatrix}e^{-2t},\boldsymbol{x}^{(2)}=\begin{bmatrix}-1\\1\end{bmatrix}e^{4t}\right\}$ is a fundamental set of solutions to the system $\boldsymbol{x}'=A\boldsymbol{x}$, where $A=\begin{bmatrix}1&-3\\-3&1\end{bmatrix}$.

Solution: In Example 2.9 we have shown that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to the differential equation above. We only need to show that these two functions form a linearly independent set. That is, we need to show that the only constants c_1,c_2 solutions of the equation below, for all $t\in\mathbb{R}$, are $c_1=c_2=0$, where

$$\boldsymbol{O}=c_1\boldsymbol{x}^{(1)}+c_2\boldsymbol{x}^{(2)}=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{4t}=\begin{bmatrix}e^{-2t}&-e^{4t}\\e^{-2t}&e^{4t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=X(t)\boldsymbol{c}$$

where $X(t)=[\boldsymbol{x}^{(1)}(t),\boldsymbol{x}^{(2)}(t)]$ and $\boldsymbol{c}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Using this matrix notation, the linear system for c_1,c_2 has the form

$$X(t)\boldsymbol{c}=\mathbf{0}.$$

We now show that matrix $X(t)$ is invertible for all $t \in \mathbb{R}$. This is the case, since its determinant is

$$\det(X(t)) = \begin{vmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{vmatrix} = e^{2t} + e^{2t} = 2e^{2t} \neq 0 \ \text{ for all } t \in \mathbb{R}.$$

Since $X(t)$ is invertible for $t \in \mathbb{R}$, the only solution for the linear system above is $\boldsymbol{c} = \boldsymbol{0}$, that is, $c_1 = c_2 = 0$. We conclude that the set $\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\}$ is linearly independent, so it is a fundamental set of solution to the differential equation above.

Example2.11 . Find the general solution to differential equation in Example 2.3 and then use this general solution to find the solution of the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x},\ \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix},\ A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

Solution: From Example 2.3 we know that the general solution of the differential equation above can be written as

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

Before imposing the initial condition on this general solution, it is convenient to write this general solution using a matrix valued function, X, as follows

$$\boldsymbol{x}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \ \Leftrightarrow \ \boldsymbol{x}(t) = X(t)\boldsymbol{c}$$

where we introduced the solution matrix and the constant vector, respectively,

$$X(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix},\ \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The initial condition fixes the vector c, that is, its components c_1, c_2, as follows,

$$\boldsymbol{x}(0) = X(0)c \; c = [X(0)]^{-1}\boldsymbol{x}(0)$$

Since the solution matrix X at $t = 0$ has the form,

$$X(0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow [X(0)]^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

introducing $[X(0)]^{-1}$ in the equation for c above we get

$$\boldsymbol{c} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 3 \end{cases}$$

We conclude that the solution to the initial value problem above is given by

$$x(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

Definition2.5 [(Kishan, 2006)]. (a) A **solution matrix** of any set of vector functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$, solutions to a differential equation $\boldsymbol{x}' = A\boldsymbol{x}$, is the $n \times n$ matrix valued function

$$X(t) = \left[\boldsymbol{x}^{(1)}(t), \cdots, \boldsymbol{x}^{(n)}(t)\right]$$

X is called a fundamental matrix iff the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a fundamental set.

(b) The **Wronskian** of the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is the function $W(t) = \det(X(t))$

Example2.12 . Find two fundamental matrices for the linear homogeneous system in Example .2.9

Solution: One fundamental matrix is simple to find, we use the solutions in Example ,

$$X = \left[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right] \;\Rightarrow\; X(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix}$$

A second fundamental matrix can be obtained multiplying by any nonzero constant each solution above. For example, another fundamental matrix is

$$\tilde{X} = \left[2\boldsymbol{x}^{(1)}, 3\boldsymbol{x}^{(2)}\right] \;\Rightarrow\; \tilde{X}(t) = \begin{bmatrix} 2e^{-2t} & -3e^{4t} \\ 2e^{-2t} & 3e^{4t} \end{bmatrix}$$

Remarks:

(a) In the case of a constant matrix A, the equation above for the Wronskian reduces to

$$W(t) = W(t_0)e^{\operatorname{tr}(A)(t-t_0)},$$

(b) The Wronskian function vanishes at a single point iff it vanishes identically for all $t \in I$.

(c) A consequence of (b): n solutions to the system $\boldsymbol{x}' = A(t)\boldsymbol{x}$ are linearly independent at the initial time t_0 iff they are linearly independent for every time $t \in I$.

Example2.13 . Compute the exponential function e^{At} and use it to express the vectorvalued function $\boldsymbol{x}$ solution to the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x},\; A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\; \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

Solution: The exponential of a matrix is simple to compute in the case that the matrix is diagonalizable. So we start checking whether matrix A above is diagonalizable. Theorem 8.3.8 says that a 2×2 matrix is diagonalizable if it has two eigenvectors not proportional to each other. In oder to find the eigenvectors of A we need to compute its eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 4$$

The roots of the characteristic polynomial are

$$(\lambda - 1)^2 = 4 \iff \lambda_\pm = 1 \pm 2 \iff \lambda_+ = 3,\ \lambda_- = -1$$

The eigenvectors corresponding to the eigenvalue $\lambda_+ = 3$ are the solutions $\boldsymbol{v}^+$of the linear system $(A - 3I_2)\boldsymbol{v}^+ = \mathbf{0}$. To find them, we perform Gauss operations on the matrix

$$A - 3I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1^+ = v_2^+ \Rightarrow \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvectors corresponding to the eigenvalue $\lambda_- = -1$ are the solutions $\boldsymbol{v}^-$of the linear system $(A + I_2)\boldsymbol{v}^- = \mathbf{0}$. To find them, we perform Gauss operations on the matrix

$$A + I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1^- = -v_2^- \Rightarrow \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Summarizing, the eigenvalues and eigenvectors of matrix A are following,

$$\lambda_+ = 3,\ \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \lambda_- = -1,\ \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then, Theorem 8.3.8 says that the matrix A is diagonalizable, that is $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \; P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Now Theorem ?? says that the exponential of At is given by

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

so we conclude that

$$e^{At} = \frac{1}{2}\begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix}$$

Finally, we get the solution to the initial value problem above,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 = \frac{1}{2}\begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

In components, this means

$$\boldsymbol{x}(t) = \frac{1}{2}\begin{bmatrix} (x_{01} + x_{02})e^{3t} + (x_{01} - x_{02})e^{-t} \\ (x_{01} + x_{02})e^{3t} - (x_{01} - x_{02})e^{-t} \end{bmatrix}$$

Homogeneous Diagonalizable Systems2.3:[**(Sons J. a., 1967)]**

A linear system $\boldsymbol{x}' = A\boldsymbol{x}$ is diagonalizable iff the coefficient matrix A is diagonalizable, which means that there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (See §8.3 for a review on diagonalizable matrices.) The solution formula in Eq. (2.13) includes diagonalizable systems. But when a system is diagonalizable there is a simpler way to solve it. One transforms the system, where all the equations are coupled together, into a decoupled system. One can solve the decoupled system, one equation at a time. The last step is to transform the solution back to the original variables. We show how this idea works in a very simple example.

Example2.14 . Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$\begin{aligned} x_1' &= x_1 - x_2, \\ x_2' &= -x_1 + x_2. \end{aligned}$$

Solution: As it is usually the case, the equations in the system above are coupled. One must know the function x_2 in order to integrate the first equation to obtain the function x_1. Similarly, one has to know function x_1 to integrate the second equation to get function x_2. The system is coupled; one cannot integrate one equation at a time. One must integrate the whole system together.

However, the coefficient matrix of the system above is diagonalizable. In this case the equations can be decoupled. If we add the two equations equations, and if we subtract the second equation from the first, we obtain, respectively,

$$(x_1 + x_2)' = 0,\ (x_1 - x_2)' = 2(x_1 - x_2).$$

To see more clearly what we have done, let us introduce the new unknowns $y_1 = x_1 + x_2$, and $y_2 = x_1 - x_2$, and rewrite the equations above with these new unknowns,

$$y_1' = 0,\ y_2' = 2y_2.$$

We have decoupled the original system. The equations for x_1 and x_2 are coupled, but we have found a linear combination of the equations such that the equations for y_1 and y_2 are not coupled. We now solve each equation independently of the other.

$$y_1' = 0 \;\Rightarrow\; y_1 = c_1,$$
$$y_2' = 2y_2 \;\Rightarrow\; y_2 = c_2 e^{2t},$$

with $c_1, c_2 \in \mathbb{R}$. Having obtained the solutions for the decoupled system, we now transform back the solutions to the original unknown functions. From the definitions of y_1 and y_2 we see that

$$x_1 = \frac{1}{2}(y_1 + y_2),\; x_2 = \frac{1}{2}(y_1 - y_2).$$

We conclude that for all $c_1, c_2 \in \mathbb{R}$ the functions x_1, x_2 below are solutions of the 2×2 differential system in the example, namely,

$$x_1(t) = \frac{1}{2}(c_1 + c_2 e^{2t}),\; x_2(t) = \frac{1}{2}(c_1 - c_2 e^{2t}).$$

The equations for x_1 and x_2 in the example above are coupled, so we found an appropriate linear combination of the equations and the unknowns such that the equations for the new unknown functions, y_1 and y_2, are decoupled. We integrated each equation independently of the other, and we finally transformed the solutions back to the original unknowns x_1 and x_2. The key step is to find the transformation from x_1, x_2 to y_1, y_2. For general systems this transformation may not exist. It exists, however, for diagonalizable systems.

Theorem . (Fundamental Matrix Expression)2.4[(Sons J. W., 1969)**].** If the $n \times n$ constant matrix A is diagonalizable, with a set of linearly independent eigenvectors $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then, the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ has a unique solution given by

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\boldsymbol{x}_0$$

where $X(t) = \left[e^{\lambda_1 t}\boldsymbol{v}^{(1)}, \cdots, e^{\lambda_n t}\boldsymbol{v}^{(n)}\right]$ is a fundamental matrix of the system.

Example2.15 . Find a fundamental matrix for the system below and use it to write down the general solution to the system

$$\boldsymbol{x}' = A\boldsymbol{x},\ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution: One way to find a fundamental matrix of a system is to start computing the eigenvalues and eigenvectors of the coefficient matrix. The differential equation in this Example is the same as the one given in Example where we found that the eigenvalues and eigenvectors of the coefficient matrix are

$$\lambda_+ = 3,\ \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \text{ and }\ \lambda_- = -1,\ \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We see that the coefficient matrix is diagonalizable, so with the eigenpairs above we can construct a fundamental set of solutions,

$$\left\{\boldsymbol{x}^{(+)}(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(-)}(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

From here we construct a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}$$

Then we have the general solution $\boldsymbol{x}_{\text{gen}}(t) = X(t)\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$, that is,

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} \iff \boldsymbol{x}_{\text{gen}}(t) = c_+ e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_+ e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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