Lecture #11

Fundamentals of Lyapunov Theory

MUSTAFA M. MUSTAFA

Assistant Lecturer

College of Engineering
Department of Electrical Engineering

Properties of systems

$\mathcal{X}_{\text{sig}}$ ≡ set of all piecewise continuous signals $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$
$\mathcal{Q}_{\text{sig}}$ ≡ set of all piecewise constant signals $q:[0,T) \to \mathcal{Q}$, $T \in (0,\infty]$

Sequence property $\equiv p : \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}} \to \{\text{false},\text{true}\}$

E.g.,

$$p(q,x) = \begin{cases} \text{true} & q(t) \in \{1,3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of <u>signals</u> $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ ***satisfies*** p if $p(q, x) = \text{true}$

A hybrid <u>automaton</u> H ***satisfies*** p (write $H \vDash p$) if

$p(q, x) = \text{true}$, for every solution (q, x) of H

"***ensemble properties***" ≡ property of the whole family of solutions
(cannot be checked just by looking at isolated solutions)
e.g., continuity with respect to initial conditions…

Lyapunov stability

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

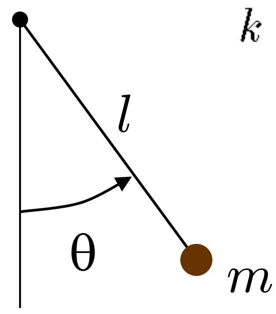
equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

thus $x(t) = x_{eq}\ \forall\, t \geq 0$ is a solution to the ODE

E.g., pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



$k \equiv$ friction coefficient

$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$

two equilibrium points:

$x_1 = 0$, $x_2 = 0$ (down)

and $x_1 = \pi$, $x_2 = 0$ (up)

Lyapunov stability

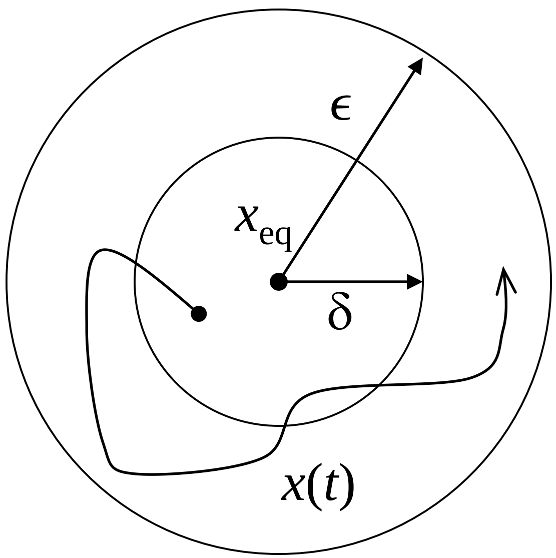
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

thus $x(t) = x_{eq}\ \forall\, t \geq 0$ is a solution to the ODE

Definition (ϵ–δ definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (***Lyapunov***) ***stable*** if

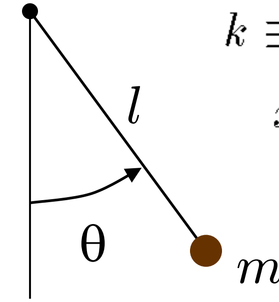
$$\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x(t_0) - x_{eq}\| \leq \delta \ \Rightarrow\ \|x(t) - x_{eq}\| \leq \epsilon\ \forall\, t \geq t_0 \geq 0$$



Example #1: Pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



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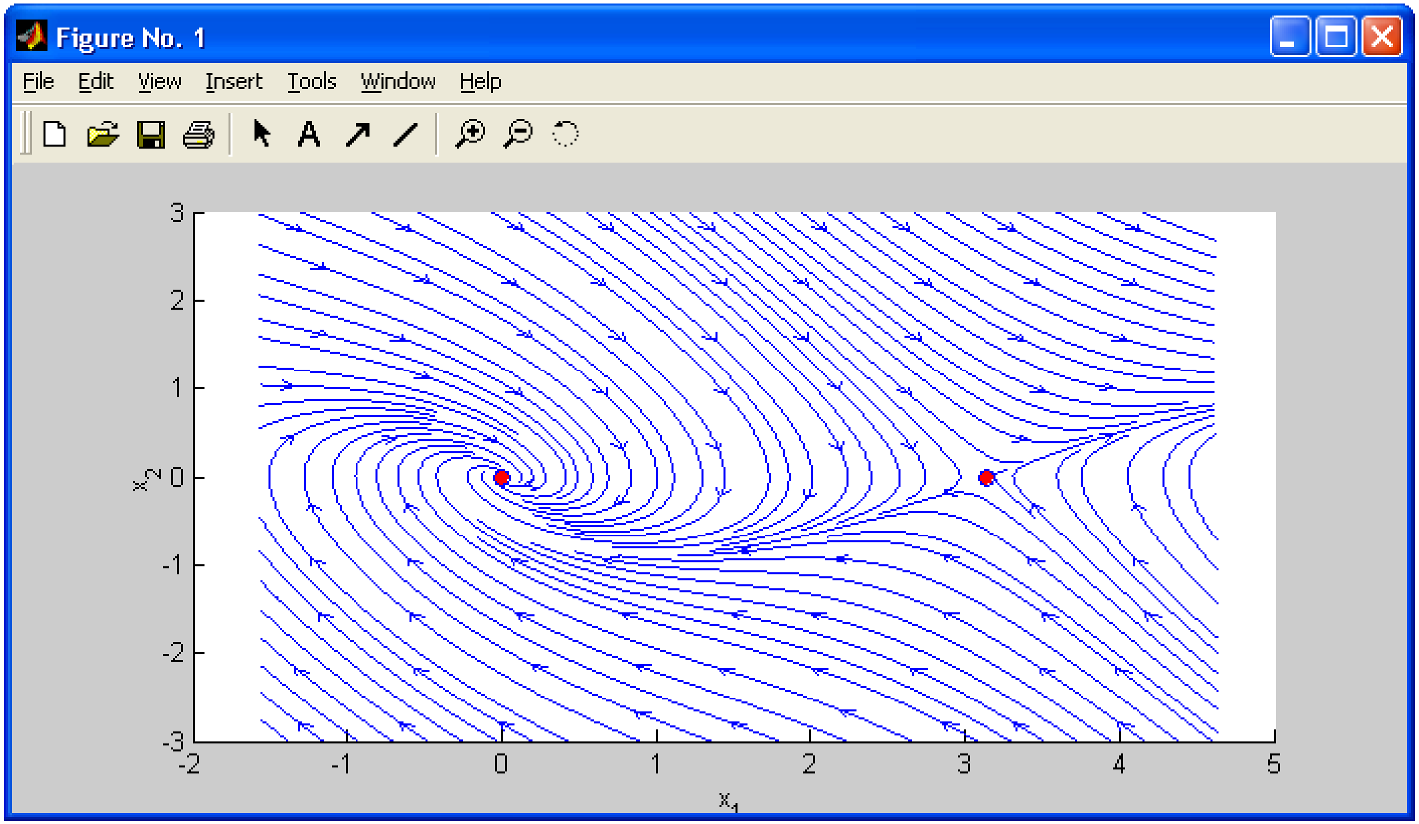
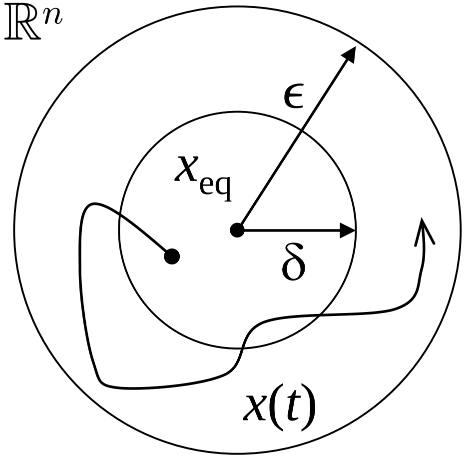


$x_{eq}=(\pi,0)$ unstable $\quad x_{eq}=(0,0)$ stable

Lyapunov stability – continuity definition

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$



$\mathcal{X}_{sig} \equiv$ set of all piecewise continuous signals taking values in $\mathbb{R}^n$

Given a signal $x \in \mathcal{X}_{sig}$, $\|x\|_{sig} := \sup_{t \geq 0} \|x(t)\|$

ODE can be seen as an operator

$$T : \mathbb{R}^n \to \mathcal{X}_{sig}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):

The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if T is continuous at x_{eq}:

$$\forall \epsilon > 0 \ \exists \ \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \ \Rightarrow \ \underbrace{\|T(x_0) - T(x_{eq})\|_{sig}}_{\sup_{t \geq 0} \|x(t) - x_{eq}\| \leq \epsilon} \leq \epsilon$$

Stability of arbitrary solutions

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals taking values in $\mathbb{R}^n$

Given a signal $x \in \mathcal{X}_{\text{sig}}$, $\|x\|_{\text{sig}} := \sup_{t \geq 0} \|x(t)\|$

signal norm

ODE can be seen as an operator

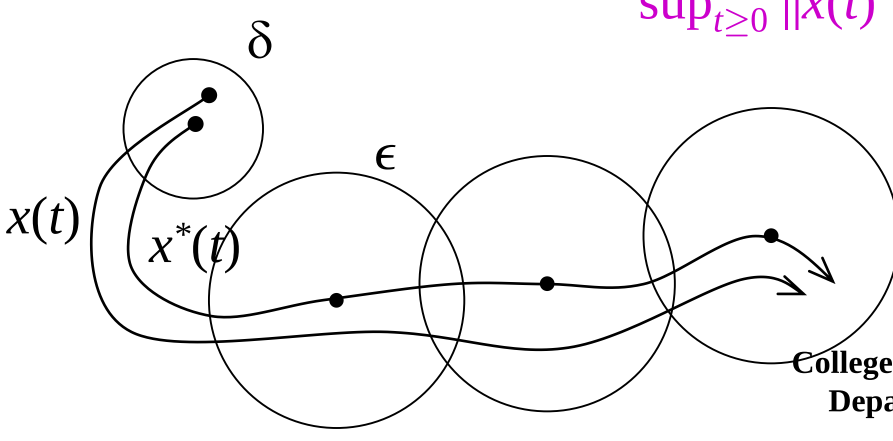
$$T : \mathbb{R}^n \to \mathcal{X}_{\text{sig}}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):

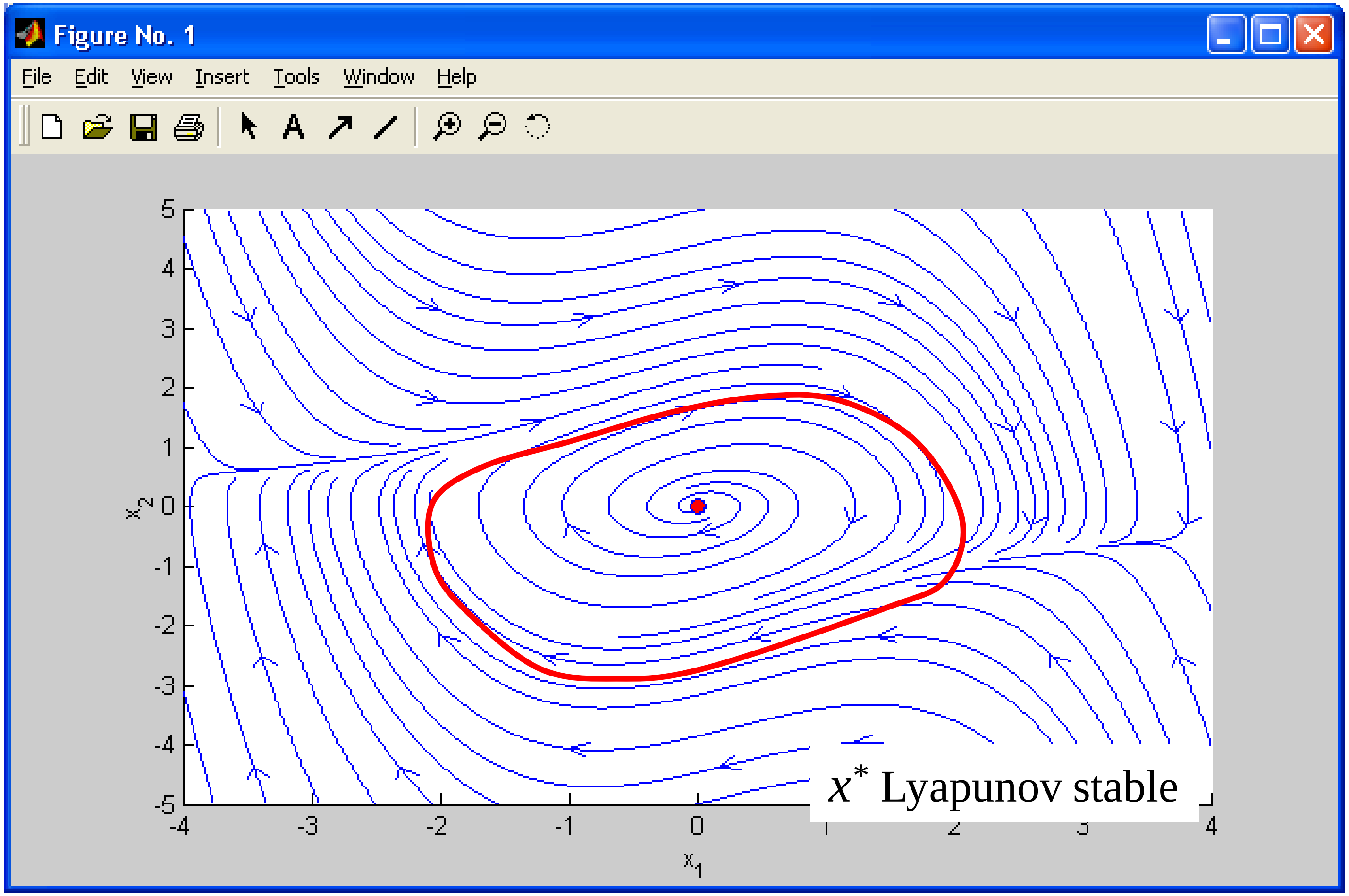
A solution $x^*:[0,T)\to\mathbb{R}^n$ is (***Lyapunov) stable*** if T is continuous at $x^*_0 := x^*(0)$, i.e.,

$$\forall \epsilon > 0 \; \exists \delta > 0 : \|x_0 - x^*_0\| \leq \delta \;\Rightarrow\; \underbrace{\|T(x_0) - T(x^*_0)\|_{\text{sig}}}_{\sup_{t\geq 0} \|x(t) - x^*(t)\| \leq \epsilon} \leq \epsilon$$



Example #2: Van der Pol oscillator

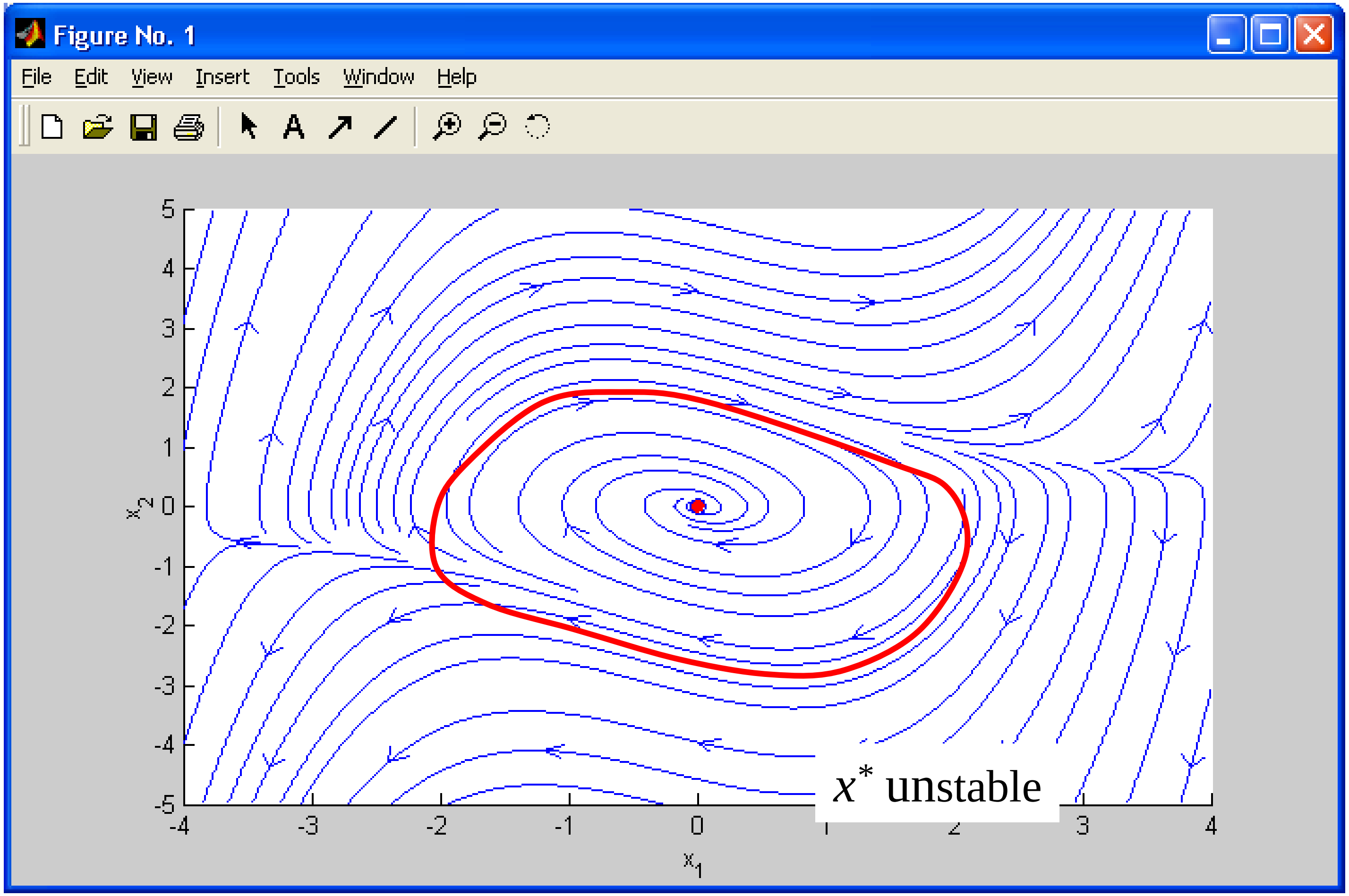
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + .5(1 - x_1^2)x_2$$



Stability of arbitrary solutions

E.g., Van der Pol oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - .5(1 - x_1^2)x_2$$



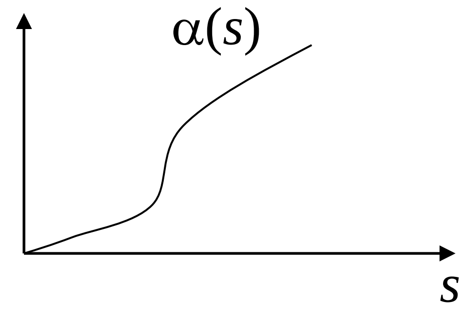
Lyapunov stability

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

class $\mathcal{K} \equiv$ set of functions $\alpha:[0,\infty)\to[0,\infty)$ that are

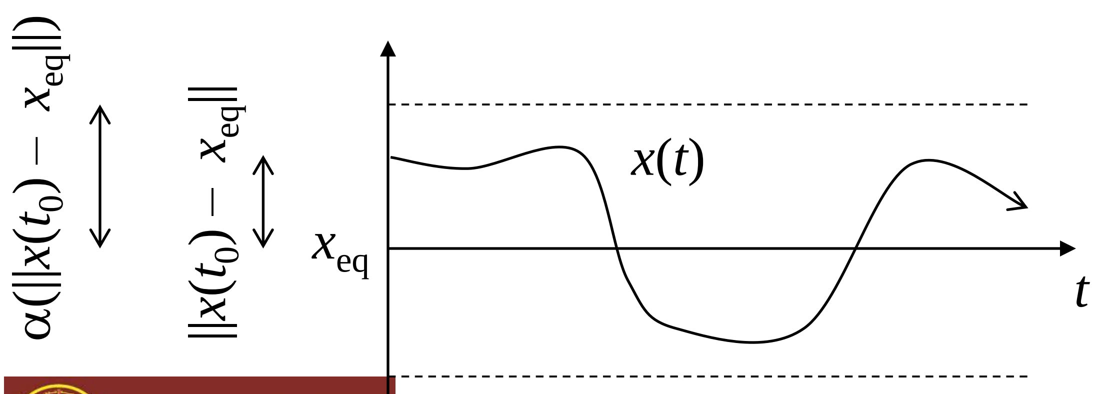
1. continuous
2. strictly increasing
3. $\alpha(0)=0$



Definition (class $\mathcal{K}$ function definition):

The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|)\ \forall\ t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$



Asymptotic stability

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

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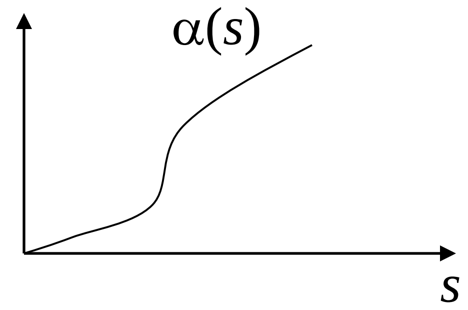
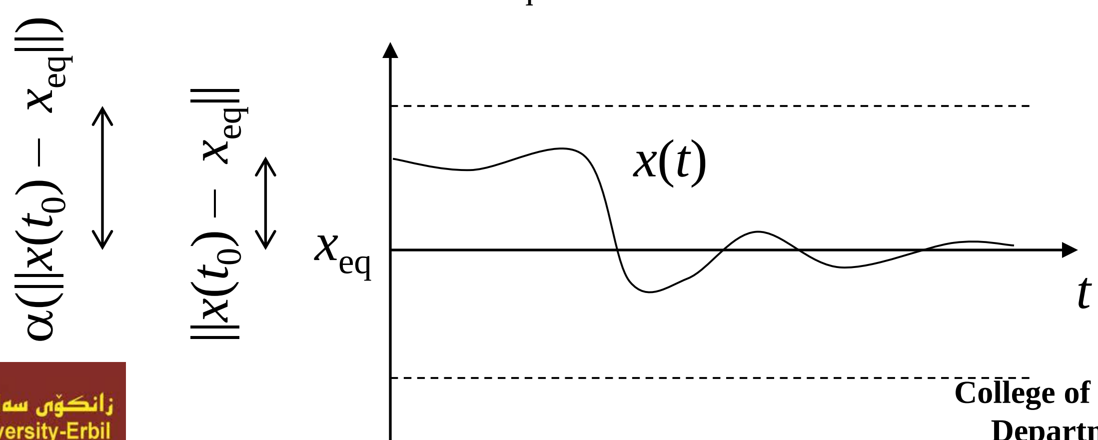


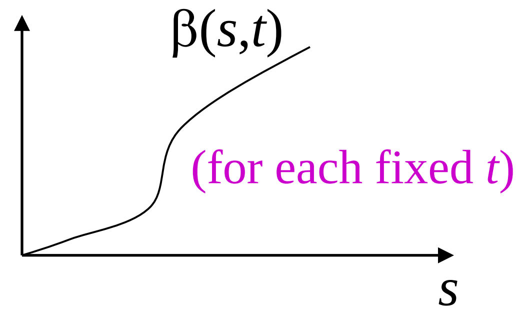
Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if
it is Lyapunov stable and for every initial state the solution exists on $[0,\infty)$ and

$$x(t) \to x_{eq} \text{ as } t\to\infty.$$



Asymptotic stability

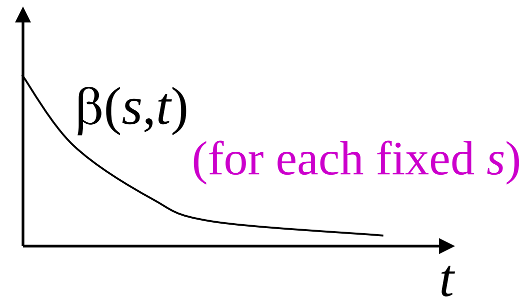
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$



equilibrium point ≡ $x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

class $\mathcal{KL}$ ≡ set of functions $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ s.t.

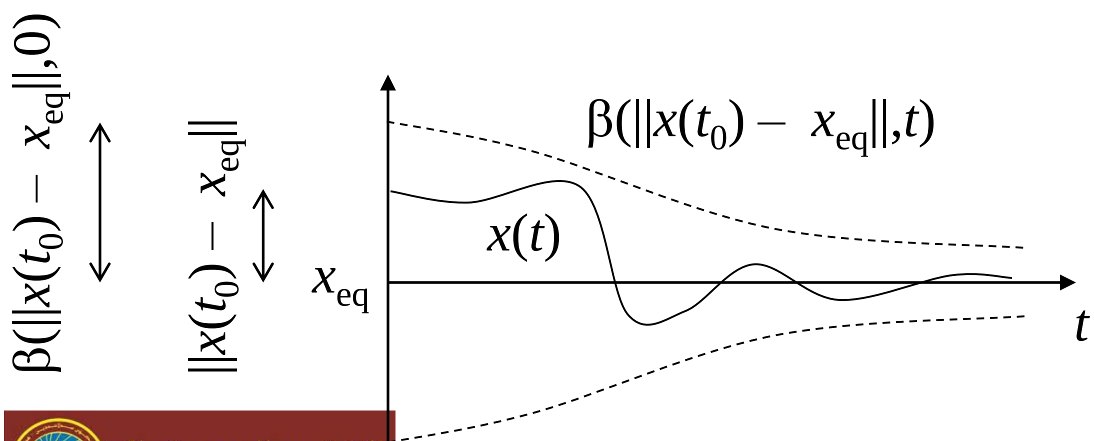
1. for each fixed t, $\beta(\cdot,t) \in \mathcal{K}$
2. for each fixed s, $\beta(s,\cdot)$ is monotone decreasing and $\beta(s,t) \to 0$ as $t\to\infty$



Definition (class $\mathcal{KL}$ function definition):

The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if $\exists\ \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \le \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\ t \ge t_0 \ge 0$$



We have ***exponential stability*** when

$$\beta(s,t) = c\, e^{-\lambda t}\, s$$

with $c,\lambda > 0$

Example #1: Pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

$k > 0$ (with friction)

$k = 0$ (no friction)

x_2

$x_{eq}=(0,0)$
asymptotically stable

$x_{eq}=(\pi,0)$
unstable

$x_{eq}=(0,0)$
stable but not asymptotically

$x_{eq}=(\pi,0)$
unstable

Example #3: Butterfly

Convergence by itself does not imply stability, e.g.,

$$\dot{x}_1 = x_1^2 - x_2^2$$
$$\dot{x}_2 = 2x_1 x_2$$

equilibrium point $\equiv$ (0,0)

all solutions converge to zero but x_{eq}= (0,0) system is not stable

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|)\ \ \forall\ t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

Suppose we could show that $\|x(t) - x_{eq}\|$ always decreases along solutions to the ODE. Then

$$\|x(t) - x_{eq}\| \leq \|x(t_0) - x_{eq}\|\ \ \forall\ t \geq t_0 \geq 0$$

we could pick $\alpha(s) = s$

We can draw the same conclusion by using other measures of how far the solution is from x_{eq}:

$V: \mathbb{R}^n \to \mathbb{R}$ positive definite $\equiv V(x) \geq 0\ \ \forall\ x \in \mathbb{R}^n$ with $= 0$ only for $x = 0$
$V: \mathbb{R}^n \to \mathbb{R}$ radially unbounded $\equiv x \to \infty \Rightarrow V(x) \to \infty$

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

V: $\mathbb{R}^n \to \mathbb{R}$ positive definite $\equiv V(x) \geq 0\ \ \forall\, x \in \mathbb{R}^n$ with $= 0$ only for $x = 0$

$$V(x - x_{\text{eq}}) \quad \begin{cases} = 0 & x = x_{\text{eq}} \\ > 0 & x \neq x_{\text{eq}} \end{cases}$$

Q: How to check if $V(x(t) - x_{eq})$ *decreases along solutions?*

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(x(t) - x_{\text{eq}}\big) = \frac{\partial V}{\partial x}\big(x(t) - x_{\text{eq}}\big)\dot{x}(t)$$

$$= \frac{\partial V}{\partial x}\big(x(t) - x_{\text{eq}}\big) f\big(x(t)\big)$$

A: $V(x(t) - x_{eq})$ *will decrease if*

$$\frac{\partial V}{\partial x}\big(z - x_{\text{eq}}\big) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Lyapunov's stability theorem

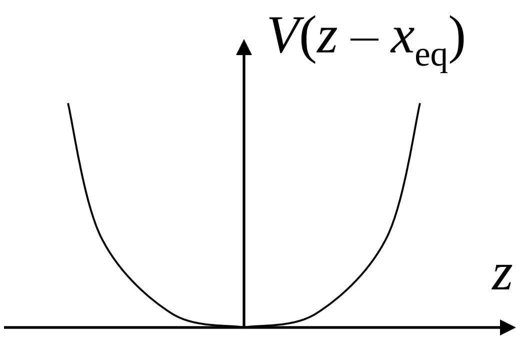
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|)\ \ \forall\ t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

Theorem (Lyapunov):
Suppose there exists a continuously differentiable, positive definite function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$



Then x_{eq} is a Lyapunov stable equilibrium.

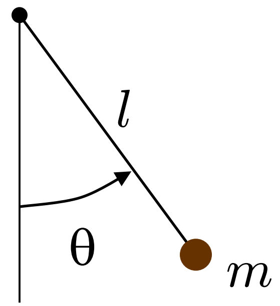
Example #1: Pendulum

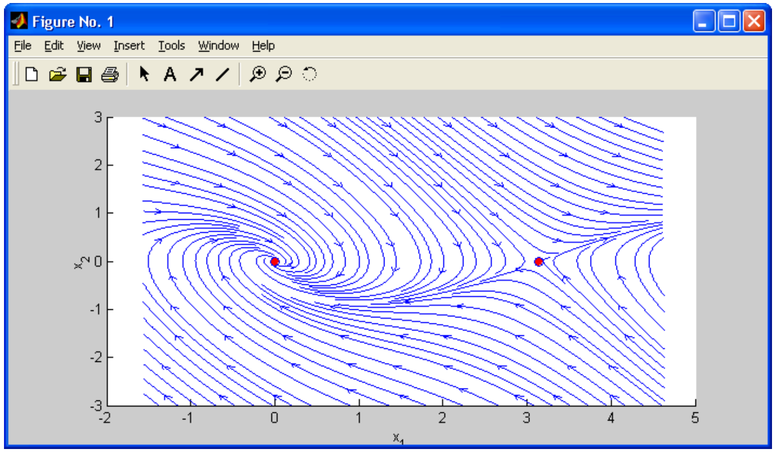
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

$k \equiv$ friction coefficient

$$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$$





$$V(x) := \frac{g}{l}(1-\cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (0,0)$

$$\frac{\partial V}{\partial x}(x - x_{eq})f(x) = \begin{bmatrix}\frac{g}{l}\sin x_1 & x_2\end{bmatrix}\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{bmatrix}$$

$$= -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$$

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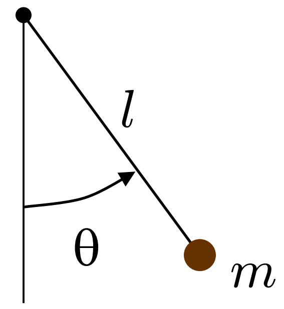

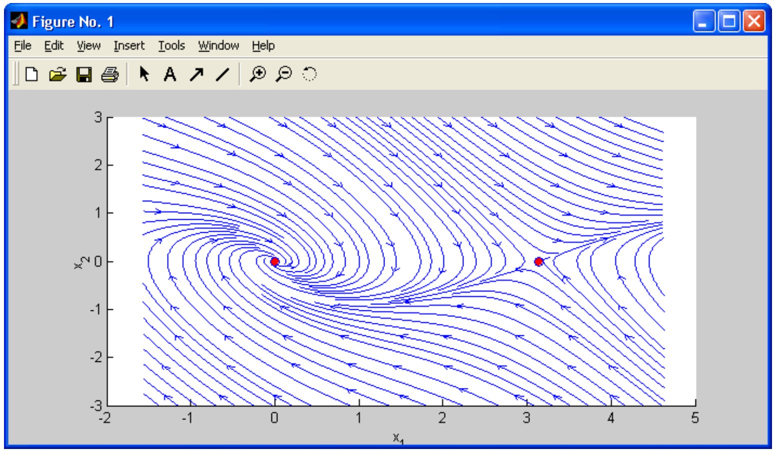

$$V(x) := \frac{g}{l}(1 - \cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (\pi, 0)$

$$\frac{\partial V}{\partial x}(x - x_{eq})f(x) = \begin{bmatrix} \frac{g}{l}\sin(x_1 - \pi) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

$$= -\frac{2g}{l}x_2 \sin x_1 - \frac{k}{m}x_2^2 \gtreqless 0$$

Salahaddin University-Erbil

College of Engineering
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Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \ \forall\ t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

Theorem (Lyapunov):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V\colon \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}})\, f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if = 0 only for $z = x_{eq}$ then x_{eq} is a (globally) asymptotically stable equilibrium.

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

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Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \rightarrow \mathbb{R}$ such that

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Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if = 0 only for $z = x_{eq}$ then x_{eq} is a (globally) asymptotically stable equilibrium.

What if

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) = 0 \qquad \forall z \in \mathbb{R}^n$$

Example #1: Pendulum

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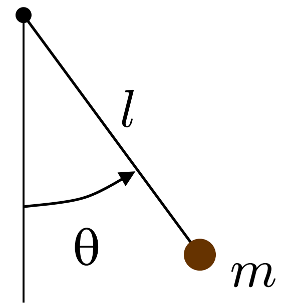

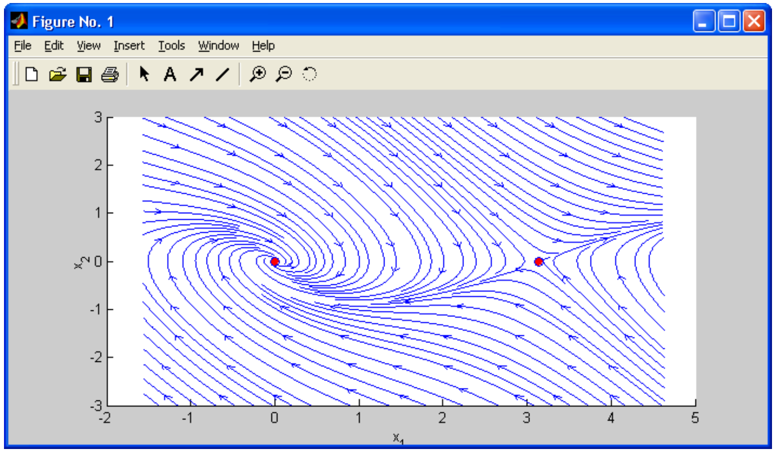

$$V(x) := \frac{g}{l}(1 - \cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (0,0)$

$$\frac{\partial V}{\partial x}(x - x_{eq})f(x) = \begin{bmatrix} \frac{g}{l}\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

$$= -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$$

LaSalle's Invariance Principle

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set $\equiv x(t_0) \in M \Rightarrow x(t) \in M \forall\ t \geq t_0$

(in the context of hybrid systems: Reach(M) $\subset M$…)

Theorem (LaSalle Invariance Principle):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\mathrm{eq}}) f(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{ z \in \mathbb{R}^n : W(z) = 0 \}$$

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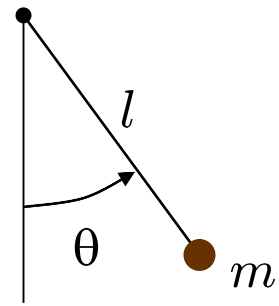
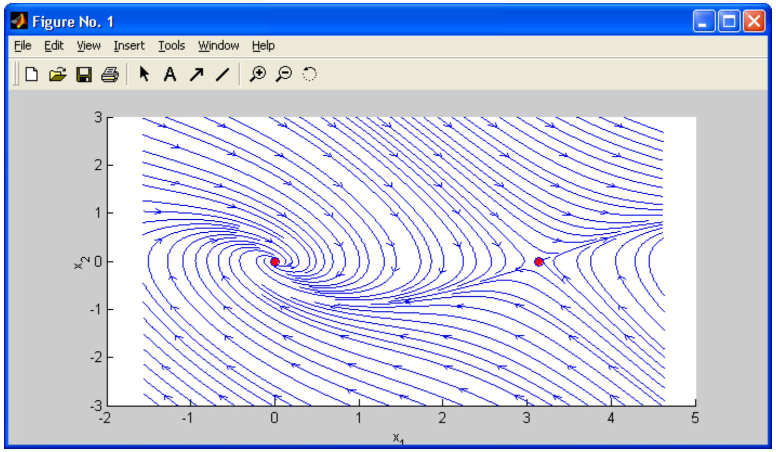
$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$





$$V(x) := \frac{g}{l}(1 - \cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (0,0)$ $\quad \frac{\partial V}{\partial x}(x - x_{\text{eq}})f(x) = -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$

$E := \{ (x_1,x_2): x_1 \in \mathbb{R} , x_2=0\}$

Inside *E*, the ODE becomes

$$\dot{x}_1 = x_2 = 0$$

$$0 = \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 = -\frac{g}{l}\sin x_1$$

} define set *M* for which system remains inside *E*

Linear systems

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is

1. Lyapunov stable if and only if all eigenvalues of A have negative or zero real parts and for each eigenvalue with zero real part there is an independent eigenvector.
2. Asymptotically stable if and only if all eigenvalues of A have negative real parts. In this case the origin is actually exponentially stable

Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix Q the equation

$$A' P + P A = -Q$$

Lyapunov equation

has a unique solutions P that is symmetric and positive definite

Recall: given a symmetric matrix P

P is positive definite $\equiv$ all eigenvalues are positive

P positive definite $\Rightarrow x' \mathrm{P}\, x > 0 \ \forall\, x \neq 0$

P is positive semi-definite $\equiv$ all eigenvalues are positive or zero

P positive semi-definite $\Rightarrow x' \mathrm{P}\, x \geq 0 \ \forall\, x$

Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix Q the equation

$$A' P + P A = -Q$$

has a unique solutions P that is symmetric and positive definite

Why?

1. P exists $\Rightarrow$ asymp. stable

   Consider the quadratic Lyapunov equation: $V(x) = x' P x$

   V is positive definite & radially unbounded because P is positive definite

   V is continuously differentiable: $\frac{\partial V}{\partial x}(x) = 2x'P$

$$\frac{\partial V}{\partial x}(x)Ax = x'(A'P + PA)x = -x'Qx < 0 \quad \forall x \neq 0$$

thus system is asymptotically stable by Lyapunov Theorem

Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!}A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix Q the equation

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Lyapunov equation

has a unique solutions P that is symmetric and positive definite

Why?

2. asympt. stable ⇒ P exists and is unique (constructive proof)

$$P := \lim_{T\to\infty} \int_0^T e^{A'\tau} Q e^{A\tau} d\tau$$

$$= \lim_{T\to\infty} \int_0^T e^{A'(T-s)} Q e^{A(T-s)} ds$$

A is asympt. stable ⇒ e^{At} decreases to zero exponentiall fast ⇒ P is well defined (limit exists and is finite)

change of integration variable $\tau = T - s$