Numerical Analysis

3rd Year physics

2022-2023

Date 1 Subject Numerical Analysis

Kinds of Number:-

There are two kinds of Numbers;

1- exact , 1, 2, 3, --- are all exact

$\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{5}$, $\sqrt{2}$, π , e also exact

2- Approximate number;

$\pi \to 3.14$, 3.141 , 3.14159 ------

$\sqrt{2} \Rightarrow 1.41$, 1.414 ------

Errors!-

Error = | True Value - Approximate Value |

Type of Errors:-

1- Inherent Errors or Input Errors

* For example in Chemistry the weighting machine has errors of 0.01 or more,
* In the experimental of simple pendulum we assume that the bob is weightless and the motion is linear.

* $y = x^2 + \frac{1}{3}x + \pi$

The Inherent errors cannot be removed Completly, but can be minimized by taking better input data and high precison Computer aids.

2- Round-off errors-

EX! 8.954266 is round off to 8.95427
or to 8.9543

Error $= |8.954266 - 8.95427| = 0.000004$

Error $= |8.954266 - 8.9543| = 0.000034$

Example:

$\frac{1}{3} = 0.333333...$ is round off to 0.33

$$\text{Error} = |0.333333 - 0.33| = 0.003333$$

or

$$\text{Error} = |0.333333 - 0.333| = 0.000333$$

3- Trancation Error:

For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Chopped off

Then it leads to error.

Similarly;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

if $x = 0.5$

First term $e^x = 1$ second term $\Rightarrow e^x = 1.5$

3rd term $e^x = 1.625$ 4th term $e^x = 1.6458\ldots$

Ex! $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots\cdots = 1$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\text{Error} = \left|1 - \frac{15}{16}\right| = 0.0625$$

$$\text{Trancation error} = \left| e^x - \left(1 + x + \frac{x^2}{2!}\right) \right|$$

Date 3 Subject

Error diffinition:

1- Absolute error
2- Relative error
3- percentage error

Example: Find the absolute error, relative error and percentage error whe the true value is 0.142857 and approximate value is 0.1429.

1- Absolute error ΔE

$$\Delta E = |\text{True value} - \text{Approximate value}|$$

$$= |0.142857 - 0.1429| = 0.000043$$

2- Relative value.

$$E_r = \frac{\Delta E}{\text{True value}} = \frac{0.000043}{0.142857} = 0.000329$$

3- Percentage error:

$$E\% = (0.000329 * 100)\% = 0.03\%$$

H.W EX1- Three approximate values of the number $\frac{1}{3}$ are given as 0.30, 0.33 and 0.34. Which of these is the best approximation.

Date 4 Subject

General formula for Error:

Let $u = f(x, y, z)$

$$\boxed{\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z}$$

Example: if $u = \frac{4x^2y^3}{z^4}$ and the errors in x, y, z is 0.001, 0.002, 0.003 respectively. Compute the absolute and relative maximum error in u when $x = y = z = 1$.

Solution –

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

Since;

$$\frac{\partial u}{\partial x} = \frac{8xy^3}{z^4}; \quad \frac{\partial u}{\partial y} = \frac{12x^2y^2}{z^4}$$

$$\frac{\partial u}{\partial z} = \frac{-16x^2y^3}{z^5}$$

$$|\delta u| = \left|\frac{8xy^2}{z^4}\delta x\right| + \left|\frac{12x^2y^2}{z^4}\delta y\right| + \left|\frac{-16x^2y^3}{z^5}\right|$$

$$|\delta u| = 8(0.001) + 12(0.002) + 16(0.003)$$

$$= 0.08$$

$$E_r = \left|\frac{\delta u}{u}\right| = \frac{0.08}{4} = 0.02$$

where $u = \frac{4x^2y^3}{z^4} = \frac{4(1)(1)}{(1)} = 4$

EX: if $u = 2m^6 - 5m$, find the percentage error in u at m = 1 if error in m is 0.05.

Solution –

Given $\boxed{u = 2m^6 - 5m}$

$$\delta u = \frac{\partial u}{\partial m}\,\delta m = (12m^5 - 5)\,\delta m$$

$$\delta u = (12(1)^5 - 5) * 0.05 = 7 * 0.05$$

$$\delta u = 0.35$$

$$u = 2m^6 - 5m = 2(1)^6 - 5(1) = -3$$

percentage error = $E_r\%$

$$E_r\% = \left|\frac{0.35}{-3} * 100\right| = 11.667\%$$

maximum percentage error

Error in series approximation –

The error committed in a series approximation can be evaluated by using the remainder after n terms.

Taylor's series for $f(x)$ at $x = a$ is given by:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n(x)$$

Where $R_n(x) = \frac{(x-a)^n}{n!}f^n(\varepsilon)$; $a < \varepsilon < x$

↳ remainder term $R_n(x) \to 0$ when $n \to \infty$

Example: The Maclaurin's expansion for e^x is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{(n-1)}}{(n-1)!} + \frac{x^n}{n!} e^{\varepsilon}, \text{ where } 0 < \varepsilon < x$$

Find the number of terms, such that their sum yields the value of e^x correct to 8 decimal places at $x = 1$.

Solution -

Given that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{(n-1)}}{(n-1)!} + \frac{x^n}{n!} e^{\varepsilon}, \quad 0 < \varepsilon < x$$

Then the remainder term is

$$R_n(x) = \frac{x^n}{n!} e^{\varepsilon}$$

When $\varepsilon = x$ gives maximum absolute error

$$E_{a(max)} = \frac{x^n}{n!} e^{x}$$

And the maximum relative error

$$E_{r(max)} = \frac{E_{a(max)}}{e^x} = \frac{\frac{x^n}{n!} \cancel{e^x}}{\cancel{e^x}} = \frac{x^n}{n!}$$

Hence at $x = 1$

$$E_{r(max)} = \frac{1^n}{n!} = \frac{1}{n!}$$

$$\frac{1}{n!} < \frac{1}{2} \times 10^{-8} \Rightarrow \boxed{n = 12}$$

Date 7 Subject

EX: the function $f(x) = \tan^{-1}(x)$ can be expanded as:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots + (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} + \cdots$$

Find n such that series determines $\tan^{-1}(1)$ correct to 8 significant digits.

Solution: if we retain n terms then,

$$(n+1)^{th} \text{ term} = (-1)^n \frac{x^{2n+1}}{2n+1}$$

For $x = 1$,

$$(n+1)^{th} \text{ term} = (-1)^n \frac{1^{2n+1}}{2n+1} = \frac{(-1)^n}{2n+1}$$

to determine a $\tan^{-1}(1)$ correct up to 8 significant digits.

$$\left| \frac{(-1)^n}{2n+1} \right| < \frac{1}{2} \times 10^{-8}$$

$$2n+1 > 2 \times 10^8$$

$$2n > 2 \times 10^8 - 1$$

$$n > 10^8 - \frac{1}{2}$$

$$\Rightarrow n > 10^8$$

Date 8 Subject

linear Regression using least square method - Line of Best Fit Equation -

Ex: write a linear equation that best fit the data in the table shown below!

x	1	2	3	4	5	6	7
y	1.5	3.8	6.7	9.0	11.2	13.6	16

$\boxed{n=7}$

(Fit equation)

x	y	xy	x^2	$\bar{y}$	DE %
1	1.5	1.5	1	1.58	5.33%
2	3.8	7.6	4	3.99	0.05
3	6.7	20.1	9	6.4	0.044
4	9.0	36	16	8.81	0.021
5	11.2	56	25	11.22	0.0017
6	13.6	81.6	36	13.63	0.0022
7	16	112	49	16.04	0.0025
$\Sigma x=28$	$\Sigma y=61.8$	$\Sigma xy=314.8$	$\Sigma x^2=140$		

$$\bar{y} = ax + b$$

slope (a) — Intercept with Y-axis (b)

$$a = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2} = \frac{7*314.8 - 28*61.8}{7*140 - (28)^2}$$

$$a = \frac{473.2}{196} = 2.414\ldots$$

$$b = \frac{\Sigma y - a\Sigma x}{n} = \frac{61.8 - (2.414)(28)}{7}$$

$$b = -0.828\ldots$$

Date 9 Subject

$\bar{Y} = aX + b$

$\bar{Y} = 2.414X - 0.828$

$\therefore \bar{Y} = 2.41X - 0.83$

$\bar{Y} = 2.41(1) - 0.83 = 1.58$

$\bar{Y} = 2.41(2) - 0.83 = 3.99$

$\bar{Y} = 2.41(3) - 0.83 = 6.4$

$\bar{Y} = 2.41(4) - 0.83 = 8.81$

$\bar{Y} = 2.41(5) - 0.83 = 11.22$

$\bar{Y} = 2.41(6) - 0.83 = 13.63$

$\bar{Y} = 2.41(7) - 0.83 = 16.04$

Error (relative error) $= \Delta E$

$$\Delta E = \frac{Y_T - \bar{Y}_A}{Y_T}$$

$$= \left| \frac{1.5 - 1.58}{1.5} \right| = 0.0533$$

Percentage error $= (\Delta E * 100)\%$

$= 0.053 * 100 = 5.33\%$

Date 10 Subject

x	1	2	3	4	5	6
y	2	4	7	9	12	14

calculate the correlation coefficient between the two variables x and y;

x	y	xy	x^2	y^2
1	2	2	1	4
2	4	8	4	16
3	7	21	9	49
4	9	36	16	81
5	12	60	25	144
6	14	84	36	196
$\sum x=21$	$\sum y=48$	$\sum xy=211$	$\sum x^2=91$	$\sum y^2=490$

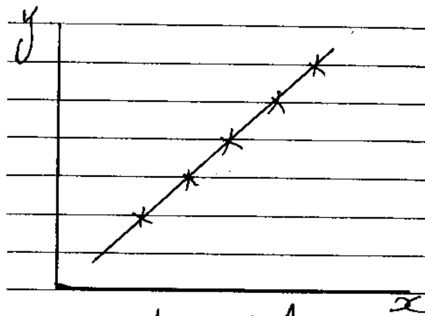
$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left[n\sum x^2 - (\sum x)^2\right]\left[n\sum y^2 - (\sum y)^2\right]}}$$

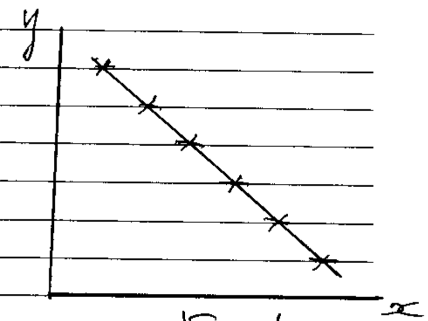
$$r = \frac{6(211) - 21(48)}{\sqrt{\left[6*91 - (21)^2\right]\left[6*490 - (48)^2\right]}}$$

$$r = \frac{1266 - 1008}{\sqrt{\left[546-441\right]\left[2940-2304\right]}}$$

$$r = \frac{258}{\sqrt{105*636}} \equiv +0.998$$

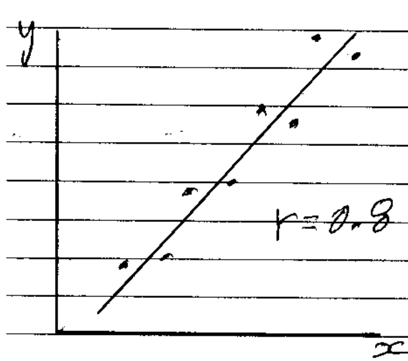
Date 11 Subject

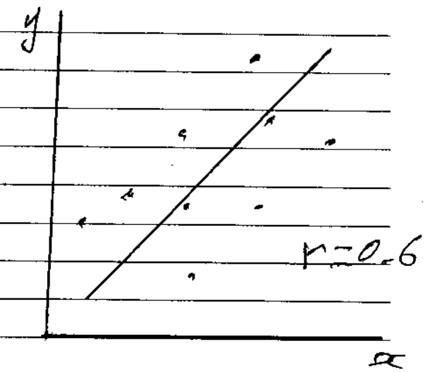




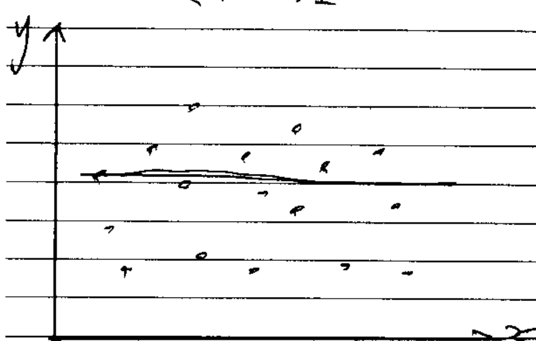
$r = +1$ $\qquad$ $r = -1$

$x\uparrow$ $y\uparrow$ $\qquad$ $x\uparrow$ $y\downarrow$





$0 < r < 1$



$r = 0$

Method of least square -

* Straight line : $y = a + bx$
* Second degree parabola : $y = a + bx + cx^2$
* Exponential equation : $y = a b^x$
* " " : $y = a e^{bx}$
* " " : $y = a x^b$

$y = \log_{10} X$

$X = 10^y \Rightarrow$ Antilog

Curve fitting Method -

Q: Find the curve of best fit $y = ae^{bx}$ to the following data by using the method of least square.

x	1	5	7	9	12
y	10	15	12	15	21

Solution

$y = ae^{bx}$

taking log on both side,

$$\log y = \log ae^{bx}$$
$$\log y = \log a + \log e^{bx}$$
$$\underbrace{\log y}_{Y} = \underbrace{\log a}_{A} + \underbrace{bx \log e}_{B}$$

$$\boxed{\log mn = \log m + \log n}$$
$$\boxed{\log m^n = n \log m}$$

$$Y^* = A + BX \Rightarrow Y^* = \log y, \; A = \log a$$
$$B = b \log e$$

$$\sum Y^* = nA + B\sum x \quad \text{---} 1$$
$$\sum xY^* = A\sum x + B\sum x^2 \quad \text{---} 2$$

x	y	$y^* = \log a$	x^2	xy
1	10	1	1	1
5	15	1.18	25	5.9
7	12	1.079	49	7.56
9	15	1.18	21	10.62
12	21	1.32	144	15.84

$\sum x = 34$ $\quad \sum Y^* = 5.76$ $\quad \sum x^2 = 300$ $\quad \sum xy = 40.92$

$\sum Y^* = nA + B\sum x$

$\sum xy^* = A\sum x + B\sum x^2$

$5.76 = 5A + 34B$ 3

$40.92 = 34A + 300B$ 4

Solving equation 3 and 4

multiply eq.(3) by 8.823

$50.82 = 44.11A + \cancel{300B}$

$\mp 40.92 = \mp 34A \mp \cancel{300B}$

$-$

$9.9 = 10.11A$

$A = \frac{9.9}{10.11} = 0.978$

$5.76 = 5A + 34B$

$5.76 = 5 \times 0.978 + 34B$

$5.76 - 4.89 = 34B$

$0.87 = 34B$

$B = \frac{0.87}{34} = 0.025$

$A = \log a \Rightarrow 0.978 = \log a$

$a = \text{Antilog}(A) \quad a = \text{Antilog}(0.978) = \underline{\underline{9.55}}$

$B = b\log e \Rightarrow b = \frac{B}{\log e} = \frac{0.025}{\log e}$

$b = \frac{0.025}{0.434} = \underline{\underline{0.057}}$

$\therefore y = ae^{bx}$

$\Rightarrow \boxed{y = 9.55e^{0.57x}}$ fit equation

Q: Using the method of least squares, find a relation of the form $y = ax^b$ that fits the data.

x	2	3	4	5
y	27.8	62.1	110	161

Solution

$$y = ax^b$$

taking $\log_{10}$ on both the sides of equality,

$$\log_{10} y = \log_{10} ax^b$$

$$\underbrace{\log_{10} y} = \underbrace{\log_{10} a} + \log_{10} x^b$$

$$Y^* = A + b\log_{10} x$$

$$Y^* = A + bX^*$$

$$Y^* = \log_{10} y$$

$$A = \log_{10} a$$

$$X^* = \log_{10} x$$

$$\sum Y^* = nA + b\sum X^* \quad \text{---} (1)$$

$$\sum \cancel{X^*} Y^* = A\sum X^* + b\sum X^{*2} \quad \text{---} (2)$$

x	y	$X^* = \log_{10} x$	$Y^* = \log_{10} y$	$X^* Y^*$	X^{*2}
2	27.8	0.3010	1.440	0.4346	0.0906
3	62.1	0.4771	1.7931	0.8555	0.2276
4	11.0	0.6021	2.0414	1.2291	0.3625
5	16.1	0.6990	2.2068	1.5426	0.4886
		$\sum X^* = 2.0792$	$\sum Y^* = 7.4853$	$\sum X^* Y^* = 4.0615$	$\sum x^{*2} = 1.1693$

$\sum Y^* = nA + b \sum X^*$

$\sum X^* Y^* = A \sum X^* + b \sum X^{*2}$

$7.4853 = 4A + 2.0792 b$ ---- 3

$4.0615 = 2.0792A + 1.1693 b$ ---- 4

multiply eq(3) by 0.5623

$4.2058 = 2.2492A + 1.1693 b$

$\mp 4.0615 = \mp 2.0792A \mp 1.1693 b$

$-$

$0.1443 = 0.17 A$

$A = \frac{0.1443}{0.17} = 0.848$

To find, b, $\Rightarrow$ $7.4853 = 4A + 2.0792 b$

$7.4853 = 4 * 0.848 + 2.0792 b$

$7.4853 = 3.392 + 2.0792 b$

$b = \frac{7.4853 - 3.392}{2.0792} = 1.96$

$a = \text{Antilog}(A) = \text{Antilog}(0.848) = 7.376$

$y = ax^b \Rightarrow \boxed{y = 7.376 x^{1.96}}$

Fitting of exponential curve -

$$y = a b^{x}$$

Taking $\log_{10}$ of both side of equation,

$$\log y = \log a b^{x}$$
$$\log y = \log a + \log b^{x}$$

$$\underset{\Downarrow}{\log_{10} y} = \underbrace{\log_{10} a}_{A} + x \underbrace{\log_{10} b}_{B} \qquad \therefore\ y^{*} = \log_{10} y$$
$$y^{*} = A + xB \qquad A = \log_{10} a$$
$$B = \log_{10} b$$

$$\sum y^{*} = nA + B\sum x \quad \text{--- (1)}$$
$$\sum x y^{*} = A\sum x + B\sum x^{2} \quad \text{--- (2)}$$

Ex: Fit a curve $y = ab^{x}$ to the following data using method of least square.

x	y	$y^{*} = \log_{10} y$	x^{2}	$x y^{*}$
2	8.3	0.9191	4	1.8382
3	15.4	1.1875	9	3.5625
4	33.1	1.5198	16	6.0792
5	62.2	1.8142	25	9.0710
6	127.4	2.1052	36	12.6312
$\sum x = 20$		$\sum y^{*} = 7.5458$	$\sum x^{2} = 90$	$\sum x y^{*} = 33.1821$

using equation 1 and 2,

$$7.5458 = 5A + 20B \qquad * \ 4$$
$$33.1821 = 20A + 90B$$

$\Longrightarrow$

$$30.1832 = 20A + 80B \quad \text{---} (3)$$
$$\mp 33.1821 = \mp 20A \mp 40B \quad \text{---} (4)$$

$$-2.9989 = -10B$$

$$B = \frac{-2.9989}{-10} = 0.29989$$

$$B = \log_{10} b \Rightarrow b = \text{Antilog}(B)$$

$$\underline{\underline{b}} = \text{Antilog}(0.29989) = \underline{\underline{1.995}}$$

From equation 3!

$$30.1832 = 20A + 80B$$
$$30.1832 = 20A + 80 * 0.29989$$
$$30.1832 = 20A + 23.991$$

$$A = \frac{30.1832 - 23.991}{20} = 0.3095$$

$$A = \log_{10} a \Rightarrow a = \text{Antilog}(A)$$

$$a = \text{Antilog}(0.3095)$$

$$a = 2.04$$

$$y = a b^{x}$$

$$y = 2.04 (1.995)^{x}$$

fit esstimate
Value

EX! Fit a parabola $y = a + bx + cx^2$ in the least square sense for the following data and hence estimate y when $x = 6$.

x	1	2	3	4	5
y	10	12	13	16	19

Solution –

We want to fit a curve $y = a + bx + cx^2$ to the given data.

Normal equations!

$$\sum y = \sum a + \sum bx + \sum cx^2$$

$$\sum y = a\sum 1 + b\sum x + c\sum x^2$$

$$\sum y = an + b\sum x + c\sum x^2 \quad \text{---} (1)$$

$$\sum 1 = 1+1+1+1 \cdots n\text{-times} = n$$

multiply equation 1 by x and then apply $\sum$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad \text{--} (2)$$

multiply equation by x^2 and then apply $\sum$.

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad \text{--} (3)$$

$\sum y = an + b\sum x + c\sum x^2$ ---- 1

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ ---- 2

$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$ ---- 3

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	13	9	27	81	39	117
4	16	16	64	256	64	256
5	19	25	125	625	95	475
$\sum x$	$\sum y$	$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum xy$	$\sum x^2y$
15	70	55	225	979	232	906

Appling equation 1, 2 and 3.

$70 = 5a + 15b + 55c$

$232 = 15a + 55b + 225c$

$906 = 55a + 225 + 979c$

Using cramer's rule

$$D = \begin{vmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{vmatrix} = 5(55*979 - 225*225) - 15(15*979 - 55*225) + 55(15*225 - 55*55)$$

$$D = 700$$

$$D_1 = \begin{vmatrix} 70 & 15 & 55 \\ 232 & 55 & 225 \\ 906 & 225 & 979 \end{vmatrix} = 70(55*979 - 2\,5*225) - 15(232*979 - 225*906) + 55(232*225 - 55*906)$$

$$D_1 = 6580$$

$$D_2 = \begin{vmatrix} 5 & 70 & 55 \\ 15 & 232 & 225 \\ 55 & 906 & 979 \end{vmatrix} = 340$$

$$D_3 = \begin{vmatrix} 5 & 15 & 70 \\ 15 & 55 & 232 \\ 55 & 225 & 906 \end{vmatrix} = 200$$

$$a = \frac{D_1}{D} = \frac{6580}{700} = 9.4$$

$$b = \frac{D_2}{D} = \frac{340}{700} = 0.4857$$

$$c = \frac{D_3}{D} = \frac{200}{700} = 0.2857$$

$$y = a + bx + cx^2$$

$$y = 9.4 + 0.4857x + 0.2857x^2$$

Put $x = 6$

$$y = 9.4 + 0.4857 * 6 + 0.2857 * (6)^2$$

$$y = 22.5994$$