Methods of solving differential equations of the first order and first degree

I *Separation of variables.*

If it is possible to re-arrange the terms of the first order and first degree differential equation in two groups, each containing only one variable, the variables are said to be separable.

When variables are separated, the differentail equation takes the form $f(x)\,dx + g(y)\,dy = 0$ in which $f(x)$ is a function of x only and $g(y)$ is a function of y only.

Then the general solution is

$$\int f(x)\,dx + \int g(y)\,dy = c \qquad (c \text{ is a constant of integration})$$

For example, consider $x\dfrac{dy}{dx} - y = 0$

$$x\frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{x} \qquad \text{(separating the variables)}$$

$\Rightarrow \quad \displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} + k$ where k is a constant of integration.

$\Rightarrow \quad \log y = \log x + k.$

The value of k varies from $-\infty$ to ∞.

This general solution can be expressed in a more convenient form by assuming the constant of integration to be $\log c$. This is possible because $\log c$ also can take all values between $-\infty$ and ∞ as k does. By this assumption, the general solution takes the form

$$\log y - \log x = \log c \ \Rightarrow \ \log\left(\frac{y}{x}\right) = \log c$$

i.e. $\dfrac{y}{x} = c \quad \Rightarrow \quad y = cx$

Example: Solve $\dfrac{dy}{dx} = 1 + e^{2x}$

Solution. $dy = (1 + e^{2x})dx$

Integrate both sides we get

$y = x + \frac{1}{2}e^{2x} + C$

Example:Solve $\frac{dy}{dx} = \sin x$

Solution. $dy = \sin x \, dx$

Integrate both sides we get

$y = -\cos x + C$ is a general solution of the given differential equation.

Example Solve $\frac{dy}{dx} = -\frac{x}{y}$ subject to $y(4) = 3$

Solution. $ydy = -xdx$

$\Rightarrow \int ydy = \int -xdx$

$x^2 + y^2 = C$ is general solution where $C = 2C_1$.

For $x = 4$ &$y = 3$

$4^2 + 3^2 = C \Rightarrow C = 25$

$\therefore x^2 + y^2 = 25$

Example :Solve $(1 + x^2)\, dy - xy\, dx = 0$

Solution. Dividing by $y(1 + x^2)$ and transposing we get

$$\frac{dy}{y} = \frac{x\, dx}{1+x^2}$$

Integrating both sides, we get $\ln y = \frac{1}{2}\ln(1 + x^2) + \ln C$

Or

$$\ln y = \ln C\, (1 + x^2)^{\frac{1}{2}}$$

Taking exponentials $y = C(1 + x^2)^{\frac{1}{2}}$

The arbitrary constant was added in the form “ln C” to facilitate the final representation.

Example :Solve $\frac{dy}{dx} = 2x(1 + y^2)e^{x^2}$

Solution. $\frac{1}{1+y^2}\, dy = 2xe^{x^2} dx$

$$\int \frac{1}{1+y^2}\, dy = \int 2xe^{x^2} dx \qquad u = x^2 \;\&\; du = 2xdx$$

$$\int \frac{1}{1+y^2}\, dy = \int e^u du$$

$$\tan^{-1} y = e^u + C$$

$$\tan^{-1} y = e^{x^2} + C$$

$y = \tan(e^{x^2} + C)$ is a general solution of diff. eq.

Example: Solve $(x+1)ydx + (x-1)(y+1)dy = 0$

Solution. Divide the differential equation by $(x-1)y$

$\frac{x+1}{x-1}\, dx + \frac{y+1}{y} dy = 0$ is separable diff. eq.

$$\int \frac{x+1}{x-1}\, dx + \int \frac{y+1}{y} dy = 0$$

$$\int \frac{x+1+1-1}{x-1}\, dx + \int \frac{y+1}{y} dy = 0$$

$$\int (1 + \frac{2}{x-1})\, dx + \int (1 + \frac{1}{y}) dy = 0$$

$$x + 2\ln(x-1) + y + \ln y = C$$

$$x + \ln(x-1)^2 + y + \ln y = C$$

$x + y + \ln(x-1)^2 y = C$ is a general solution.

Example :Solve $(xy + y)dx + (y^2x - y^2 - x + 1)dy = 0$

Solution. $(x+1)ydx + y^2(x-1) - (x-1)dy = 0$

$$(x+1)ydx + (y^2-1)(x-1)dy = 0$$

Divide the differential equation by $(x-1)y$

$\frac{x+1}{x-1}\, dx + \frac{y^2-1}{y} dy = 0$ is separable diff. eq.

$$\int \frac{x+1}{x-1}\, dx + \int \frac{y^2-1}{y} dy = 0$$

$$\int \frac{x+1+1-1}{x-1}\, dx + \int (y - \frac{1}{y}) dy = 0$$

$$\int (1 + \frac{2}{x-1})\, dx + \int (y - \frac{1}{y}) dy = 0$$

$$x + 2\ln(x-1) + \frac{1}{2}y^2 - \ln y = C$$

$$x + \ln(x-1)^2 + \frac{1}{2}y^2 - \ln y = C$$

$$x + \frac{1}{2}y^2 + \ln\frac{(x-1)^2}{y} = C$$ is a general solution.

Exercises:

1) $\frac{dy}{dx} = \sin 5x$ $\quad$ *Ans.* $y = -\frac{1}{5}\cos 5x + C$

2) $dx + e^{3x}dy = 0$ $\quad$ *Ans.* $y = \frac{1}{3}e^{-3x} + C$

3) $(1+x)\frac{dy}{dx} = x + 6$ $\quad$ *Ans.* $y = x + 5\ \ln|x+1| + C$

4) $xy' = 4y$ $\quad$ *Ans.* $y = Cx^4$

5) $\frac{dy}{dx} = \frac{y^3}{x^2}$ $\quad$ *Ans.* $y^{-2} = 2x^{-1} + C$

6) $\frac{dx}{dy} = \frac{x^2\ \ y^2}{1+x}$ $\quad$ *Ans.* $-3 + 3x\ln x = xy^3 + 3xC$

7) $\frac{dy}{dx} = e^{(3x+2y)}$ $\quad$ *Ans.* $-3e^{-2y} = 2e^{3x} + C$ *where* $C = 6C1$

8) $(4y + y\,x^2)dy - (2x + xy^2)dx = 0$ $\quad$ *Ans.* $2 + y^2 = C(4 + x^2)$ *where* $C = e^{2C1}$

9) $2y(x+1)dy = xdx$ $\quad$ *Ans.* $y^2 = x - \ln|x+1| + C$

10) $y\ \ln x\ \frac{dx}{dy} = (\frac{y+1}{x})^2$ $\quad$ *Ans.* $\frac{x^3}{3}\ln x - \frac{x^3}{9} = \frac{y^2}{2} + 2y + \ln|y| + C$

11) $sec^2x\ dy + cscy\ \ dx = 0$ $\quad$ *Ans.* $4\cos y = 2x + \sin 2x + 4C$

12) $e^y\ \sin 2x\,dx + \cos x(\ e^{2y} - y)\,dy = 0$ $\quad$ *Ans.* $-2\ \cos x + e^y - ye^{-y} - e^{-y} = C$

13) $(e^y + 1)^2e^{-y}dx + (e^x + 1)^3e^{-x}dy = 0$ $\quad$ *Ans.* $(e^x + 1)^{-2} + 2\ \ (e^y + 1)^{-1} = C$

Definition: Homogenous equations:

A function is called *homogeneous of degree* n if $f(tx, ty) = t^n f(x, y)$ for all x, y, t.

Example: $f(x, y) = x - 3\sqrt{xy} + 5y$

Solution: $f(tx, ty) = tx - 3\sqrt{(tx)(ty)} + 5ty$

$$= tx - 3t\sqrt{xy} + 5ty$$

$$= t(x - 3\sqrt{xy} + 5y)$$

$$f(tx, ty) = tf(x, y)$$

Thus the given function is homogeneous With degree one.

Example:$f(x, y) = \sqrt{\ x^3 + y^3}$

Solution: $f(tx, ty) = \sqrt{\ t^3x^3 + t^3y^3}$

$$= \ t^{3/2}\sqrt{\ x^3 + y^3}$$

$$= t^{3/2}f(x, y)$$

Thus $f(x, y)$is homogeneous With degree $3/2$.

Example:$f(x, y) = x^2 + y^2 + 1$

Solution: $f(tx, ty) = (tx)^2 + (ty)^2 + 1$

$$= t^2x^2 + t^2y^2 + 1 \neq t^n f(x, y)$$

$f(x, y)$is not homogeneous

Example: $f(x, y) = \ln x^2 - 2\ln y = 2\ln x - 2\ln y$

Solution: $f(tx, ty) = 2\ln tx - 2\ln ty$

$$= 2\ln t + 2\ln x - 2\ln t - 2\ln y$$

$$= 2\ln x - 2\ln y$$

$$f(tx, ty) = t^0 f(x, y)$$

Thus the given function is homogeneous With degree zero.

Example: $f(x, y) = x^2y + \ y^3\sin\frac{y}{x}$

Solution: $f(tx, ty) = t^2x^2ty + \ t^3y^3\sin\frac{ty}{tx}$

$$= t^3(x^2y + \ t^3y^3\sin\frac{y}{x}\ )$$

$$f(tx, ty) = t^3 f(x, y)$$

The function $f(x, y)$ is homogeneous of degree 3

II Homogeneous differential equations. The ODE $M(x,y)+N(x,y)y'=0$ is said to be *homogeneous of degree* n if both $M(x,y)$ and $N(x,y)$are homogeneous of degree n.

If we write the above DE as $y'=f(x,y)$, where $f(x,y)=-\frac{M(x,y)}{N(x,y)}$. Then f (x,y) is
homogeneous of degree 0.

To solve the DE

$$y'=f(x,y)$$

where f is homogeneous of degree 0, we use the substitution $y=ux$. Then

$$\frac{dy}{dx}=u+x\frac{du}{dx}$$

Thus the DE becomes

$$u+x\frac{dv}{dx}=f(x,ux)=x^0f(1,u)=f(1,u)$$

Consequently, the variables can be separated to yield

$$\frac{du}{f(1,u)-u}=\frac{dx}{x} \quad ,$$

and integrating both sides will give the solution.

Or if we write the differential equation as $M(x,y)dx+N(x,y)dy=0$ is said to be *homogeneous of degree* n if both $M(x,y)$ and $N(x,y)$are homogeneous of degree n.

To solve the DE $M(x,y)dx+N(x,y)dy=0$

We use the substitution $y=ux$. Then $dy=udx+xdy$ and the differential equation takes the form:

$$M(x,ux)dx+N(x,ux)[udx+xdu]=0$$

Now by homogeneity property ,we can write

$$x^nM(1,u)dx+x^nN(1,u)[udx+xdu]=0$$

Or $$\frac{dx}{x}+\frac{N(1,u)du}{M(1,u)+uN(1,u)}=0$$

Note: 1)The subst. $y = ux$ can be used for every homo. Diff. eq..In practice we try $x = vy$ when the function. $M(x,y)$is simpler than $N(x,y)$,where v new dependent variables.

2) The choice of subst. usually depends on which coefficient is simpler.

Example: Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$

Solution. We cannot separate the variables,

$$M(tx,ty) = tx - ty = t(x-y)$$

$$M(tx,ty) = tM(x,y)$$

The function $M(x,y)$ is homogeneous of degree 1

$$N(tx,ty) = ty + tx = t(y+x)$$

$$N(tx,ty) = tN(x,y)$$

The function $N(x,y)$ is homogeneous of degree 1

Then the differential equation is homogeneous of degree 1.

$$(y-x)dx - (x+y)dy = 0$$

Substituting

$$y = ux \quad \text{and} \quad \frac{dy}{dx} = u + x\frac{du}{dx} \quad \text{or} \quad dy = udx + xdu$$

we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \quad \Rightarrow \quad u + x\frac{du}{dx} = \frac{u-1}{u+1}$$

$$x\frac{du}{dx} = \frac{u-1}{u+1} - u \Rightarrow x\frac{du}{dx} = \frac{u-1-u^2-u}{u+1}$$

$$x\frac{du}{dx} = -\frac{1+u^2}{u+1} \quad \Rightarrow \quad \frac{(u+1)du}{u^2+1} + \frac{dx}{x} = 0$$

$$\int \frac{u}{u^2+1}du + \int \frac{du}{u^2+1} + \int \frac{dx}{x} = 0$$

$$\frac{1}{2}\ln(u^2+1)+\tan^{-1}u+\ln x=\ln C$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{y}{x^2}^2+1\right)+\tan^{-1}\frac{y}{x}+\ln x=\ln C$$ is a general solution

$$\frac{1}{2}\ln(y^2+x^2)-\ln(x^2)^{\frac{1}{2}}+\tan^{-1}\frac{y}{x}+\ln x=\ln C$$

$$\Rightarrow \frac{1}{2}\ln(y^2+x^2)+\tan^{-1}\frac{y}{x}=\ln C$$

Or

Another way let $x=vy \Rightarrow v=\frac{x}{y}$

$\frac{dx}{dy}=v+y\frac{dv}{dy}$ or $dx=vdy+ydv$

we have

$$\frac{dy}{dx}=\frac{y-x}{y+x}\Rightarrow \frac{dx}{dy}=\frac{y+x}{y-x} \quad \Rightarrow \frac{dx}{dy}=\frac{1+\frac{x}{y}}{1-\frac{x}{y}}$$

$$\Rightarrow \quad v+y\frac{dv}{dy}=\frac{v+1}{v-1}$$

$$y\frac{dv}{dy}=\frac{v+1}{v-1}-v\Rightarrow y\frac{dv}{dy}=\frac{1+v-v+v^2}{1-v}$$

$$\Rightarrow y\frac{dv}{dy}=\frac{1+v^2}{1-v} \quad \Rightarrow \frac{(1-v)dv}{1+v^2}+\frac{dy}{y}=0$$

$$\Rightarrow \int\frac{dv}{1+v^2}dv-\int\frac{v}{1+v^2}dv+\int\frac{dy}{y}=0$$

$$\tan^{-1}v-\frac{1}{2}\ln(1+v^2)-\ln|y|=C$$

$$\Rightarrow \tan^{-1}\frac{x}{y}-\frac{1}{2}\ln\left(1+\frac{x^2}{y^2}\right)-\ln|y|=C$$

$$\Rightarrow \tan^{-1}\frac{x}{y}-\frac{1}{2}\ln(y^2+x^2)+\ln|y|-\ln|y|=C$$

$$\Rightarrow \tan^{-1}\frac{x}{y}-\frac{1}{2}\ln(y^2+x^2)=C$$ is a general solution

Example: Solve $(x^2 - y^2)dx + 2xydy = 0$

Solution. We cannot separate the variables,

$$M(tx, ty) = (tx)^2 - (ty)^2 = t^2(x^2 - y^2)$$

$$M(tx, ty) = t^2 M(x, y)$$

The function $M(x, y)$ is homogeneous of degree 2

$$N(tx, ty) = 2t^2xy = t^2(2xy)$$

$$N(tx, ty) = t^2 N(x, y)$$

The function $N(x, y)$ is homogeneous of degree 2

Then the differential equation is homogeneous of degree 2.

Substituting

$$y = ux \qquad \text{and} \qquad dy = udx + xdu$$

we get

$$(1 - u^2)dx + 2u(udx + xdu) = 0$$

Separating the variables gives

$$\frac{2udu}{u^2+1} = -\frac{dx}{x}$$

Integrating we get

$$\ln(u^2 + 1) = -\ln x + \ln C$$

Taking exponentials we obtain

$$x(u^2 + 1) = C$$

Finally, since $= \frac{y}{x}$, this becomes

$$x^2 + y^2 = Cx$$

Example: Solve $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Solution. Inspection of M(x, y) and N(x, y) shows that these coefficients are homogeneous functions of degree 2.

If we let $y = ux$ and $dy = udx + xdu$ so that, after substituting, the given equation becomes

$$(x^2 + x^2u^2)dx + (x^2 - ux^2)[udx + xdu] = 0$$

$$x^2(1+u)dx + x^3(1-u)du = 0$$

$$\frac{1-u}{1+u}du + \frac{dx}{x} = 0$$

$$(\frac{2}{1+u} - 1)du + \frac{dx}{x} = 0$$

After integration the last line becomes

$$-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|C|$$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x+y)^2}{Cx}\right| = \frac{y}{x} \quad \text{or,equivalentiy,} \quad (x+y)^2 = Cxe^{y/x}.$$

Example : Solve the O.D.E. $y' = \dfrac{y}{x} + \sin(\dfrac{y-x}{x})$

Solution: We cannot separate the variables,

$$f(tx, ty) = \frac{ty}{tx} + \sin\frac{ty-tx}{tx} = \frac{y}{x} + \sin\frac{y-x}{x}$$

$$f(tx, ty) = t^0 f(x, y)$$

Then the differential equation is homogeneous of degree 0

Let $= ux$. Hence we

$$u + x\frac{du}{dx} = u + \sin(u-1) \quad \text{or} \quad x\frac{du}{dx} = \sin(u-1)$$

Separating the variables, we get $\dfrac{1}{\sin(u-1)}du = \dfrac{dx}{x}$

Integrating we get $\csc(u-1) - \cot(u-1) = Cx$