1.1 Definition A set is collection of distinct objects. These objects are called the elements, or members.

Important Sets of Real Numbers

The set $\mathbb{N}$ of natural numbers is define by

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

The set $\mathbb{Z}$ of integers is define by

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The set $\mathbb{Q}$ of rational numbers is define by

$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}.$$

Non-periodic decimal fractions are called irrational numbers and denoted by *Irr*.

For example, $\sqrt{2}, \sqrt{3}, \pi$

Real numbers

Real Numbers are made up of rational numbers and irrational numbers and denoted by $\mathbb{R}$.

The Number Line

We may use the number line to represent all the real numbers graphically; each real number corresponds to exactly one point on the number line. ∞ and $-\infty$ are not real numbers because there is no point on the number line corresponding to either of them.

$\mathbb{C}$, denoting the set of all complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

For example, $1 + 2i \in \mathbb{C}$.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements.

Types of intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Absolute value

1.11 Definition The ***absolute value*** of a number x, denoted by is defined by the formula

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}.$$

1.12 Example $|2| = 2,\ |-5| = -(-5) = 5.$

Some properties of the absolute value

Let a, b and x be any real numbers then:

1) $|x| = \sqrt{x^2}$
2) $|ab| = |a||b|$
3) $|a + b| \leq |a| + |b|$
4) $|a - b| \geq ||a| - |b||$
5) $|x| \leq a$ if and only if $x \leq a$ and $x \geq -a$ $(\,or\ -a \leq x \leq a)$
6) $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

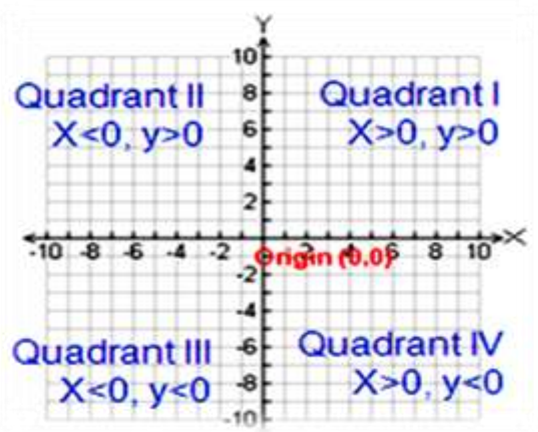
Cartesian product

Definition Let A and B be any two non empty sets, the Cartesian product of A with B denoted by $A \times B$ is defined by

$$A \times B = \{(a, b): a \in \mathrm{A}\ \ and\ b \in \mathrm{B}\}$$
$$B \times A = \{(a, b): a \in \mathrm{B}\ \ and\ b \in \mathrm{A}\}$$

Remark

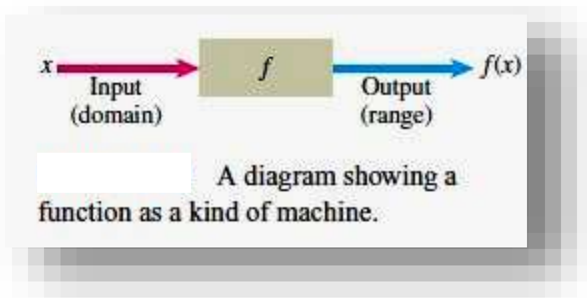
1) $A \times B \neq B \times A$
2) $A \times B = B \times A$ iff $A = B$
3) If A contains m elements and B contains n elements, then $A \times B$ contains $m \times n$ elements.
4) $A \times B = \phi$ iff $A = \phi$ or $B = \phi$
5) The Cartesian product of $\mathbb{R}$ with itself is $\mathbb{R} \times \mathbb{R}$ denoted by $\mathbb{R}^2$, $\mathbb{R}^2 = \{(x, y): x, y \in \mathbb{R}\}$, $\mathbb{R}^2$ denotes the Cartesian plane.



The Function

Definition Let A and B be any two non empty sets then a function (denoted by f) from A to B is a relation from A to B provided that for each $x \in$ A there exist only a unique $y \in$ B such that $(x, y) \in f$ and f can be written as:

$$f: A \to B,\ y = f(x) \text{ or } \ y \xrightarrow{f} x.$$



A diagram showing a function as a kind of machine.

Example Let $f: \mathbb{R} \to \mathbb{R}$. Does the following f are functions or not?

1) $f(x) = \sqrt{x}$ **2)** $f(x) = x^2$ **3)** $f(x) = 3$

Solution:

1. Is not a function because $\sqrt{-1}$ is undefined.

2. Is a function since for all x there exist y such that $(x, y) \in f$.

3. is a function since for all x there exist y such that $(x, y) \in f$.

Example Let $f: \mathbb{R}^+ \to \mathbb{R},\ x = y^2$. Is not a function because

$$(4, -2) \in f \text{ and } (4, 2) \in f$$

Example Let $f: \mathbb{R}^+ \to \mathbb{R}^+,\ x = y^2$. Is a function

Definitions

1) The set A of all possible input values is called the ***domain*** of the function.

That means $D_f = \{x: x \in A \text{ and } y = f(x) \text{ for a unique } y \in B\}$

2) The set of all values of $f(x)$ as x varies throughout A is called the ***range*** of the function, i.e. $R_f = \{y: y \in B \text{ and } y = f(x) \text{ for at least one } x \in A\}$

The range may not include every element in the set Y.

Examples Find the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

H.W. Find the domains and ranges of these functions.

1) $y = f(x) = \frac{x+3}{x^3+1}$ **2)** $y = f(x) = \sqrt{\frac{x+1}{x}}$

Definition (The Graph of Function)

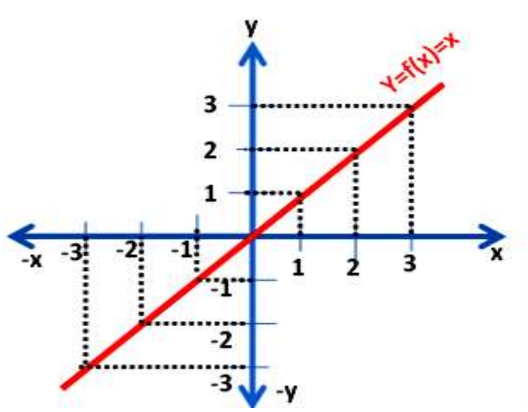
The graph of the function $y = f(x)$ is the set of all points (x, y) in the Cartesian plane $X \times Y$ such that (x, y) satisfies the function $y = f(x)$.

That means the graph is $\{(x, y): y = f(x)\}$.

Example Find the graph of this function $y = f(x) = x$

Solution:

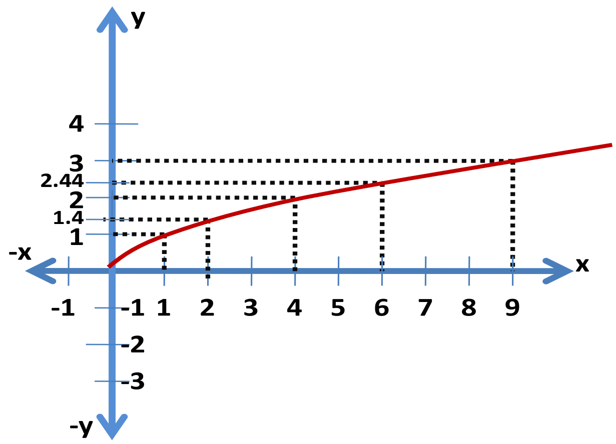
x	1	2	3	0	-1	-2	-3
$y = f(x)$	1	2	3	0	-1	-2	-3
(x, y)	$(1,1)$	$(2,2)$	$(3,3)$	$(0,0)$	$(-1,-1)$	$(-2,-2)$	$(-3,-3)$



1.34 Example : Graph this function $y = f(x) = \sqrt{x}$

Solution:

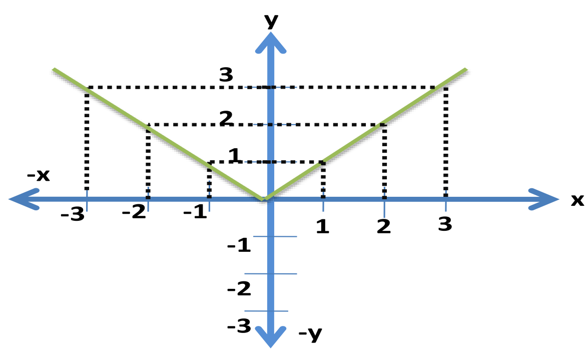
x	y=f(x)	(x, y)
0	**0**	**(0,0)**
1	**1**	**(1,1)**
2	**1.4**	**(2,1.4)**
4	**2**	**(4,2)**
6	**2.44**	**(6,2.44)**
9	**3**	**(9,3)**
0	**0**	**(0,0)**



1.35 Example : graph of this function $y = f(x) = |x|$

Solution:

x	y=f(x)	(x, y)
1	1	(1,1)
2	2	(2,2)
3	3	(3,3)
0	0	(0,0)
-1	1	(-1,1)
-2	2	(-2,2)
-3	3	(-3,3)



Note Let $f(x)$ and $g(x)$ be two functions having D_f and D_g as a domain respectively. Then

1) $D_{f+g} = D_{f-g} = D_{f\times g} = D_f \cap D_g$

2) $D_{f/g} = D_f \cap D_g - \{x: g(x) = 0\}$

Example Find the domain for the function $K(x) = \frac{x+1}{[\![x]\!]-1}$

Solution: Let $f(x) = x + 1$ and $g(x) = [\![x]\!] - 1$

$D_f = \mathbb{R}$ and $D_g = \mathbb{R}$

$\Longrightarrow D_K = \mathbb{R} \cap \mathbb{R} - \{x: [\![x]\!] - 1 = 0\} = \mathbb{R} - \{x: [\![x]\!] = 1\} = \mathbb{R} - [1,2)$

Definition Type of Functions

1) ***Constant function*** $y = f(x) = c$ where $c \in \mathbb{R}$ is called the constant function.

2) ***Identity function*** $y = f(x) = x$ is called the identity function.

3) ***Polynomial function*** $y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_i \in \mathbb{R}$, $i = 0,1,\ldots,n$. For $a_0 \neq 0$ and $n \geq 0$ an integer is called a polynomial of degree n.

For example $y = f(x) = x^6 + x - 6$.

4) ***Definition*** **The greatest integer function.** The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function**, or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$.

5) ***Even function*** The fu...

$f(-x) = f(x)\ \forall\ x \in L$

For example $y = f(x) = x^2 + 1$ is an even function.

6) ***Odd function*** The function $y = f(x)$ is called an even function if

$f(-x) = -f(x)\ \forall\ x \in D_f$.

For example $y = f(x) = x^3$ is an odd function.

7) ***Injective function (1-1 one to one)*** The function $y = f(x)$ is said to be a one to one function if for any $x_1, x_2 \in D_f$, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

For example $y = f(x) = x + 1$ is one to one function. Since for any $x_1, x_2 \in \mathbb{R}$, if

$$f(x_1) = f(x_2) \Longrightarrow x_1 + 1 = x_2 + 1 \Longrightarrow x_1 = x_2.$$

Remark (Horizontal Line Test) A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example Is the function $y = x^2$ from $\mathbb{R}$ *to* $\mathbb{R}$ one−to−one?
Solution¹: This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

Solution²: From Figure 1.21 we see that there are horizontal lines that intersect the graph of more than once. Therefore, by the Horizontal Line Test, is not one-to-one.

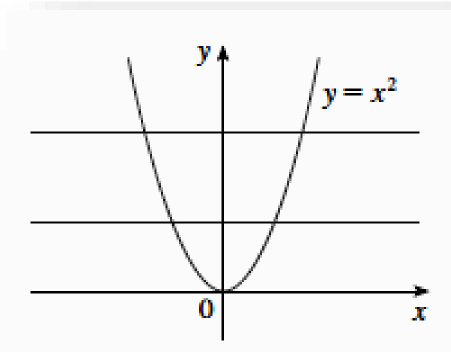


Figure 1.21

8) ***Surjective function (onto)*** The function $y = f(x)$ is said to be an on to function if for any $y \in R_f$ there exist at least one value of $x \in D_f$ such that $y = f(x)$.

For example $y = f(x) = x + 1$ is one to one function. Since if we put

$$x = y - 1 \Longrightarrow f(x) = f(y - 1) = y - 1 + 1 = y$$

9) bijective function The function $y = f(x)$ is said to be bijective function if f are both Surjective and Injective function.

For example $y = f(x) = x + 1$.

10) ***Inverse function*** Let $f: A \to B$ be a bijective function, the inverse function of $f(x)$ denoted by $f^{-1}(x)$ is defined by $f^{-1}: B \to A$ satisfying $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.

$i.e.$ The composition of any function with its inverse is the identity function and if $(x, y) \in f$ then $(y, x) \in f$.

Note **1**) $f^{-1}(x) \neq \frac{1}{f(x)}$ **2**) $(f^{-1}(x))^{-1} = f(x)$

3) Any function is symmetric with its inverse about line $y = x$.

Example If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$

$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.