Numerical Analysis

Part 4

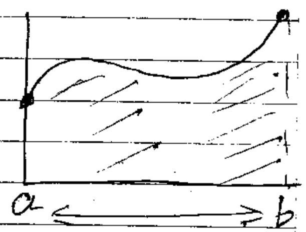
3rd Year physics

2022-2023

Numerical Integration

The area bounded by the curve $f(x)$ and x-axis is;

$$I = \int_a^b f(x)\,dx$$



$$\therefore I = \frac{h}{2}\left[f(a) + f(b)\right]$$

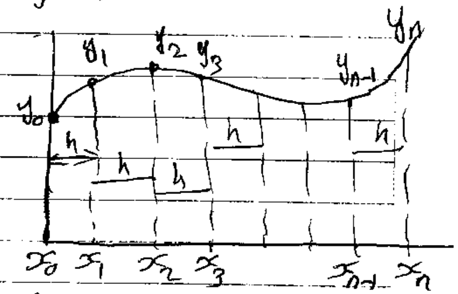
$h = b - a$

→ Trapezoidal Rule when $\boxed{n=2}$

* Let us divide the interval $[a,b]$ into "n" equal parts

$n > 2$

$$h = \frac{b-a}{n}$$



$$I = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots y_{n-1})\right]$$

→ Trapezoidal rule for $n > 2$

Simpson's $\frac{1}{3}$ rule [even # of interval]

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[y_0 + y_n + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\Big]$$

Simpson's $\frac{3}{8}$ rule ("n" multiple of 3)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\Big[y_0 + y_n + 3(y_1 + y_2 + y_4 + \quad) + 2(y_3 + y_6 + y_9 + y_{12} + \cdots)\Big]$$

Q: using Trapezoidal simpson's $\frac{1}{3}$ and simpson's $\frac{3}{8}$ rule, evaluate the definite integral $\int_1^7 f(x)\,dx$

by using following data:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	1	2	3	4	5	6	7
$f(x)$	2.1	2.8	3.6	4.6	5.8	7.4	9.4
	y_0	y_1	y_2	y_3	y_4	y_5 (y_{n-1})	y_6 (y_n)

Solution

Trapezoidal rule

$$I=\int_1^7 f(x)\,dx=\frac{h}{2}\left[y_0+y_n+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$n=6 \quad \therefore h=\frac{b-a}{n}=\frac{7-1}{6}=1$$

$$I=\frac{1}{2}\left[2.1+9.4+2(2.8+3.6+4.6+5.8+7.4)\right]$$

$$I=\frac{1}{2}[59.9]=29.95$$

* Simpson's $\frac{1}{3}$ rule

$$I=\int_1^7 f(x)\,dx=\frac{h}{3}\left[y_0+y_n+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$\therefore I=\frac{1}{3}\left[2.1+9.4+4(2.8+4.6+7.4)+2(3.6+5.8)\right]$$

$$I=\frac{1}{3}\left[11.5+59.2+18.8\right]=\frac{1}{3}[89.5]=29.8\dot{3}$$

Simpson's $\frac{3}{8}$ rule

$$I = \int_1^7 f(x)\,dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$I = \frac{3}{8}\left[2.1 + 9.4 + 3(2.8 + 3.6 + 5.8 + 7.4) + 2(4.6)\right]$$

$$I = \frac{3}{8}\left[11.5 + 58.8 + 9.2\right]$$

$$I = \frac{3}{8}\left[79.5\right] = 29.81$$

Q! Evalute the integral $\int_{100}^{200} \frac{dx}{\ln x}$

Taking $n=4$ using Trapezoidal rule and simpson's $\frac{1}{3}$ rule.

Solution!—

$n=4 \Rightarrow h = \frac{b-a}{n} = \frac{200-100}{4}$

$h = 25$

$a = 100$
$b = 200$

	x_0	x_1	x_2	x_3	x_4
X	100	125	150	175	200
$f(x) = \frac{1}{\ln x}$	0.217	0.207	0.199	0.193	0.188
	y_0	y_1	y_2	y_3	y_4
				$\downarrow$ y_{n-1}	$\downarrow$ y_n

Trapezoidal rule -

$$I = \int_{100}^{200} \frac{dx}{\ln x} = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{25}{2}\left[0.217 + 0.188 + 2(0.207 + 0.199 + 0.193)\right]$$

$$= \frac{25}{2}\left[0.217 + 0.188 + 1.198\right]$$

$$= \frac{25}{2}\left[1.603\right] = 20.037$$

Simpson's $\frac{1}{3}$ rule –

$$I = \int_{100}^{200} \frac{dx}{\ln x} = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3) + 2(y_2)\right]$$

$$I = \frac{25}{3}\left[0.217 + 0.188 + 4(0.207 + 0.193) + 2(0.199)\right]$$

$$I = \frac{25}{3}\left[0.217 + 0.188 + 1.6 + 0.398\right]$$

$$I = \frac{25}{3}\left[2.403\right]$$

$$= 20.025$$

* Simpson's $\frac{3}{8}$ rule –

$$I = \int_{100}^{200} \frac{dx}{\ln x} = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2) + 2(y_3)\right]$$

$$= \frac{3 \times 25}{8}\left[0.217 + 0.188 + 3(0.207 + 0.199) + 2(0.193)\right]$$

$$I = \frac{3 \times 25}{8}\left[0.217 + 0.188 + 1.21 + 0.386\right]$$

$$= \frac{75}{8}\left[2.001\right] = 18.759$$

Q: Evaluate $\int_0^1 x^4 dx$

when $n=2$ and $n=5$

(1) Solution

$n=2$

	y_0	y_n
x	0	1
$f(x)=x^4$	0	1

$$I = \frac{h}{2}\left[y_0 + y_n\right]$$

$$= \frac{1}{2}\left[0+1\right] = 0.5$$

$h = b-a$
$= 1-0$
$= 1$

(2) $n=5$

$$h = \frac{b-a}{n-1} = \frac{1-0}{5-1} = 0.25$$

	x_0	x_1	x_2	x_3	x_4
x	0	0.25	0.5	0.75	1
$f(x)=x^4$	0	$(0.25)^4$	$(0.5)^4$	$(0.75)^4$	1^4
	y_0	y_1	y_2	y_3	y_4

$$I = \int_0^1 x^4 dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{1}{8}\left[0 + 1^4 + 2\left((0.25)^4 + (0.5)^4 + (0.75)^4\right)\right]$$

$$= 0.2207$$

Simpson's $\frac{1}{3}$ rule

$$I = \int_0^1 x^4 dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{0.25}{3}\left[0 + 1 + 4\left((0.25)^4 + (0.75)^4\right) + 2(0.5)^4\right]$$

$$= ??$$

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Q evaluate $\int_0^1 \frac{dx}{1+x^2}$

when $\boxed{n=4}$

Solution:

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	x_0 0	x_1 0.25	x_2 0.5	x_3 0.75	x_4 1
$f(x) = \frac{1}{1+x^2}$	1	0.94	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	$y_4 = n$

Trapezoidal rule

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3)\right]$$

$$I = \frac{0.25}{2}\left[1 + 0.5 + 2(0.94 + 0.8 + 0.64)\right]$$

$$I = \frac{0.25}{2}\left[6.26\right] = 0.782$$

– – evaluate using Simpson's $\frac{1}{3}$ and Simpson's $\frac{3}{8}$.

* Numerical solution of ordinary differential equation by Rang-Kutta method of fourth order.

Let the given diff. equ. be

$\frac{dy}{dx} = f(x,y)$ with initial condition $y = y_0$ when $x = x_0$

the To find value of $\boxed{y = y_0 + K}$ at $x = x_0 + h$ we calculated following terms.

$K_1 = h \cdot f(x_0, y_0)$

$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = h \cdot f(x_0 + h, y_0 + K_3)$

Lastly;

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Then;

$$\boxed{y = y_0 + K}$$

Ex: Apply R-K method of 4th order to find approximate value of y at $x=0.2$ if $\frac{dy}{dx} = x + y^2$ given that $y=1$ when $x=0$ in steps of $h=0.1$

Solution:

$$\frac{dy}{dx} = x + y^2$$

$$f(x,y) = x + y^2$$

$$x_0 = 0 \qquad y_0 = 1 \qquad h = 0.1$$

$$f(x_0, y_0) = x_0 + y_0^2$$

$$K_1 = h \cdot f(x_0, y_0) = 0.1(x_0 + y_0^2) = 0.1(0+1)$$

$$\boxed{K_1 = 0.1}$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_2 = h \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$K_2 = h\ f(0.05, 1.05)$$

$$K_2 = 0.1\left(x + y^2\right)$$

$$K_2 = 0.1\left(0.05 + (1.05)^2\right)$$

$$K_2 = 0.11525$$

$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_3 = h \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11525}{2}\right)$

$K_3 = h \cdot f(0.05, 1.05762)$

$K_3 = 0.1 * [x + y^2]$

$K_3 = 0.1 * [0.05 + (1.05762)^2]$

$K_3 = 0.11686$

$K_4 = h \cdot f(x_0 + h, y + K_3)$

$K_4 = h \cdot f(0 + 0.1, 1 + 0.11686)$

$K_4 = h \cdot f(0.1, 1.11686)$

$K_4 = 0.1 [0.1 + (1.11686)^2]$

$K_4 = 0.13474$

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$K = \frac{1}{6}[0.1 + 2*(0.11525) + 2*(0.11686) + 0.13474]$

$\boxed{K = 0.1165}$

Then the approximation Value is,

$y = y_0 + K \Rightarrow y = 1 + 0.1165$

$\boxed{y = 1.1165}$

numerical ordinary differential equation

* Taylor's series method.

Let $y = f(x)$ be a solution of the equation

$\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

Expanding it by Taylor's series about the point x_0, we get,

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \frac{(x-x_0)^4}{4!} f''''(x_0) + \cdots$$

This may be written as

$$y = f(x) = y_0 + \frac{(x-x_0)}{1!} y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \frac{(x-x_0)^4}{4!} y''''_0 + \cdots$$

Ex: Given $\frac{dy}{dx} = x - y^2$ with the initial condition that $y_0 = 1$ at $x_0 = 0$ $\Rightarrow \boxed{y(0) = 1}$

Compute $y(0.1)$ correct to four places of decimal by using Taylor's series method.

solution: Taylor's equation is

$$f(x) = y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0'''' + \cdots \quad (1)$$

$$\frac{dy}{dx} = y_0' = x_0 - y_0^2 = 0 - (1)^2 = \underline{-1}$$

---- (2)

$$y_0'' = 1 - 2yy'$$
$$= 1 - 2(1)(-1) = \underline{3}$$

diff of eq. 2,

$$y''' = -2[yy'' + y'y'] \quad (3)$$
$$y_0''' = -2[(1)(3) + (-1)(-1)]$$
$$y_0''' = \underline{-8}$$

diff. of eq. 3

$$y_0'''' = -2[yy''' + y''y' + y''y'' + y'y'']$$
$$y_0^4 = -2[(1)(-8) + (3)(1) + (-1)(3) + (-1)(-1)]$$
$$y_0^4 = 34$$

$\Rightarrow$

$$x_0 = 0$$

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Using Taylor's Series;

$$f(x) = y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0'''' + \cdots$$

$$f(x) = 1 + x*(-1) + \frac{x^2}{2}*(3) + \frac{x^3}{6}*(-8) + \frac{x^4}{24}*(34)$$

$$f(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4$$

$$f(0.1) = 1 - (0.1) + \frac{3}{2}(0.1)^2 - \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4$$

$$f(0.1) = 1.1053425$$

correct to four decimal place

$$\boxed{f(0.1) = 1.1053}$$

HW Example: Solve $\frac{dy}{dx} = x + y$, with Initial Condition that $f(0) = 1$. Compute $y(0.5)$ correct to 3 places of dicimal by using Taylor's method.

HW Example! Apply the Taylor's series method to Find the value of $y(1.1)$ and $y(1.2)$ correct to three dicimal places given that $\frac{dy}{dx} = xy^{\frac{1}{3}}$, with initial condition $y(1) = 1$ taking the first three terms of the Taylor's series method.

Ex: Find a Taylor series for the function $f(x) = \ln(x)$ centered at $x_0 = 1$

Sol.

$$f(x) = f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)}{2!} + f'''(x_0)\frac{x-x_0}{3!} + f''''(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

$$f(x_0) = \ln(x_0) \implies f(1) = \ln 1 = 0$$

$$f'(x_0) = \frac{1}{x_0} \implies f'(1) = \frac{1}{1} = 1$$

$$f''(x_0) = \frac{-1}{x^2} \implies f''(1) = \frac{-1}{(1)^2} = -1$$

$$f'''(x_0) = \frac{2}{x^3} \implies f'''(1) = \frac{2}{(1)^2} = 2$$

$$f''''(x_0) = \frac{-6}{x^4} \implies f''''(1) = \frac{-6}{(1)^4} = -6$$

$$f(x) = f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + f(x_0)\frac{(x-x_0)^3}{3!} + f''''(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

$$f(x) = \ln(x) = 0 + (1)(x-1) + (-1)\frac{(x-1)^2}{2!} + (2)\frac{(x-1)^2}{3!} + (-6)\frac{(x-1)^4}{4!}$$

$$f(x) = \ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

EX: Find a Taylor's series for the function $f(x) = e^{x}$, centered at $\boxed{x_0 = 3}$

Solution:

$$f(x_0) = f(x_0) + f'(x_0)\cdot(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + f''''(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

$$f(x_0) = e^{x} = f'(x_0) = f''(x_0) = f'''(x_0) = f''''(x_0)$$

$$f(x_0) = e^{3}$$

$$f(x) = e^{x} = e^{3} + e^{x}(x-3) + e^{x}(x-3)^2 + e^{x}(x-3)^3 + e^{x}(x-3)^4 + \cdots$$

$$e^{x} = \sum_{n=0}^{\infty} e^{3}\frac{(x-3)^n}{n!}$$

EX: Find the Taylor (Maclaurin) series for the function

$f(x) = \sin(x) \rightarrow \boxed{x_0 = 0}$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + f''''(x_0)\frac{(x-x_0)}{4!} + \cdots$$

$f(x_0) = \sin x \Rightarrow f(0) = \sin(0) = 0$

$f'(x_0) = \cos x \Rightarrow f'(0) = \cos(0) = 1 = f^5$

$f''(x_0) = -\sin x \Rightarrow f''(0) = -\sin(0) = 0 = f^6$

$f'''(x_0) = -\cos x \Rightarrow f'''(0) = -\cos(0) = -1 = f^7$

$f''''(x_0) = \sin x \Rightarrow f''''(0) = \sin(0) = 0 = f^8$

$$f(x) = 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{-x^7}{7!} + \cdots$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)}$$

$\sin x$

Power = 1 ($n=0$), 3 ($n=1$), 5 ($n=2$), 7 ($n=3$), 9 ($n=4$), 11 ($n=5$); $d = 2$

Note:

$a_n = a_1 + (n-1)d$

$a_n = 3 + (n-1)(2)$

$= 3 + 2n - 2$

$\boxed{a_n = 2n+1}$

Ex!

Find the maclaurin series for the function $\boxed{f(x) = \cos x}$ using the maclaurin series $f(x) = \sin x$. when $x_0 = 0$

Sol!

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{d}{dx}[\sin x] = \frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right.$$

$$\cos x = 1 - \frac{\cancel{3}x^2}{\cancel{3}\cdot 2!} + \frac{\cancel{5}x^4}{\cancel{5}\cdot 4!} - \frac{\cancel{7}x^6}{\cancel{7}\cdot 6!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}(-1)^n$$

$$\frac{d}{d}[\sin x] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n\right]$$

$$\cos x = \sum_{n=0}^{\infty} \frac{\cancel{(2n+1)}\, x^{2n}}{\cancel{(2n+1)}(2n)!}(-1)^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}(-1)^n$$

$a_n = a_1 + (n-1)d$

$a_n = 2 + (n-1)(2$

pow	1	2	4	6	$\boxed{a_n = 2n}$
	$n=0$	$n=1$	$n=2$	$n=3$	

EX1

Find the Maclaurin series for $\cos x^2$.

Sol.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{2n!}$$

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2n!} \Longleftrightarrow$$

EX1 Find the Maclaurin series for

$f(x) = x\cos x$

$$x\cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$x\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x \cdot x^{2n}}{2n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{2n!}$$

EX Find Maclaurin series for the function $f(x) = x^2 e^{-x}$

Solu

$$f(x) = x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^2 \cdot x^n}{n!}$$

$$x^2 e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(n+2)}}{n!}$$

EX Find the Maclaurin series for the function $f(x) = \cos^2 x$

Sol.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$\cos^2 x = \frac{1}{2} [1 + \cos 2x]$$

$$\cos^2 x = \frac{1}{2} \left[1 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\right]$$

Ex: Find the Maclaurin series for $f(x) = \ln(1+x)$.

Solution

$f(x_0) = \ln(1+x_0) \qquad f(0) = \ln(1+0) = \ln 1 = 0$

$f'(x_0) = \frac{1}{(1+x_0)} \Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x_0) = \frac{-1}{(1+x_0)^2} \Rightarrow f''(0) = \frac{-1}{1^2} = -1$

$f'''(x_0) = \frac{(-1)(-2)}{(1+x_0)^3} \Rightarrow f'''(0) = \frac{2}{(1)^3} = 2$

$f''''(x_0) = \frac{(-1)(-2)(-3)}{(1+x_0)^4} \Rightarrow f''''(0) = \frac{-6}{1^4} = -6$

$$\ln(1+x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + f''''(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

$$\ln(1+x) = 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$f(x) = \ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$$

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EX1 Given $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$

at $x = 1$, $y = 0$

$y' = 2$

use Taylor series to find a Solution to the differential equation up to term $(x-1)^3$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!}$$

$y' = 2$

$\frac{d^2y}{dx^2} = x_0 - y\frac{dy}{dx}$

$y'' = x_0 - y \cdot y'$

$y'' = 1 - (0)(2)$

$y'' = 1$

$y''' = 1 - (yy'' + y'y')$

$y''' = 1 - [(0)(1) + (2)(2)]$

$y''' = 1 - 4 = -3$

$f(x) = 0 + 2(x-1) + (1)\frac{(x-1)^2}{2!} - 3\frac{(x-1)^3}{3!}$

$f(x) = 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3$

* Euler Method -

$$y_{i+1} = y_i + h\,\bar{y}_i \qquad \bar{y} = \frac{dy}{dx}$$

$i = 0, 1, 2, \ldots\ldots$

if $\boxed{h = x_{i+1} - x_i}$

EX: use Euler method to solve

$$\bar{y} = x - y$$

if $x_1 = 0.1$ and $y(\underset{x_0}{0}) = \underset{y_0}{1}$

Sol.

$h = x_1 - x_0$

$h = 0.1 - 0$

$\boxed{h = 0.1}$

$\bar{y} = x - y$

$i=0 \quad \bar{y}_0 = x_0 - y_0$

$\bar{y}_0 = 0 - 1$

$\boxed{\bar{y}_0 = -1}$

$y_1 = y_0 + h\,\bar{y}_0$

$y_1 = 1 + (0.1)(-1)$

$= 1 - 0.1$

$\boxed{y_1 = 0.9}$

$i=1 \quad y'_1 = x_1 - y_1$

$= 0.1 - 0.9$

$y'_1 = -0.8$

$y_2 = y_1 + h\,\bar{y}_1$

$y_2 = 0.9 + (0.1)(-0.8) = 0.82$

Ex! Use Euler method to solve

$$y' = 1 - 2xy$$

if $h = 0.1$ and $y(\overset{x_0}{1}) = \overset{y_0}{0.538}$

Solution:

$i = 0$

$$y' = 1 - 2x_0 y_0$$

$$y' = 1 - 2(1)(0.538)$$

$$y' = -0.076$$

$$y_{i+1} = y_i + h y'_i$$

$$y_1 = y_0 + h y'_0$$

$$y_1 = 0.538 + 0.1 * (-0.076)$$

$$\boxed{y_1 = 0.5304}$$

$i = 1$ $\quad y'_1 = 1 - x_1 y_1$

$$h = x_1 - x_0$$

$$x_1 = h + x_0$$

$$x_1 = 0.1 + 1$$

$$\boxed{x_1 = 1.1}$$

$$y'_1 = 1 - (1.1)(0.5304)$$

$$y'_1 = -0.16688$$

$y_2 = y_1 + h\, y'_1$

$y_2 = 0.5304 + (0.1)(-0.16688)$

$y_2 = 0.513712$

$\boxed{i = 2}$

$y'_2 = 1 + 2 x_2 y_2$

H-W

H-W

Ex: using Runge-Kutta method of order 4, find y for $x = 0.1$, 0.2, 0.3 given that

$$\frac{dy}{dx} = xy + y^2$$

$y(0) = 1$

H-W

EX: Solve $y' = x - y^2$ by Euler's method for $x = 0.2$ with $h = 0.2$ initially $x = 0$, $y = 1$

EX: Solve the diff. equ.

$$\frac{dy}{dx} = 2y + 3e^{x}$$

with $x_0 = 0$, $y_0 = 0$ using Taylor's series method to obtain y at $x = 0.1$, 0.2

EX: Solve by Euler's method the following differential equation $x = 0.1$, correct to four decimal places

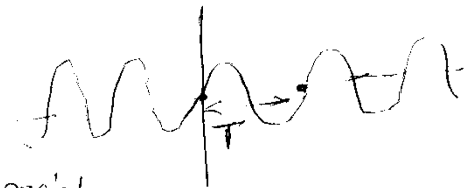
$$\frac{dy}{dx} = \frac{y - x}{y + x}$$ with initial condition $y(0) = 1$

EX: Use Runge-Kutta method to approximate y when $x = 0.1$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = x + y$

Fourier series Representation of periodic signals.

$$f(x) = a_0 + \left[\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$



where $\boxed{2L = T}$ = period

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

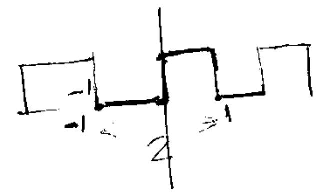
$n = 1, 2, 3, \ldots$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

general function

EXP: suppose that

$$f(t) = \begin{cases} 0 & -1 \le t \le 0 \\ 1 & 0 < t < 1 \end{cases}$$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$2L = T$
$2L = 2$
$L = 1$

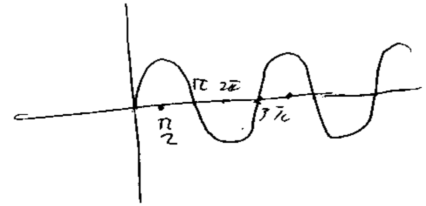
$$a_0 = \frac{1}{2}\int_{0}^{1} 1\,dt = \frac{1}{2}t\Big|_0^1$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}$$

$$a_n = 1\int_0^1 1 * \cos(n\pi t)\,dt$$

$$a_n = \frac{\sin n\pi t}{n\pi}\Big|_0^1$$



$$a_n = \frac{\cancel{\sin n\pi}^{\,0} - \sin 0}{n\pi}$$

$\sin n\pi = 0$

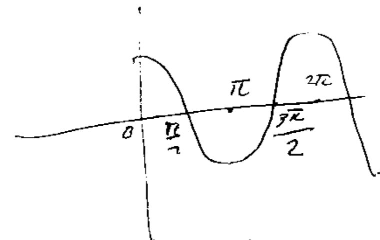
$n = 1, 2, 3, \ldots$

0

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{L} * \int \ldots$$

$$b_n = \int_0^1 \sin n\pi t\, dt$$



$$b_n = \frac{-\cos n\pi t}{n\pi}\Big|_0^1$$

$$= \frac{-(-1)^n + 1}{n\pi}$$

-1

1

-1

1

$(-1)^n$

$b_n = \frac{2}{n\pi}$ $\quad n = \text{odd}$

$= 0$ $\quad n = \text{even}$

$$f(t) = a_0 + \sum_{n}^{\infty} (\overset{=0}{a_n} \cos n\pi t + b_n \sin n\pi t)$$

$$= \frac{1}{2} + \sum_{\substack{n=1 \\ \text{odd}}} \frac{2}{n\pi} * \sin n\pi t$$

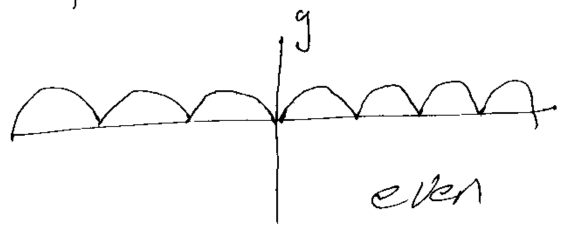
$$= \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \ldots$$

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \sin(2n+1)\pi t$$

$\Rightarrow$ in term of series form.

$f(x) = f(-x)$ even

$f(x) = -f(x)$ odd $\Rightarrow$ depend on function



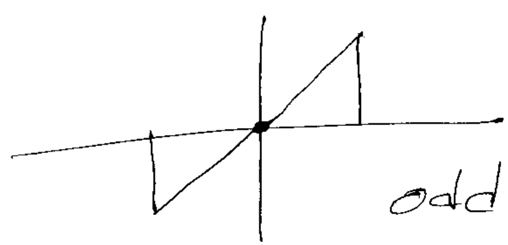
even

symmetric on both side of y-axis

$a_0 = \checkmark$

$a_n = \checkmark$

$\boxed{b_n = 0}$



odd

symmetric on origin point

$a_0 = a_n = 0$

$b_n = \checkmark$