**Chapter one**

**Differential Equations (DEs.).**

**Equation**: Equations describe the relations between the dependent and independent variables. An equal sign "=" is required in every equation.

**Differential Equations**: Equations that involve dependent variables and their derivatives with respect to the independent variables are called differential equations.

**Differential Equation**:- Is an equation consist algebraic function or non-algebraic function or both of them which contains derivative. Divide DE. in two types

1- Ordinary DEs. .

2- Partial DEs.

**Ordinary Differential Equations**: Differential equations that involve only ONE independent variable are called ordinary differential equations.

**Partial Differential Equations**: Differential equations that involve two or more independent variables are called partial differential equations.

**Examples:**



****

****

****

****

**Order of ODE:-**Is an order of the highest derivative in which occurs.

**Degree of ODE:-**Is the highest power of the highest derivative in which occurs.

**Examples:**



****

****

****

****

**Note:** if the O.D.E containing the roots or rational power then to find the degree of this ODE. can be reduces this roots or rational powers.

**Example:1)** +6 

Square both side of the equation

**Order 3, degree e2**

**2) **(Ordinary DE.) **Or. 2 D.4**

**Linear O.D.E**

A.D.E. in any order is said to be linear if satisfies:-

1) The dep.v .is exist and of the first degree.

2) The derivatives  exist and each of them of the first degree.

3) The dep.v. and the derivatives not multiply by each other.

**Note:** Ifone of these conditions is not satisfied, then the equation considerate non- Linear.

**Examples:**

 (L.)

2) (Non.L.)

3) (Non.L.)

4) (Non.L.)

**The Solutions of ODE**

**Solutions**: A functional relation between the dependent variable y and the independent variable x that satisfies the given ODE in some interval J is called a solution of the given ODE on J .

**Type of solutions:**

1) The general solution, denoted by yG

2) The particular solution, denote1d by yp.

3) The singular solution, denoted by S.

Description: C:\Users\Public\Videos\Desktop\General Terms of Ordinary Differential Equations_files\section_bar_1.png

**General Solution**: Solutions obtained from integrating the differential equations are called general solutions. The general solution of aDescription: nthorder ordinary differential equation contains Description: narbitrary constants resulting from integrating Description: ntimes.

**Example:solve:** , (y)

Lin y=2x+c y= , suppose =k

y=k Isthe G. solution.

**Particular Solution**: Particular solutions are the solutions obtained by assigning(giving) specific values to the arbitrary constants in the general solutions.

**Example**/pervious example

Choose (x, y) = (0, 1) 

y=is the particular solution.

**Note**: To find the particular solution of O.D.E in the G. solution by giving the value of arbitrary constant as follows.

By giving the value of dependent variable and the value of independent variable (Represent the integral curve).Obtained the value of arbitrary constant and substituted in the G. solution ,we get the particular solution .

(These values of (x, y) is called initial conditions or boundary conditions).

**Example:**  ,

**Singular Solutions**: Solutions that cannot be expressed by the general solutions are called singular solutions.

**Example: **If y=0 then  is undefined, then y =0

is the singular solution.

**Elimination the arbitrary constants (Finding the O.D.E if the G. Solution is Exist)**

1) Differential the G. solution n-times (the number of arbitrary constant =the order of O.D.E)

2) We obtain (n+1) of equations.   
3) we can solve these equations by simultaneous way (or by determinant way).

4) Substituted the value of arbitrary constants in the G. solution, we get the O.D.E.

**Three kind of method to solve the problems**

1. **Elimination method**

**Example**: find the D.E if the G. solution is y=A3+B2+C ------ (1)

**Solution**: =3A+ 2B +c ------ (2)

=6A+2B

=6A

A= , B=-, C=+ - 

Subset value of A, B and C in the G. solution , we get the O.D.E of third order

Y = + (-) + (+-)

1. **Simultaneous method**

**Example**: find the D.E if the G. solution is

**Solution**

-------- (1)

-------- (2)

-------- (3)

From 1 & 2 we get

------ (\*)

Put (\*) in (2) we get

--------(\*\*)

Put (\*) & (\*\*) in (1) we get ODE

1. **determinant method**

**Example**: find the D.E if the G. solution is

**Solution**

= 0

= 0

= 0

= 0

is DE

Q/ prove that is G.S is DE

**Example**: find the O.D.E if the G. solution is, ( H.W)

1. y=A+B
2. y=C1sin x +C2 cos
3. y=A+B+C 

**Chapter Two**

**Methods for Solving the O.D.E in the first order and first degree**

1) Separation variable (separable).

2) Substitution method.

3) Homogenous D.E.

4) Non-homogenous D.E. of linear coefficient.

5) Exact D.E.

**1) Separable DEs**

**Separable Function**: A function F(x, y) is called separable if can be written of the form.

F(x, y) = g(x).h(y) or = ; 0.

Where g is a function of (x) only and h is a function of (y )only.

Ex1: F(x, y) =y SF

Ex2: F(x, y) = y Non-SF

**(S.D.E.)**: A D.E. M(x, y) + N(x, y)  =0 is called separable if both M and N are S. functions.

**Example 1):** solve: sinx cosy dx+siny cosx dy = 0.

Solution: 



Take ***exp*** to both side we get

let

is G.S.

**Example 2) :**( -) +(x+1) =0 Non-S.D.E

**Example 3)** : solve: 



You finish it (one mark for any one solve it )🡪 ln(y)- (1/2)(tanh-1((x-2)/2)) = k.



**Example 4):** solve :









**H.w/ Solve:**

1) x3 dx + (y+1)2dy = 0

2) x2 (y+1) dx + y (x-1)dy = 0

3)

**2) Substitution method.**

**If** the D.E. of the form

Non-Separable D.E., then

Suppose 







By subset .equation (2) and (4) in D.E (1) we get.



Is S.D.E can be solved by previous way.

**Example**: Solve:-  ---------(1)

**Solution**: Suppose + =­----- (2)

1+==-1----- (3)

By put equ (3) and equ (2) in equ (1) we get -1=

=is SDE 

 is G.S.

**Example**: solve:-

**Solution**:

------ (1)

Suppose ­----- (2)

-----(3)

Put (2) & (3) in (1) we get

Take integration to both side we get

------(4)

Take for (4)

is G.S

**Example (H.W.):- Solve**: =x2 -8xy+16y2

**3) Homogenous D.E**

**Homogenous function**: A function F(x, y) is called Homogenous function of n-th degree if satisfy the relation.



**Example**: 1)Test the function

f(x, y) =, tR

F (tx, ty) = (tx)2(ty)

= (t2x2) (ty)

=t3f(x, y)  F is H.F. of 3-rd degree



**Example**: 2)Test the function

F(x, y= (



- degree.

**Example**: 3) Test the function





F is H.F. of first degree.

**.**

**Homogeneous D.E :**A D.E. M(x, y) dx+ N(x, y) dy=0 is called H.D.E if both functions M and N are H. functions of the same degree (i.e.)[the H. degree of M=the H. degree of N]

**Example**: 1).

M(x, y) =

M () =is H.F. of 2-nd degree.

N () =

N = 

= N is H.F. of 2-nd degree

DE is H. of 2-nd degree .

**Example:-**Solve:-

. H.W

. H.W

**Note**: Every homogeneous D. E of n-th degree can be reduced into separable D.E by using the relation.

**Example**: solve: -  is HDE of 2-nd degree.

**Solution**: suppose 

 (2)



Sub. equ. (2) inequ. (1), we get the S.D.E.





(  )











where e4c =c1

Subst.





**Examples:-**Solve the following DEs**. ( H.W.)**

1- 2 

2- 

3- 

**Example**:Test the functions **in DE.(1)**





is non-homog.func.

N(x,y)=x

degree

DE (1) is non-Homog. DE.

**4) Non-Homogenous DE With Linear Coefficients:-**

Th general form of non-homog DE with linear coefficients is



whereare constants.

To changing the Non-H.D.E. into H.D.E.or S.DE. , there exist two cases:-

**Case1**:- if (two lines are intersected).







(h, k) the intersection point



 (4)

Suppose and 

and (5)

Subst .equ (5) in D.E (1), we get the H.D.E 



is H.D.E.

equ. (6) can be solving by homogenous method by supposes



and (7)



Substequ. (7) inequ (6) we get the separable D.E., we can solving by integration immediately we get the G.solution .

In which contains two variable subst. values of ,

, and 

**Example**: solve 

**Solution**:





,



2y – x - 5 = 0





Suppose and 

and (A)

and

Substequ. (A) inequ (1) we get the H. D. E.





is HDE

Suppose

and (B)



Substequ. (B) inequ (2) we get the separable DE



Is SDE



**(H.W) solve** 

**Case 2**: if (the two lines are not intersected at the point, but are parallel.

Then suppose 



 (k)

Subst. equ. (k) in D.E. (1) we get the S.D.E.



Can be solved, by integration immediately finally subst. the value of (z), we get G.solution.

m= is multiple of (z)

**Example**: solve: (3x-6y+5) dx+ (12x-24y-2) dy =0 -----(1)

**Solution**:



Then suppose





 (2)

Subst. equ. (2) in D.E. (1) we get the S.D.E.

is S.D.E.

**Example:-**Solve the following DEs.(H.W.)

1) (x + y + 2) dx+ (- x – y + 2) dy =0

2) (6x-8y-5) dy= (3x-4y-2) dx

**Note / If the function of the form f (x y) or the DE.of theform y f(x y) dx + x g(x y) dy=0**

**(1) Suppose yx=v**

y=

 (2)

Subst. equ.(2) in the DE.(1) ,we get 

**Example**: solve:-

**Solution** :

Suppose yx=v

y =

 (2)

Subst. equ.(2) in the D.E.(1)







**5) Exact D.E.**

**Exact DE**: A D.E. M(x, y) dx+ N(x, y) dy=0 ----- (1) is called exact D.E. if  such that 

(by def. of exact function)

and

**\*\*\*\*\*** The necessary and sufficient conditions of D.E 

is Exact D.E. if = such that M, N, Mx, My, Nx,Ny,… are continuous in R.

**Example: solve:** (x+y) dx +(x-y) dy =0 ------- (1)

**Solution**: 

and

Since = than D.E. (1) is exact

=(x+y) dx +(x-y) dy =0 s.t

-----(1)  
-----(2)

Integration both side of equ. (1) with respect to (x) and choosing the arbitrary function of (y) only

=,

Differentiation equ.(4) with respect to

------- (5)

Put (1) in (5) we get

x-y = 

-y =-----(6)

Integrate both side of equ. (6) with respect to (y)

, put in (4) we get G.S as follow

=

**Example: solve**: 

**Solution**: =, =

Since = D.E. is exact ,=

----- (2) and -------(3)

Integration both side of equ. (3) with respect to (y) and choosing the arbitrary function of (x) only

=+ (x) -------(\*)

Differentiation equ.(\*) with respect to

----- (4) .In both equation (2) and (4) ,we get

==-------(5)

Integrate both side of equ. (5) with respect to (x)



=be the G.solution.

H.W.: Solve the following DES:-

(1) .

(2)(3x2+3xy2)dx+(3x2y – 3y2+2y)dy =0 .

(3)(2ye2x + 2xcosy)dx +(e2x – x2siny)dy =0

**Linear First Order Ordinary D.E**

The general form of first order and first degree of ordinary DE is ; where a(x)0 and a,b,c are functions of x only.



is the standard form of **LFODE**

 (non-homog. & non-exact )

 , Or 

Is the general solution of non-homog.and non-exact D.E(liner first order )

**Example**: solve: ------(1)

**Solution**: divide equ (1) by, we obtain,

By comparison with 

=, =

I.F = =



**Example**: solve: 

**Solution**: 

; =&=





**Bernoulli's Equation**

**Def**: A DE is said to be Bernoulli's Equation such that p, and are functions of (x) only and n0, 1(n):

If n=0 is (F.O.L.D.E)

If n=1 is (S.D.E)

----- (1) (Bernolli's Equation)

To solve DE (1) we will divided it by () we obtain 



Suppose =u --- (3)





Subst both equ (3) &equ (4) in equ (2); we get an FOLDE



[] 



equ (5) can be solving by method of FOLDE. we get the value of u .Finally subst. the value of =u, we get the general solution of Bernoulli's equ.

**Example**: solve: 

**Solution**: 

----- (1) Bernoulli's equ.

Divide equ (1) by  we obtain



Let 

Differentiation equ (3) with respect to x



Substequ (3) &equ (4)inequ.(2), we obtain a 



is a =&=









**Example: solve**

**Solution**: 

Suppose = u --- (3) ( --- (4) , Subst (3) & (4) in equ(2) we obtain 

[] ()

is a 





**H.W**: solve the following equations.

1)

2)

3)

**Simultaneous Ordinary Differential Equation**

Is a set of equations which contains only one independent variable and the number of equations are equal to the number of the dependent variables , as fallows:



We can solving system (\*) by choosing the D.E in which contains only one dependent variable say equ.(1) & solving by integration immediately or by previous ways , we get the value of dependent variable and subst. in equ. (2) , and solving by previous way , we get the G. solution of system (\*)

**Example: solve**: 

**Solution** :



 (A)

To find the general Solution, we can choose the first equ.



 [FOLDE]









subst. the value of (x) in equ. (2) in system A





is F.O.L.D.E

**Chapter Three**

**Reduction of Higher Order Ordinary D.E.**

Consider a D.E of higher order is 0 can be reduced into first order by using substitution. We can study this D.E of second order, then equ. (1) becomes equ(2) can be solved by two cases:

**Cases 1**: If the dependent variable (y) does not (explicity) or (appears) in the D.E (2), then equ(2), becomes 

Suppose 





Subst. both equ (4) and (5) in D.E (3) we get a relation between 

can be solving by previous way and we get (p) subst. (p) by () and by integration immediately we get the g.solution

**Cases 2**:If the independent variable (x) does not (explicity) or(appears) in the D.E (2) ,then equ(2), becomes 

Suppose 

 (by chain rule)

Subst. both equ (\*\*) and (\*\*\*) in D.E (\*) we get a relation between 

can be solving by previous way and we get (p) subst. (p) by () and by integration immediately we get the g.solution

**Examples**: solve the following D.E.

1)----------(1)

2)

3)  H.W

4) 

5)  H.W.

**Solution 1**) since not appears.



 (2) Subst. equ (2) in equ (1) we get 





 Suppose 





**Solution** 2) since not appears in equ.(\*)

Then Suppose 

 (\*\*)



Subst. both equ (\*\*) and (\*\*\*) in D.E (\*) we get a relation between 









let





**Solution** 3) since  don’s not appears in D.E (1), then solving by any cases. Suppose we can solving by first case.





 (2)

















 (4)

Substequ (4) in D.E (3), we get









be the g.solution of D.E (1)

**Example: solve** H.W

1) 

2)

3)

4)

**Higher Degree of ordinary D.E**

O.D.E of the first order but of the higher degree. The general form of higher degree of O.D.E is where  are constant and

()---(1). To solve equ.(1) there exist three cases:

**Case1**: equ. (1) Solvable for (p). ifequ.(1) can be written of the form 





Continue in this way, we get .

Then the general solution of this case is ;where c is constant.

**Example: solve:**

**Solution:**  Suppose 

, or 

and

The general solution of D.E.O is  where c is a constant.

**Solves the follows equation**. (H.W)

(1) 

(2)

**Case 2**: equ.(1) solvable for y

Equ.(1): can be written of the form differentiation equ.(2) with respect to x and subst.  we obtain a relation between can be solved by previous way and obtained the value of p.

Finally subst. the value of p in equ.(2) , we get the g. solution, and the singular

**Example: 1) solve**: 

Solution: 







**2) Solve**: 

**Solution**: 











is the g. solution or 

**H.W: solve the following D.E**

(1)

(2)

(3)

**Case 3**: equ.(1) solvable for (x) equ.(1)  can be written of the form  differentiation equ.(2) with respect to y and subst.  we obtain the a relation between can be solved by previous way and obtain the value of p. finally subst. the value of p in equ.(2) we get the g. solution .

**Example: solve**: 

**Solution**: since equ.(1) solvable for x, then













and be the general solution.

or

**H.W: solve** (1)

(2)

(3)

**Clairt's Equation**: an equation of the form where f is a function of (p) only is called clairut’s equation and we can obtained the general solution by subst. p by c as follows: is the g. solution of equ.(1) where c is a constant differentiation equ.(1) with respect to x and subst.  , we obtain 



, 

or .

**Example: solve**: 

**Solution**: since equ.(1) is clairut’sequ. Then 



**Example: solve:**

**Solution**:



isclairut’sequ.



is the general solution

or

be a singular solution .

**H.W: solve the following D.E**.

(1)

(2)

(3)

**Higher order ordinary D.E with constant coefficients:**

The general form of higher order O.D.E is where  are constants.

An 

If , then equ.(1) is called non-Homog. D.E with constant coefficient if  ,thenequ.(1) becomes( an  is called a H.D.E with c.c .

If at least one of the coefficient  is a function of x only then equ.(1) becomes anequ.(3) is called NON.H.D.E with variable coefficients or if  in equ.(3) , then equ.(3) becomes 

Equ.(4) is called a H.D.E with V.C.

**Operators: (D)**

Is a differentiation of any dependent variable with respect to independent variable.



Or 

**Properties:**

(1)

(2)

(3)

(4)if C is constant then 

(5)where a,b are constants .

(6)where at least one of them (a) or (b) is a function of (x) only.

**Ex:**.

**Ex:**

**Proof (5)** :







Where a,b are constants .

**Proof (6):**







Since equ(1)equ(2) then



Note: we can written higher order O.D.E with constant coefficient or with the variable coefficient (Homog.or.non-Homog) by using the operator







 Where



Homog.L.D.E

With C.C. the  is called the characteristic equation.

non-Homg. with V.C

Or Homog. With V.C.

**Reduction of higher order O.D.E in to first order D.E with c.c.**



Where  are constants and Q is a function of (x) only we study the second order and non-Homog.D.E with C.C as follows:







Whereand



Suppose 

Subst. eqn.(3) in eqn.(2), we get a.F..L.D.E















Subst. equ (3) in equ.(4), we get L.F.O.D.E.

1







be the g. solution where  are constants.

**Ex.:** solve: 

**Solution:**







Suppose 

Subst. equ.(3) in equ.(2) , we get  is (F.O.L.D.E)













where c is constant



Subst, equ(4) in equ.(3) , we get



 (L.F.O.D.E)













is the g. solution where  is constant.

**Example:**

**Solution:**





Suppose 

Substequ.(3) in equ.(2) , we get .



is S.D.E







Substequ.(4) in equ.(3) , we get



Suppose 

Substequ.(6) in equ.(5) , we get

 L.F.O.D.E









Substequ.(6) in equ.(7) , we get



L.F.O.D.E

**Solve: H.W:**

**Homogenous linear D.E. with C.C.**

The general from of higher-order O.D.E with C.C. is



We can study the L.H.D.E of the second order with C.C.





Note: if  (\*)

Is non-Homog. L.D.E. with C.C. we can find the general solution of equ.(\*) such that contains the complementary solution of  H.D.E with C.C. in which denoted by (yc) , and the particular solution of equ.(\*) (non-Homog),denoted by(yp),then .

We study the H.D.E. with C.C.



is a characteristic equ.







Or 



 Such that

are roots





Suppose 

Substequ.(4) in equ.(3) , we get S.D.E.









,where c is constant



 where c\* is constant

Substequ.(4) in equ.(5) , we get



L.F.O.D.E









We find the relation between two equ.(2)&(3). There exist three cases:

**Case1**: if  then there exists two distinct (different) real roots  in equ (7)







where



Be a complementary solution of H.L.D.E with C.C in which contains two real distinct roots.

**Case 2:**

If  then there exist two (repeated) real roots

. In equ.(7)







 where

Or



Or



Be a complementary solution of H.L.D.E with C.C in which contains two equal roots (repeated real roots).

**Case 3**: if  then there exist two complex roots





In first case







By Euler’s formula









Where  and 

be a complementary solution of H.L.D.E with C.C in which contains two complex roots .

Or  


**Examples:**

Solve the following H.D.E.(find the g. solution or the complementary solution)

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

**Solution (1):**















**Solution (2):**









Or



Solution (\*): 





Solution (3): 













**Solution (4):**









Or





**Solution (5):**















**Solution (6):**















**Solution (7):**















Or



**Solution (8):**











**Solution (9):**









H.W: solve

(1)

(2)

(3)

(4)

(5)

**How to find a particular solution of L. Non-Homog D.E. with C.C.:**

Consider 

BE A L.NON-H. D.E With C.C, where Constants and Q is a function of (x) only.

There exist three methods:

(1) the variation of parameters method

(2) the operators method

(3) Un determinant coefficients method

**(1) The variation of parameters method:**

We study non-H.L.D.E of second order with C.C.:





To find the general solution of D.E (1), we obtain (yc) and (yp)

to find (yc) (be a complementary solution ) of equ.(1) suppose 



(whereare two linearly independent. Solutions) since  are two linearly independent solution. Solution then



We can find a particular solution of D.E(1) by changing two arbitrary constants  in to 

be a particular solution of D.E(1)



Suppose 





Subst. equ (4, 5 and 6) in D.E (1), we get









We can solve both equ. A and B by grammer’s method.









And









**Example**: - solve 

**Solution:-**















=1

=



==

==









H.W.

Solve th**e**following D.Es.

1)

2) 

3) 

4) 

**Operator’s method**

To study n-th order O.D.E with C.C by using operators method (D) dependent on the type of Q(x) and by using some theorems.

**Theorem:** - 1) 

2) where b is constant.

3) 

4) 

**Proof (1):-** where  are constant



=

=

=

=

=

=

=

=

=

**.**

**.**

**.**

=

=







**Type of Q(x)**

|  |
| --- |
| 1) |
| 2) or |
| 3) |
| 4) or or |
| 5)  or |
| 6)  or |

There exist six cases.

**Case1**:- If Q(x) =then the particular solution is[]



**i)** If then  (By using theorem)

**Example 1):** - solve: 

**Solution:**





Suppose 

















**Example 2):-** solve: 





















**H.W: solves the following.**

1) 

2) 

**ii)** If  then undefined, and 

Where b is a root in the complementary solution and (r) is the number of repeated root g(D)

the remainder terms in f(D).

By using theorem





 , where 

 , where =

It is particular solution.

**Examples:**- 

**Solution:-**





****

****

















**Example:- **

**Solution:-**















**H.W:** Solve the following solution.





**Case 2:**- If  then 



By using theorem or 

There exist two branches

i)If then 

**Example:**-

**Solution:-**













**Example:**-

**Solution:-** 



























**H.W:** Solve the following solution.



**ii)**Ifthen 

can be changing  into Euler’s formula



of the 1-st cases

Can be solved by first cases. Finally  changed into () then choose the real part if the problem contains () or choose the imaginary part if the problem contains ()

**Example:**-

**Solution:-** 





























**Case 3:**- If polynomials function i.e.  can be solved by using the series 

Or 

**Example:-** solve : 

**Solution:-** 





















is particular solution .



**Case 4:-** if then can be solved by using the theorem



can be solving by second cases or can be solved by third cases

**Example: -** solve: 

**Solution:-** 

























**Case 5**:- If  then 

We can solved by changing sinbx or cosbxinto Euler’s formula ()

 ----(\*)

can be solving by third cases. Finally we can changing  into  and chooses the real part if equ ( \*) contain (cosbx),but if equ(\*) contain (sinbx) ,then choose the imaginary part.

**Example:-**

**Solution:-**= H.W



















