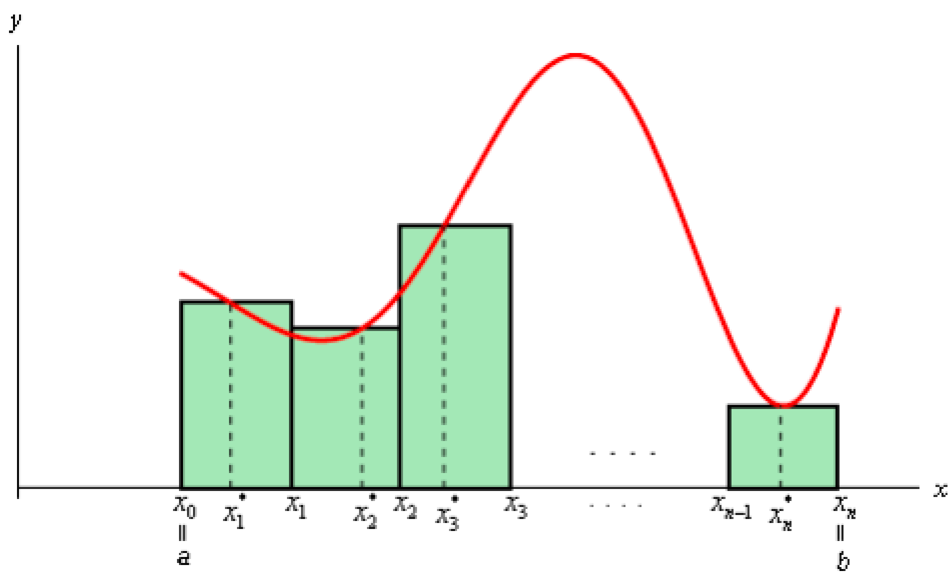
Double Integrals

Before starting on double integrals let's do a quick review of the definition of a definite integrals for functions of single variables. First, when working with the integral,

$$\int_a^b f(x)\,dx$$

we think of x's as coming from the interval $a \le x \le b$. For these integrals we can say that we are integrating over the interval $a \le x \le b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \le x \le a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval $a \le x \le b$ into n subintervals of width Δx and choose a point, x_i^*, from each interval as shown below,



Each of the rectangles has height of $f\left(x_i^*\right)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

In this section we want to integrate a function of two variables, $f(x,y)$. With functions of one variable we integrated over an interval (*i.e.* a one-dimensional space) and so it makes some sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (two-dimensional space).

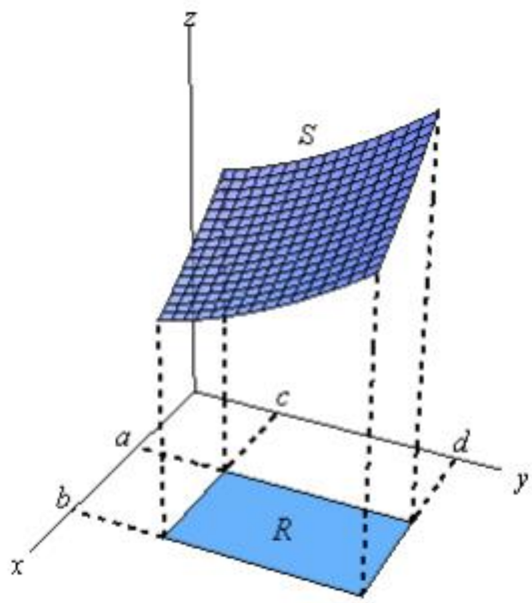
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

This means that the ranges for x and y are $a \le x \le b$ and $c \le y \le d$.

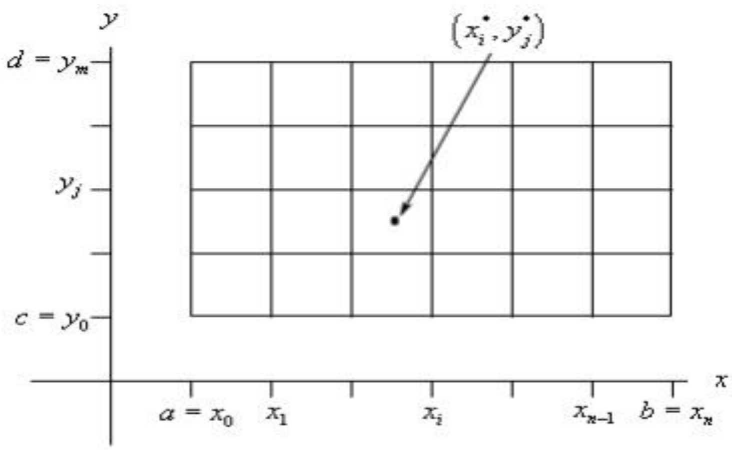
Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case.

Let's start out with the graph of the surface S give by graphing $f(x,y)$ over the rectangle R.

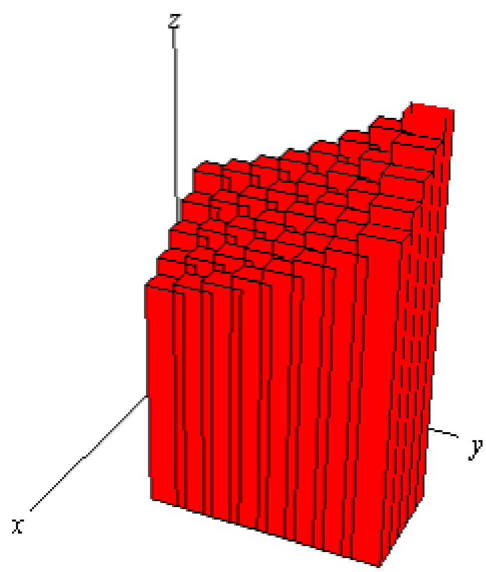


Now, just like with functions of one variable let's not worry about integrals quite yet. Let's first ask what the volume of the region under S (and above the xy-plane of course) is.

We will first approximate the volume much as we approximated the area above. We will first divide up $a \le x \le b$ into n subintervals and divide up $c \le y \le d$ into m subintervals. This will divide up R into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*). Here is a sketch of this set up.



Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.



Each of the rectangles has a base area of ΔA and a height of $f\left(x_i^*, y_j^*\right)$ so the volume of each of these boxes is $f\left(x_i^*, y_j^*\right)\Delta A$. The volume under the surface *S* is then approximately,

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

We will have a *double* sum since we will need to add up volumes in both the *x* and *y* directions.

To get a better estimation of the volume we will take *n* and *m* larger and larger and to get the exact volume we will need to take the limit as both *n* and *m* go to infinity. In other words,

$$V = \lim_{n,\, m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

Now, this should look familiar. This looks a lot like the definition of the integral of a function of single variable. In fact this is also the definition of a double integral, or more exactly an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region R as well as the notation that we'll use for it.

$$\iint\limits_R f(x,y)\, dA = \lim_{n,\, m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

Note the similarities and differences in the notation to single integrals. We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well. Note that the differential is dA instead of the dx and dy that we're used to seeing. Note as well that we don't have limits on the integrals in this notation. Instead we have the R written below the two integrals to denote the region that we are integrating over.

Note that one interpretation of the double integral of $f(x,y)$ over the rectangle R is the volume under the function $f(x,y)$ (and above the xy-plane). Or,

$$\text{Volume} = \iint\limits_R f(x,y)\, dA$$

We can use this double sum in the definition to estimate the value of a double integral if we need to. We can do this by choosing $\left(x_i^*, y_j^*\right)$ to be the midpoint of each rectangle. When we do this we usually denote the point as $\left(\overline{x}_i, \overline{y}_j\right)$. This leads to the **Midpoint Rule**,

$$\iint\limits_R f(x,y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$

In the next section we start looking at how to actually compute double integrals.

In the previous section we gave the definition of the double integral. However, just like with the definition of a single integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. We will continue to assume that we are integrating over the rectangle

$$R = [a,b] \times [c,d]$$

We will look at more general regions in the next section.

The following theorem tells us how to compute a double integral over a rectangle.

Fubini's Theorem

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

These integrals are called **iterated integrals**.

Note that there are in fact two ways of computing a double integral and also notice that the inner differential matches up with the limits on the inner integral and similarly for the out differential and limits. In other words, if the inner differential is dy then the limits on the inner integral must be y limits of integration and if the outer differential is dy then the limits on the outer integral must be y limits of integration.

Now, on some level this is just notation and doesn't really tell us how to compute the double integral. Let's just take the first possibility above and change the notation a little.

$$\iint\limits_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

We will compute the double integral by first computing

$$\int_c^d f(x,y)\,dy$$

and we compute this by holding x constant and integrating with respect to y as if this were an single integral. This will give a function involving only x's which we can in turn integrate.

We've done a similar process with partial derivatives. To take the derivative of a function with respect to y we treated the x's as constants and differentiated with respect to y as if it was a function of a single variable.

Double integrals work in the same manner. We think of all the x's as constants and integrate with respect to y or we think of all y's as constants and integrate with respect to x.

Let's take a look at some examples.

Example 1 Compute each of the following double integrals over the indicated rectangles.

(a) $\displaystyle\iint\limits_R 6xy^2\,dA$, $R=[2,4]\times[1,2]$ [Solution]

(b) $\displaystyle\iint\limits_R 2x-4y^3\,dA$, $R=[-5,4]\times[0,3]$ [Solution]

(c) $\displaystyle\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA$, $R=[-2,-1]\times[0,1]$ [Solution]

(d) $\displaystyle\iint\limits_R \frac{1}{(2x+3y)^2}\,dA$, $R=[0,1]\times[1,2]$ [Solution]

(e) $\displaystyle\iint\limits_R x\mathrm{e}^{xy}\,dA$, $R=[-1,2]\times[0,1]$ [Solution]

Solution 2

In this case we'll integrate with respect to x first and then y. Here is the work for this solution.

$$\begin{aligned}\iint\limits_R 6xy^2\,dA &= \int_1^2\int_2^4 6xy^2\,dx\,dy\\ &= \int_1^2 \left(3x^2y^2\right)\Big|_2^4\,dy\\ &= \int_1^2 36y^2\,dy\\ &= 12y^3\Big|_1^2\\ &= 84\end{aligned}$$

Sure enough the same answer as the first solution.

So, remember that we can do the integration in any order.

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(b) $\displaystyle\iint\limits_R 2x-4y^3\,dA\,,\quad R=[-5,4]\times[0,3]$

For this integral we'll integrate with respect to y first.

$$\begin{aligned}\iint\limits_R 2x-4y^3\,dA &= \int_{-5}^{4}\int_0^3 2x-4y^3\,dy\,dx\\ &= \int_{-5}^{4}\left(2xy-y^4\right)\Big|_0^3\,dx\\ &= \int_{-5}^{4}6x-81\,dx\\ &= \left(3x^2-81x\right)\Big|_{-5}^{4}\\ &= -756\end{aligned}$$

Remember that when integrating with respect to y all x's are treated as constants and so as far as the inner integral is concerned the $2x$ is a constant and we know that when we integrate constants with respect to y we just tack on a y and so we get $2xy$ from the first term.

(c) $\displaystyle\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA\,,\quad R=[-2,-1]\times[0,1]$

In this case we'll integrate with respect to x first.

$$\begin{aligned}\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA &= \int_0^1\int_{-2}^{-1} x^2y^2+\cos(\pi x)+\sin(\pi y)\,dx\,dy\\ &= \int_0^1\left(\frac{1}{3}x^3y^2+\frac{1}{\pi}\sin(\pi x)+x\sin(\pi y)\right)\Big|_{-2}^{-1}\,dy\\ &= \int_0^1\frac{7}{3}y^2+\sin(\pi y)\,dy\\ &= \frac{7}{9}y^3-\frac{1}{\pi}\cos(\pi y)\Big|_0^1\\ &= \frac{7}{9}+\frac{2}{\pi}\end{aligned}$$

(d) $\displaystyle\iint\limits_R \frac{1}{(2x+3y)^2}\,dA$, $R=[0,1]\times[1,2]$

In this case because the limits for x are kind of nice (*i.e.* they are zero and one which are often nice for evaluation) let's integrate with respect to x first. We'll also rewrite the integrand to help with the first integration.

$$\begin{aligned}
\iint\limits_R (2x+3y)^{-2}\,dA &= \int_1^2\int_0^1 (2x+3y)^{-2}\,dx\,dy \\
&= \int_1^2 \left(-\frac{1}{2}(2x+3y)^{-1}\right)\Bigg|_0^1 dy \\
&= -\frac{1}{2}\int_1^2 \frac{1}{2+3y}-\frac{1}{3y}\,dy \\
&= -\frac{1}{2}\left(\frac{1}{3}\ln|2+3y|-\frac{1}{3}\ln|y|\right)\Bigg|_1^2 \\
&= -\frac{1}{6}(\ln 8-\ln 2-\ln 5)
\end{aligned}$$

(e) $\iint\limits_R x\mathbf{e}^{xy}\,dA$, $R = [-1,2]\times[0,1]$

Now, while we can technically integrate with respect to either variable first sometimes one way is significantly easier than the other way. In this case it will be significantly easier to integrate with respect to y first as we will see.

$$\iint\limits_R x\mathbf{e}^{xy}\,dA = \int_{-1}^{2}\int_0^1 x\mathbf{e}^{xy}\,dy\,dx$$

The y integration can be done with the quick substitution,

$$u = xy \qquad\qquad du = x\,dy$$

which gives

$$\begin{aligned}\iint\limits_R x\mathbf{e}^{xy}\,dA &= \int_{-1}^{2}\mathbf{e}^{xy}\Big|_0^1\,dx\\ &= \int_{-1}^{2}\mathbf{e}^{x}-1\,dx\\ &= \left(\mathbf{e}^{x}-x\right)\Big|_{-1}^{2}\\ &= \mathbf{e}^{2}-2-\left(\mathbf{e}^{-1}+1\right)\\ &= \mathbf{e}^{2}-\mathbf{e}^{-1}-3\end{aligned}$$

So, not too bad of an integral there provided you get the substitution. Now let's see what would happen if we had integrated with respect to x first.

$$\iint\limits_R x\mathbf{e}^{xy}\,dA = \int_0^1\int_{-1}^{2} x\mathbf{e}^{xy}\,dx\,dy$$

In order to do this we would have to use integration by parts as follows,

$$u = x \qquad\qquad dv = \mathbf{e}^{xy}\,dx$$

$$du = dx \qquad\qquad v = \frac{1}{y}\mathbf{e}^{xy}$$

The integral is then,

$$\begin{aligned}\iint\limits_R x\mathbf{e}^{xy}\,dA &= \int_0^1\left(\frac{x}{y}\mathbf{e}^{xy}-\int\frac{1}{y}\mathbf{e}^{xy}\,dx\right)\Bigg|_{-1}^{2}\,dy\\ &= \int_0^1\left(\frac{x}{y}\mathbf{e}^{xy}-\frac{1}{y^2}\mathbf{e}^{xy}\right)\Bigg|_{-1}^{2}\,dy\\ &= \int_0^1\left(\frac{2}{y}\mathbf{e}^{2y}-\frac{1}{y^2}\mathbf{e}^{2y}\right)-\left(-\frac{1}{y}\mathbf{e}^{-y}-\frac{1}{y^2}\mathbf{e}^{-y}\right)dy\end{aligned}$$

We're not even going to continue here as these are very difficult integrals to do.

Fact

If $f(x,y) = g(x)h(y)$ and we are integrating over the rectangle $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

So, if we can break up the function into a function only of x times a function of y then we can do the two integrals individually and multiply them together.

Let's do a quick example using this integral.

Example 2 Evaluate $\iint\limits_R x\cos^2(y)\,dA$, $R = [-2,3] \times \left[0, \frac{\pi}{2}\right]$.

Solution

Since the integrand is a function of x times a function of y we can use the fact.

$$\begin{aligned}\iint\limits_R x\cos^2(y)\,dA &= \left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos^2(y)\,dy\right)\\ &= \left(\frac{1}{2}x^2\right)\Bigg|_{-2}^{3}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} 1+\cos(2y)\,dy\right)\\ &= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y+\frac{1}{2}\sin(2y)\right)\Bigg|_0^{\frac{\pi}{2}}\right)\\ &= \frac{5\pi}{8}\end{aligned}$$

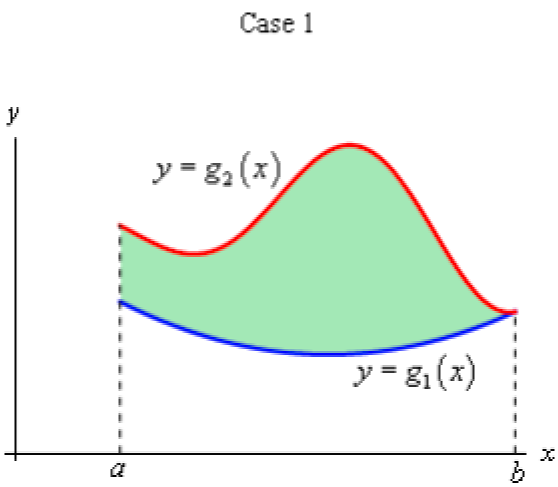
Double Integrals Over General Regions

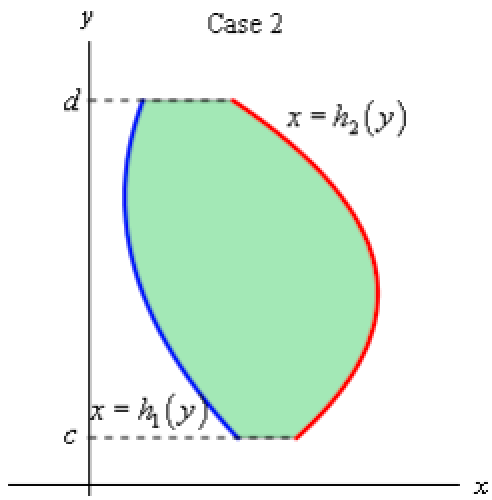
In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint_D f(x,y)\,dA$$

where D is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.





We will often use *set builder notation* to describe these regions. Here is the definition for the region in Case 1

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

and here is the definition for the region in Case 2.

$$D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$$

This notation is really just a fancy way of saying we are going to use all the points, (x,y), in which both of the coordinates satisfy the two given inequalities.

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ the integral is defined to be,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

In Case 2 where $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$ the integral is defined to be,

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Here are some properties of the double integral that we should go over before we actually do some examples. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Properties

1. $\iint\limits_D f(x,y)+g(x,y)\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$

2. $\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$, where c is any constant.

3. If the region D can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA$$

Let's take a look at some examples of double integrals over general regions.

Example 1 Evaluate each of the following integrals over the given region D.

(a) $\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D=\left\{(x,y)\,|\,1\le y\le 2,\ y\le x\le y^3\right\}$ [Solution]

(b) $\iint\limits_D 4xy-y^3\,dA$, D is the region bounded by $y=\sqrt{x}$ and $y=x^3$. [Solution]

(c) $\iint\limits_D 6x^2-40y\,dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$. [Solution]

Solution

(a) $\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D=\left\{(x,y)\,|\,1\le y\le 2,\ y\le x\le y^3\right\}$

Okay, this first one is set up to just use the formula above so let's do that.

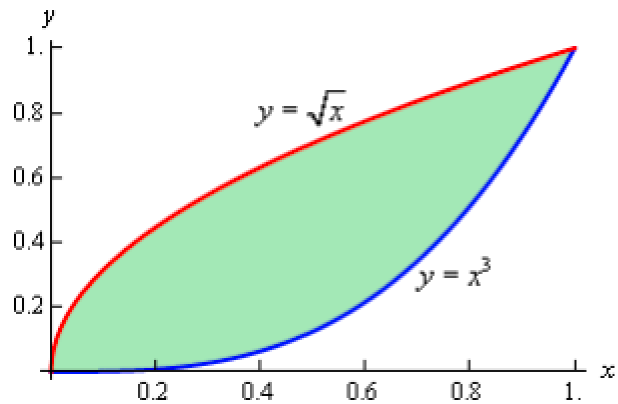
$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA = \int_1^2\int_y^{y^3} \mathbf{e}^{\frac{x}{y}}\,dx\,dy = \int_1^2 \left. y\mathbf{e}^{\frac{x}{y}}\right|_y^{y^3} dy$$

$$= \int_1^2 y\mathbf{e}^{y^2} - y\mathbf{e}^1\,dy$$

$$= \left.\left(\frac{1}{2}\mathbf{e}^{y^2} - \frac{1}{2}y^2\mathbf{e}^1\right)\right|_1^2 = \frac{1}{2}\mathbf{e}^4 - 2\mathbf{e}^1$$

(b) $\iint\limits_D 4xy - y^3\,dA$, ***D*** **is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$**.**

In this case we need to determine the two inequalities for *x* and *y* that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that that two inequalities are,

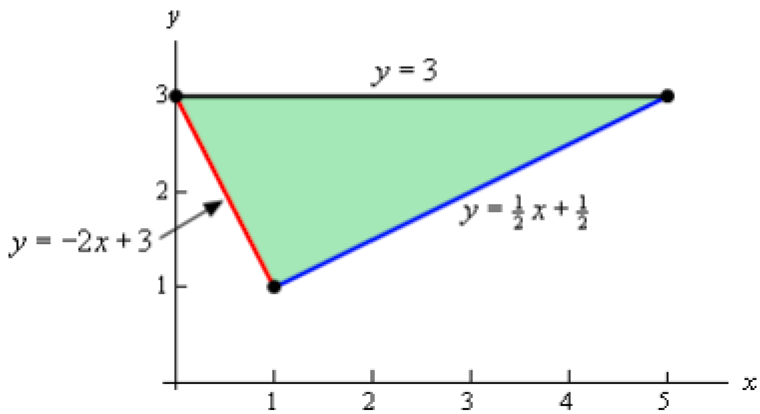
$$0 \le x \le 1 \qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral,

$$\begin{aligned}\iint\limits_D 4xy - y^3\,dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\,dy\,dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4\right)\Bigg|_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12}\,dx \\ &= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13}\right)\Bigg|_0^1 = \frac{55}{156}\end{aligned}$$

(c) $\iint\limits_D 6x^2 - 40y\,dA$, ***D*** **is the triangle with vertices** $(0,3)$, $(1,1)$, **and** $(5,3)$**.**

We got even less information about the region this time. Let's start this off by sketching the triangle.



Since we have two points on each edge it is easy to get the equations for each edge and so we'll leave it to you to verify the equations.

Now, there are two ways to describe this region. If we use functions of x, as shown in the image we will have to break the region up into two different pieces since the lower function is different depending upon the value of x. In this case the region would be given by $D = D_1 \cup D_2$ where,

$$D_1 = \{(x,y) \mid 0 \le x \le 1,\ -2x+3 \le y \le 3\}$$

$$D_2 = \left\{(x,y) \mid 1 \le x \le 5,\ \frac{1}{2}x + \frac{1}{2} \le y \le 3\right\}$$

Note the $\cup$ is the "union" symbol and just means that D is the region we get by combing the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions.

To avoid this we could turn things around and solve the two equations for x to get,

$$y = -2x + 3 \qquad \Rightarrow \qquad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad \Rightarrow \qquad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \left\{(x,y) \mid -\frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1,\ 1 \le y \le 3\right\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Either way should give the same answer and so we can get an example in the notes of splitting a region up let's do both integrals.

Solution 1

$$\iint_D 6x^2 - 40y\,dA = \iint_{D_1} 6x^2 - 40y\,dA + \iint_{D_2} 6x^2 - 40y\,dA$$

$$= \int_0^1 \int_{-2x+3}^3 6x^2 - 40y\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 6x^2 - 40y\,dy\,dx$$

$$= \int_0^1 \left(6x^2 y - 20y^2\right)\Big|_{-2x+3}^3 dx + \int_1^5 \left(6x^2 y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^3 dx$$

$$= \int_0^1 12x^3 - 180 + 20(3-2x)^2\,dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^2 dx$$

$$= \left(3x^4 - 180x - \tfrac{10}{3}(3-2x)^3\right)\Big|_0^1 + \left(-\tfrac{3}{4}x^4 + 5x^3 - 180x + \tfrac{40}{3}\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^3\right)\Big|_1^5$$

$$= -\frac{935}{3}$$

That was a lot of work. Notice however, that after we did the first substitution that we didn't multiply everything out. The two quadratic terms can be easily integrated with a basic Calc I substitution and so we didn't bother to multiply them out. We'll do that on occasion to make some of these integrals a little easier.

Solution 2

This solution will be a lot less work since we are only going to do a single integral.

$$\iint_D 6x^2 - 40y\,dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} 6x^2 - 40y\,dx\,dy$$

$$= \int_1^3 \left(2x^3 - 40xy\right)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 100y - 100y^2 + 2(2y-1)^3 - 2\left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^3 dy$$

$$= \left(50y^2 - \tfrac{100}{3}y^3 + \tfrac{1}{4}(2y-1)^4 + \left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^4\right)\Big|_1^3$$

$$= -\frac{935}{3}$$

So, the numbers were a little messier, but other than that there was much less work for the same result. Also notice that again we didn't cube out the two terms as they are easier to deal with using a Calc I substitution.

Example 2 Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$ [Solution]

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$ [Solution]

Solution

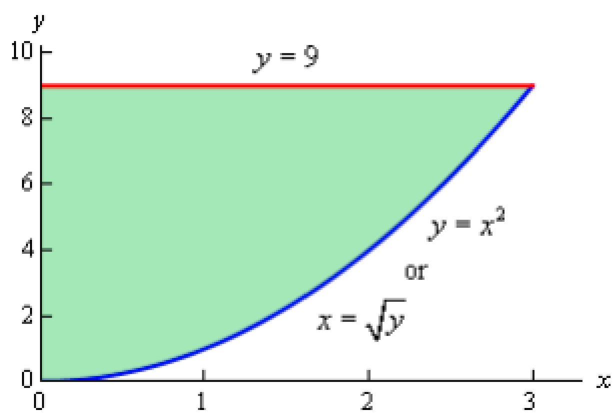
(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

First, notice that if we try to integrate with respect to y we can't do the integral because we would need a y^2 in front of the exponential in order to do the y integration. We are going to hope that if we reverse the order of integration we will get an integral that we can do.

So, let's see how we reverse the order of integration. The best way to reverse the order of integration is to first sketch the region given by the original limits of integration. From the integral we see that the inequalities that define this region are,

$$0 \le x \le 3$$
$$x^2 \le y \le 9$$

These inequalities tell us that we want the region with $y = x^2$ on the lower boundary and $y = 9$ on the upper boundary that lies between $x = 0$ and $x = 3$. Here is a sketch of that region.



Since we want to integrate with respect to x first we will need to determine limits of x (probably in terms of y) and then get the limits on the y's. Here they are for this region.

$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 9$$

Any horizontal line drawn in this region will start at $x = 0$ and end at $x = \sqrt{y}$ and so these are the limits on the x's and the range of y's for the regions is 0 to 9.

The integral, with the order reversed, is now,

$$\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy$$

and notice that we can do the first integration with this order. We'll also hope that this will give us a second integral that we can do. Here is the work for this integral.

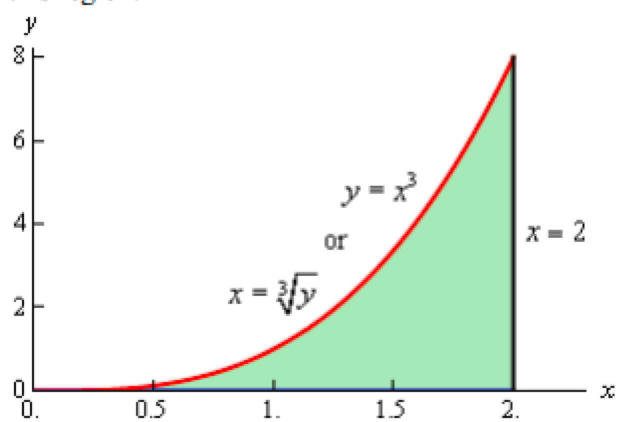
$$\begin{aligned}
\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx &= \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy \\
&= \int_0^9 \frac{1}{4} x^4 \mathbf{e}^{y^3} \Big|_0^{\sqrt{y}}\, dy \\
&= \int_0^9 \frac{1}{4} y^2 \mathbf{e}^{y^3}\, dy \\
&= \frac{1}{12} \mathbf{e}^{y^3} \Big|_0^9 \\
&= \frac{1}{12}\left(\mathbf{e}^{729} - 1\right)
\end{aligned}$$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dx\, dy$

As with the first integral we cannot do this integral by integrating with respect to x first so we'll hope that by reversing the order of integration we will get something that we can integrate. Here are the limits for the variables that we get from this integral.

$$\sqrt[3]{y} \le x \le 2$$
$$0 \le y \le 8$$

and here is a sketch of this region.



So, if we reverse the order of integration we get the following limits.

$$0 \le x \le 2$$
$$0 \le y \le x^3$$

The integral is then,

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy = \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\, dy\, dx$$
$$= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3} dx$$
$$= \int_0^2 x^3\sqrt{x^4+1}\, dx = \frac{1}{6}\left(17^{\frac{3}{2}} - 1\right)$$

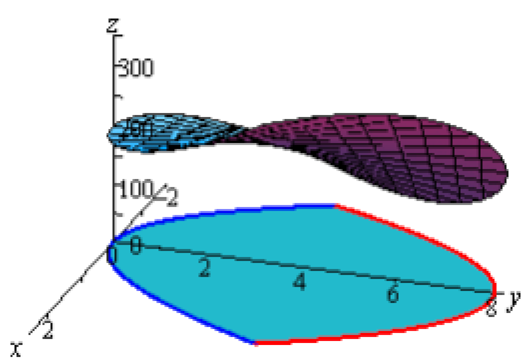
The volume of the solid that lies below the surface given by $z = f(x, y)$ and above the region D in the xy-plane is given by,

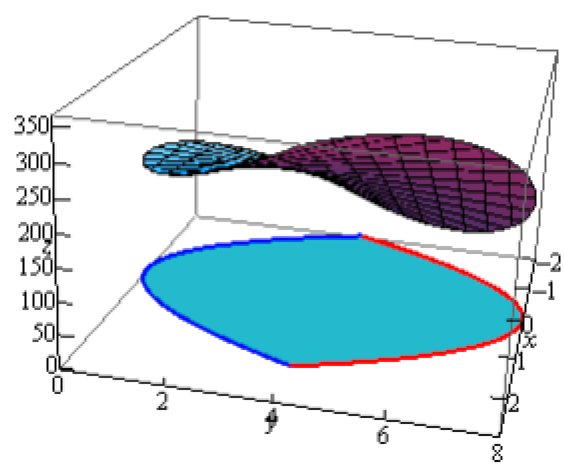
$$V = \iint_D f(x,y)\, dA$$

Example 3 Find the volume of the solid that lies below the surface given by $z = 16xy + 200$ and lies above the region in the xy-plane bounded by $y = x^2$ and $y = 8 - x^2$.

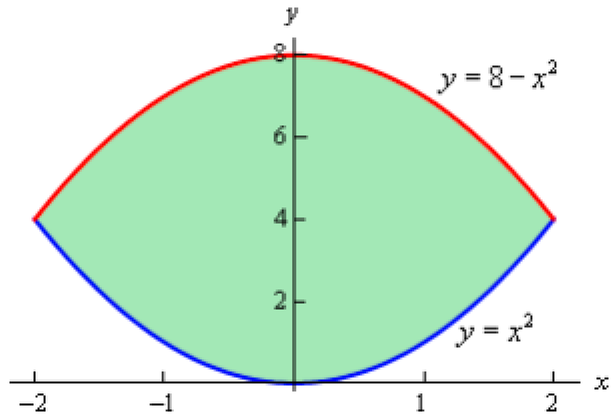
Solution

Here is the graph of the surface and we've tried to show the region in the xy-plane below the surface.





Here is a sketch of the region in the xy-plane by itself.



By setting the two bounding equations equal we can see that they will intersect at $x = 2$ and $x = -2$. So, the inequalities that will define the region D in the xy-plane are,

$$-2 \le x \le 2$$
$$x^2 \le y \le 8 - x^2$$

The volume is then given by,

$$\begin{aligned} V &= \iint_D 16xy + 200\, dA \\ &= \int_{-2}^{2} \int_{x^2}^{8-x^2} 16xy + 200\, dy\, dx \\ &= \int_{-2}^{2} \left(8xy^2 + 200y\right)\Big|_{x^2}^{8-x^2} dx \\ &= \int_{-2}^{2} -128x^3 - 400x^2 + 512x + 1600\, dx \\ &= \left(-32x^4 - \frac{400}{3}x^3 + 256x^2 + 1600x\right)\Big|_{-2}^{2} = \frac{12800}{3} \end{aligned}$$

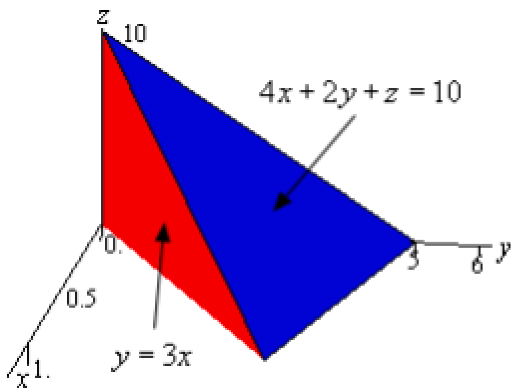
Example 4 Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, $x = 0$.

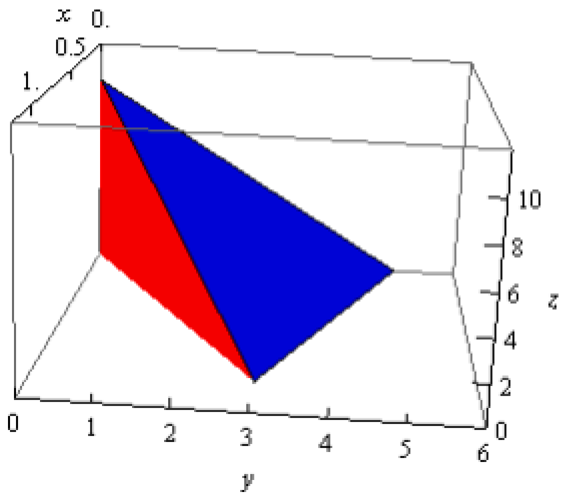
Solution This example is a little different from the previous one. Here the region D is not explicitly given so we're going to have to find it. First, notice that the last two planes are really telling us that we won't go past the xy-plane and the yz-plane when we reach them.

The first plane, $4x + 2y + z = 10$, is the top of the volume and so we are really looking for the volume under,

$$z = 10 - 4x - 2y$$

and above the region D in the xy-plane. The second plane, $y = 3x$ (yes that is a plane), gives one of the sides of the volume as shown below.

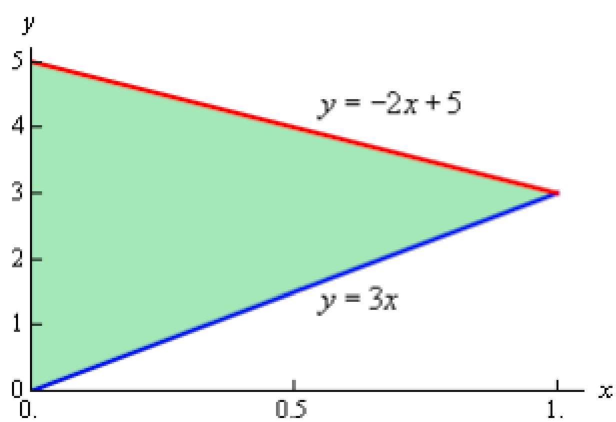




The region D will be the region in the xy-plane (*i.e.* $z = 0$) that is bounded by $y = 3x$, $x = 0$, and the line where $z + 4x + 2y = 10$ intersects the xy-plane. We can determine where $z + 4x + 2y = 10$ intersects the xy-plane by plugging $z = 0$ into it.

$$0 + 4x + 2y = 10 \qquad \Rightarrow \qquad 2x + y = 5 \qquad \Rightarrow \qquad y = -2x + 5$$

So, here is a sketch the region D.



The region D is really where this solid will sit on the xy-plane and here are the inequalities that define the region.

$$0 \le x \le 1$$
$$3x \le y \le -2x + 5$$

Here is the volume of this solid.