Definition 1: A ***differential equation*** is an equation that involves unknown function and any of its derivatives.

Definition 2.Ordinary differential equation: An equation that involving ordinary derivatives of one or more dependent variable with respect to a single independent variable is called an ordinary differential equation.

Definition 3.Partial differential equation: A differential equation involving partial derivative of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Examples of differential equations.

$$1)\left(\frac{dy}{dx}\right)^2 - 3\frac{dy}{dx} + 2y = e^x \qquad 2)\ \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$$

$$3)\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = k\frac{d^2y}{dx^2} \qquad 4)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$5)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 6)\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y$$

(1),(2) and (3) are ordinary differential equations and
(4),(5) and (6) are partial differential equations

Definition 4 :**Order**// The ***order*** of a differential equation is the largest derivative present in the differential equation.

For example, consider the differential equation

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{d^3y}{dx^3}\right)^2 + 7\frac{dy}{dx} - 4y = 0$$

The orders of $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ are 3, 2 and 1 respectively. So the highest order is 3. Thus the order of the differential equation is 3.

Definition 5:**Degree**// The ***degree*** of the differential equation is the power of highest ordered derivative occurring in the equation.

Thus the degree of

$$x^2\left(\frac{d^2y}{dx^2}\right)^3+3\left(\frac{d^3y}{dx^3}\right)^2+7\frac{dy}{dx}-4y=0 \quad \text{is } 2$$

Example : determine the order and degree of the following differential equations

(i) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)+y=3e^x$ **(ii)** $\left(\frac{d^2y}{dx^2}\right)^3+7\left(\frac{dy}{dx}\right)^4=3\sin x$

(iii) $\frac{d^2x}{dy^2}+a^2x=0$ **(iv)** $\left(\frac{dy}{dx}\right)^2-3\frac{d^3y}{dx^3}+7\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)-\log x=0$

(v) $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=4x$ **(vi)** $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}}=\frac{d^2y}{dx^2}$

(vii) $\frac{d^2y}{dx^2}-\sqrt{\frac{dy}{dx}}=0$ **(viii)** $\sqrt{1+x^2}=\frac{dy}{dx}$

Solution :

The order and the degree respectively are,

(i) 1 ; 3 (ii) 2 ; 3 (iii) 2 ; 1 (iv) 3 ; 1

(v) 1 ; 2 (vi) 2 ; 3 (vii) 2 ; 2 (viii) 1 ; 1

Note

Before ascertaining the order and degree in (v), (vi) & (vii) we made the differential coefficients free from radicals and fractional exponents.

Note: Not every differential equation has a degree

Example : $\sin\left(\frac{dy}{dx}\right)=3x$ has no degree

Definition 5//**Linear Differential Equations** is a differential equation that can be written in the following form.

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x) \qquad (6)$$

The important thing to note about linear differential equations is that there are no products of the function, $y\,(x)$, and its derivatives and neither the function nor its derivatives occur to any power other than the first power.

The coefficients $a_0(x) \ldots a_n(x)$and $g(x)$ can be zero or non-zero functions, constant or no constant functions, linear or non-linear functions. Only the function, $y(x)$, and its derivatives are used in determining if a differential equation is linear.
If a differential equation cannot be written in the form, (6) then it is called a ***non-linear*** differential equation.

Definition1.6// ***Solution*****:** A function $y = \varphi(x)$is called a solution of
$\dot{y}(x) = f(x, y(x))$ if it satisfies

$$\dot{\varphi}(x) = f(x, \varphi(x))\ .$$

To verify that a function $y = \varphi(x)$ is a solution of the ODE, is a solution, we substitute the function into both sides of the differential equation.

If the solution of a differential equation contains as many arbitrary constants of integration as its order, then the solution is said to be the **general solution** of the differential equation.

Definition1.7// ***Particular solution***:

The solution obtained from the general solution by assigning particular values for the arbitrary constants, is said to be a **particular solution** of the differential equation.
For example,

Differential equation	General solutuion	Particular solution
(i) $\frac{dy}{dx} = \sec^2 x$	$y = \tan x + c$ (c is arbitrary constant)	$y = \tan x - 5$
(ii) $\frac{dy}{dx} = x^2 + 2x$	$y = \frac{x^3}{3} + x^2 + c$	$y = \frac{x^3}{3} + x^2 + 8$
(iii) $\frac{d^2 y}{dx^2} - 9y = 0$	$y = Ae^{3x} + Be^{-3x}$	$y = 5e^{3x} - 7e^{-3x}$

Example: Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$,where c_1 and c_2 are arbitrary constants, a solution of $y'' + 4y = 0$?

Differentiating y,we find $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$ and

$y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$ .Hence,

$$\begin{aligned} y'' + 4y &= (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x) \\ &= (-4c_1 + 4c_1)\sin 2x + (-4c_2 + 4c_2)\cos 2x \\ &= 0 \end{aligned}$$

Thus, $y = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of x and is a solution on the interval $(-\infty, \infty)$.

Example: Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$

Note that left side of diff. eq. nonnegative for every real function$y(x)$ $and\ any\ x$, while there right side is negative. Since no function $y(x)$will satisfy this eq., the given diff. eq. has no solution.

Exercise:

1. Show that $y(x) = 3\,e^{2x} - e^{-2x}$is a solution to $y'' - 4y = 0$
2. Determine whether $y(x) = 2\,e^{-x} + xe^{-x}$ is a solution of $y'' + 2y' + y = 0$

Family of curves

Sometimes a family of curves can be represented by a single equation. In such a case the equation contains an arbitrary constant c .By assigning different values for c. we get a family of curves. In this case c is called the parameter or arbitrary constant of the family.

Examples

i) $y = mx$ represents the equation of a family of straight lines through the origin. Where m is the parameter.
ii) $x^2 + y^2 = a^2$ represents the equation of a family of concentric circles having the origin as center. Where a is the parameter.
iii) $y = mx + c$ represents the equation of a family of straight lines in a plane. Where m and c are parameters.

Formation of Ordinary Differential Equation

Consider the equation $y = mx + \lambda$ ---------(1)
where m is a constant and λ is the parameter.

This represents one parameter family of parallel straight lines having same slope m.

Differentiating (1) with respect to x, we get, $\frac{dy}{dx} = m$

This is the differential equation representing the above family of straight lines.

Similarly for the equation $y = Ae^{5x}$, we form the differential equation $\frac{dy}{dx} = 5y$ by eliminating the arbitrary constant A.

The above functions represent one-parameter families. Each family has a differential equation. To obtain this differential equation differentiate the equation of the family with respect to x, treating the parameter as a constant. If the derived equation is free from parameter then the derived equation is the differential equation of the family.

Note

(i) The differential equation of a two parameter family is obtained by differentiating the equation of the family twice and by eliminating the parameters.

(ii) In general, the order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

Example :

Form the differential equation of the family of curves $y = A\cos 5x + B\sin 5x$ where A and B are parameters.

Solution :

Given $y = A\cos 5x + B\sin 5x$

$$\frac{dy}{dx} = -5A\sin 5x + 5B\cos 5x$$

$$\frac{d^2y}{dx^2} = -25\,(A\cos 5x) - 25\,(B\sin 5x) = -25y$$

$$\therefore \quad \frac{d^2y}{dx^2} + 25y = 0.$$

Example :

Form the differential equation of the family of curves $y = ae^{3x} + be^{x}$ where a and b are parameters.

Solution :

$$y = ae^{3x} + be^{x} \qquad \text{------------(1)}$$

$$\frac{dy}{dx} = 3ae^{3x} + be^{x} \qquad \text{------------(2)}$$

$$\frac{d^2y}{dx^2} = 9ae^{3x} + be^{x} \qquad \text{------------(3)}$$

$$(2) - (1) \Rightarrow \frac{dy}{dx} - y = 2ae^{3x} \qquad \text{------------(4)}$$

$$(3) - (2) \Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 6ae^{3x} = 3\left(\frac{dy}{dx} - y\right) \qquad \text{[using (4)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Example :

Find the differential equation of a family of curves given by $y = a\cos(mx+b)$, a and b being arbitrary constants.

Solution :

$$y = a\cos(mx+b) \qquad \text{------------}(1)$$

$$\frac{dy}{dx} = -ma\sin(mx+b)$$

$$\frac{d^2y}{dx^2} = -m^2a\cos(mx+b) = -m^2y \qquad [\text{using }(1)]$$

$\therefore \quad \dfrac{d^2y}{dx^2} + m^2y = 0$ is the required differential equation.

Example :

Find the differential equation by eliminating the arbitrary constants a and b from $y = a\tan x + b\sec x$.

Solution :

$$y = a\tan x + b\sec x$$

Multiplying both sides by $\cos x$ we get,

$$y\cos x = a\sin x + b$$

Differentiating with respect to x we get

$$y(-\sin x) + \frac{dy}{dx}\cos x = a\cos x$$

$$\Rightarrow \quad -y\tan x + \frac{dy}{dx} = a \qquad \text{-----------}(1)$$

Differentiating (1) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx}\tan x - y\sec^2 x = 0$$

Example :

Show that $y = \ln x$ is a solution of $xy'' + y' = 0$ on $\mathcal{I} = (0, \infty)$ but is not a solution on $\mathcal{I} = (-\infty, \infty)$.

On $(0, \infty)$ we have $y' = 1/x$ and $y'' = -1/x^2$. Substituting these values into the differential equation, we obtain

$$xy'' + y' = x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = 0$$

Thus, $y = \ln x$ is a solution on $(0, \infty)$.

Note that $y = \ln x$ could not be a solution on $(-\infty, \infty)$, since the logarithm is undefined for negative numbers and zero.

EXERCISE

1) Find the order and degree of the following :

(i) $x^2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + y = \cos x$ (ii) $\dfrac{d^3y}{dx^3} - 3\left(\dfrac{d^2y}{dx^2}\right)^2 + 5\dfrac{dy}{dx} = 0$

(iii) $\dfrac{d^2y}{dx^2} - \sqrt{\dfrac{dy}{dx}} = 0$ (iv) $\left(1 + \dfrac{d^2y}{dx^2}\right)^{\frac{1}{2}} = \dfrac{dy}{dx}$

(v) $\left(1 + \dfrac{dy}{dx}\right)^{\frac{1}{3}} = \dfrac{d^2y}{dx^2}$ (vi) $\sqrt{1 + \dfrac{d^2y}{dx^2}} = x\dfrac{dy}{dx}$

(vii) $\left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}} = \left(\dfrac{dy}{dx}\right)^2$ (viii) $3\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3 - 3y = e^x$

(ix) $\dfrac{d^2y}{dx^2} = 0$ (x) $\left(\dfrac{d^2y}{dx^2} + 1\right)^{\frac{2}{3}} = \left(\dfrac{dy}{dx}\right)^{\frac{1}{3}}$

2) Find the differential equation of the following

(i) $y = mx$ (ii) $y = cx - c + c^2$

(iii) $y = mx + \dfrac{a}{m}$, where m is arbitrary constant

(iv) $y = mx + c$ where m and c are arbitrary constants.

3) Find the differential equation of all circles $x^2 + y^2 + 2gx = 0$ which pass through the origin and whose centres are on the x-axis.

4) Form the differential equation of $y^2 = 4a\,(x + a)$, where a is the parameter.

5) Find the differential equation of the family of curves $y = ae^{2x} + be^{3x}$ where a and b are parameters.

6) Form the differential equation for $y = a \cos 3x + b \sin 3x$ where a and b are parameters.

7) Form the diffrential equation of $y = ae^{bx}$ where a and b are the arbitrary constants.

8) Find the differential equation for the family of concentric circles $x^2 + y^2 = a^2$, a is the paramter.

Initial-Value Problem and boundary-Value Problems

Definition1.11//Initial value problem:: an initial value problem for an n th order differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0\ ,$$

We mean :Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0 \\ \frac{dy}{dx}(x_0) &= y_1 \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}, \end{aligned}$$

Where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.
When condition for the given differential equation related to two or more *x values*. The condition are called boundary conditions or boundary values. The differential eq. with its boundary conditions is called a ***boundary value problm***.

Example:

Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\frac{d^2y}{dx^2} + y = 0\ ; \qquad y(0) = -1\ , \qquad \frac{dy}{dx}(0) = 1\ .$$

Observe that $\phi(x) = \sin x - \cos x$, $d\phi/dx = \cos x + \sin x$, and $d^2\phi/dx^2 = -\sin x + \cos x$ are all defined on $(-\infty, \infty)$. Substituting into the differential equation gives

$$(-\sin x + \cos x) + (\sin x - \cos x) = 0\ ,$$

which holds for all $x \in (-\infty, \infty)$. Hence $\phi(x)$ is a solution to the differential equation on $(-\infty, \infty)$. When we check the initial conditions, we find

$$\phi(0) = \sin 0 - \cos 0 = -1\ ,$$

$$\frac{d\phi}{dx}(0) = \cos 0 + \sin 0 = 1\ ,$$

which meets the requirements of . Therefore $\phi(x)$ is a solution to the given initial value problem.

Example:

Find a solution to the boundary-value problem $y'' + 4y = 0$; $y(\pi/8) = 0$, $y(\pi/6) = 1$, if the general solution to the differential equation is $y(x) = c_1 \sin 2x + c_2 \cos 2x$.

Note that

$$y\left(\frac{\pi}{8}\right) = c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right) = c_1\left(\frac{1}{2}\sqrt{2}\right) + c_2\left(\frac{1}{2}\sqrt{2}\right)$$

To satisfy the condition $y(\pi/8) = 0$, we require

$$c_1\left(\frac{1}{2}\sqrt{2}\right) + c_2\left(\frac{1}{2}\sqrt{2}\right) = 0 \qquad (1)$$

Furthermore,

$$y\left(\frac{\pi}{6}\right) = c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \cos\left(\frac{\pi}{3}\right) = c_1\left(\frac{1}{2}\sqrt{3}\right) + c_2\left(\frac{1}{2}\right)$$

To satisfy the second condition, $y(\pi/6) = 1$, we require

$$\frac{1}{2}\sqrt{3}c_1 + \frac{1}{2}c_2 = 1 \qquad (2)$$

Solving (*1*) and (*2*) simultaneously, we find

$$c_1 = -c_2 = \frac{2}{\sqrt{3} - 1}$$

Substituting these values into $y(x)$, we obtain

$$y(x) = \frac{2}{\sqrt{3} - 1}(\sin 2x - \cos 2x)$$

as the solution of the boundary-value problem.