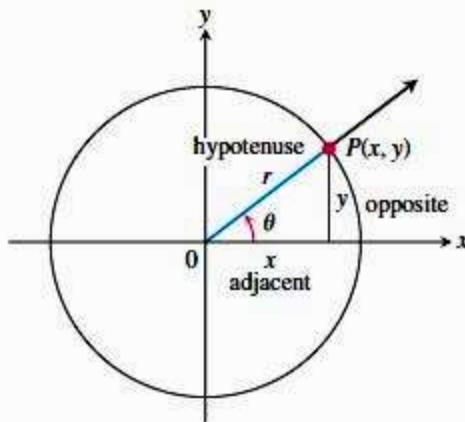


Trigonometric Function

$$\sin^2\theta + \cos^2\theta = 1 \quad \dots (1)$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x} \text{ and } \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{x}{y}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{r}{x} \text{ and } \csc \theta = \frac{1}{\sin \theta} = \frac{r}{y}$$



$$\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta \Rightarrow \boxed{\sin(-\theta) = -\sin \theta}$$

$$\cos(-\theta) = \frac{x}{r} = \cos \theta \Rightarrow \boxed{\cos(-\theta) = \cos \theta}$$

Divide equation (1) by $\cos^2\theta$ we get:

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Divide equation (1) by $\sin^2\theta$ we get:

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

Since the circumference of the circle is 2π and one complete revolution of a circle is 360° , the relation between radians and degrees is given by

$$\pi \text{ radians} = 180^\circ.$$

For example, 45° in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30^\circ.$$

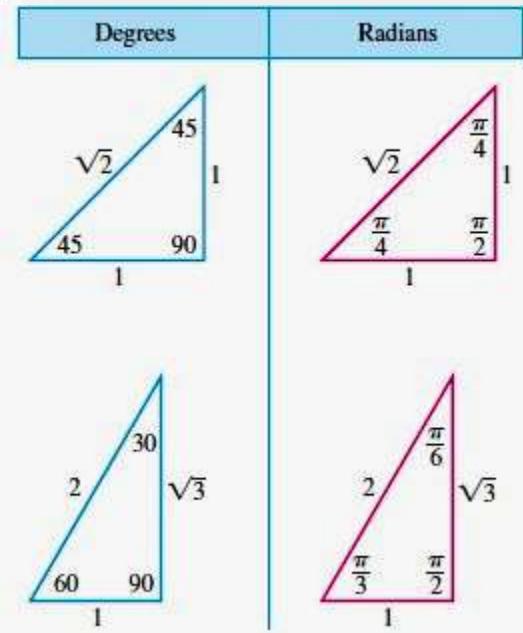
Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$



Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Periods of Trigonometric Functions

Period π :	$\tan(x + \pi) = \tan x$
	$\cot(x + \pi) = \cot x$
Period 2π :	$\sin(x + 2\pi) = \sin x$
	$\cos(x + 2\pi) = \cos x$
	$\sec(x + 2\pi) = \sec x$
	$\csc(x + 2\pi) = \csc x$

Some Important Identities

1. $\sin(\alpha \mp \beta) = \sin \alpha \cos \beta \mp \sin \beta \cos \alpha$

2. $\cos(\alpha \mp \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta$

3. $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

4. $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

5. $\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}$

6. $\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$

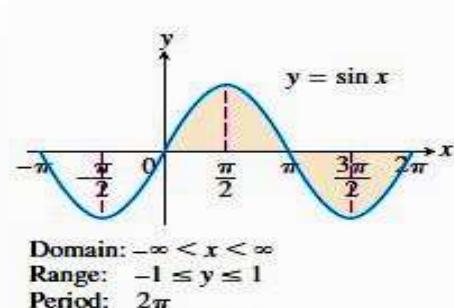
1. Sine Function

It is a function $f: \mathbb{R} \rightarrow [-1, 1]$ defined by $f(x) = \sin x$.

$\sin x = 0$ iff $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \dots = n\pi, n \in \mathbb{Z}$.

$\sin x = \mp 1$ iff $x = \mp \frac{\pi}{2}, \mp \frac{3\pi}{2}, \dots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$.

Since $\sin(-x) = -\sin x$, therefore the sine is an odd function.



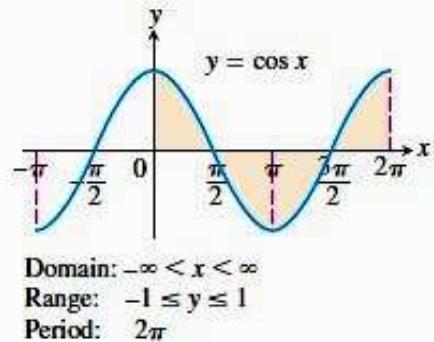
2. Cosine Function

It is a function $f: \mathbb{R} \rightarrow [-1, 1]$ defined by $f(x) = \cos x$.

$$\cos x = 0 \text{ iff } x = \mp \frac{\pi}{2}, \mp \frac{3\pi}{2}, \dots = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}.$$

$$\cos x = \mp 1 \text{ iff } x = 0, \mp \pi, \mp 2\pi, \mp 3\pi, \dots = n\pi, n \in \mathbb{Z}$$

Since $\cos(-x) = \cos x$, therefore the cosine is an even function.

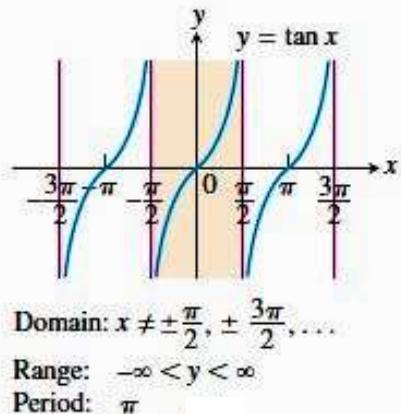


3. Tangent Function

It is defined by $\tan: D_{\tan} \rightarrow \mathbb{R}$,

$$f(x) = \tan x = \frac{\sin x}{\cos x}, \cos x \neq 0.$$

$$\cos x = 0 \text{ if } x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z},$$

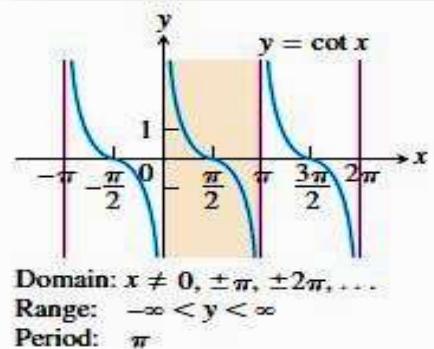


4. Cotangent Function

It is defined by $\cot: D_{\cot} \rightarrow \mathbb{R}$,

$$f(x) = \cot x = \frac{\cos x}{\sin x}, \sin x \neq 0.$$

$$\sin x = 0 \text{ if } x = n\pi, n \in \mathbb{Z},$$



5. Secant Function

It is a function $f: D_{\sec} \rightarrow (-\infty, -1] \cup [1, \infty)$, and

defined by $f(x) = \sec x = \frac{1}{\cos x}$, $\cos x \neq 0$.

$\cos x = 0$ if $x = (n + \frac{1}{2})\pi, n \in \mathbb{Z}$,

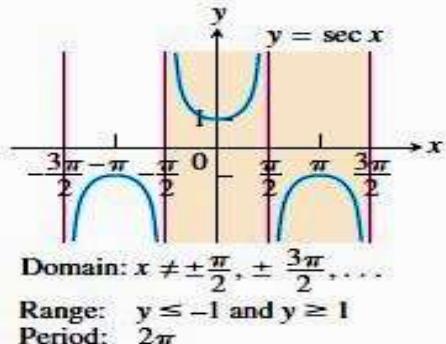


Figure 2.9

6. Cosecant Function

It is a function $f: D_{\csc} \rightarrow (-\infty, -1] \cup [1, \infty)$,

and defined by $f(x) = \csc x = \frac{1}{\sin x}$, $\sin x \neq 0$.

$\sin x = 0$ if $x = n\pi, n \in \mathbb{Z}$,

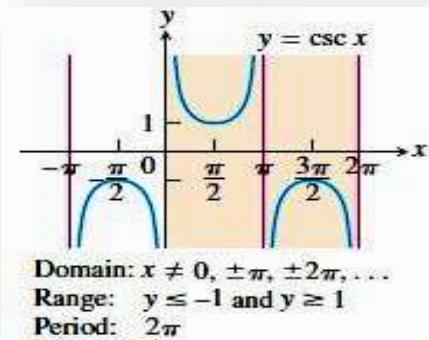


Figure 2.10

The Inverse of Trigonometric Functions

1. The inverse of sine function

It's denoted by \sin^{-1} , and defined to be the inverse of

the sine function for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

$$f(x) = \sin^{-1} x, \sin^{-1}: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$$

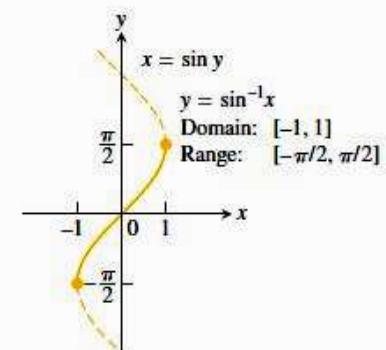
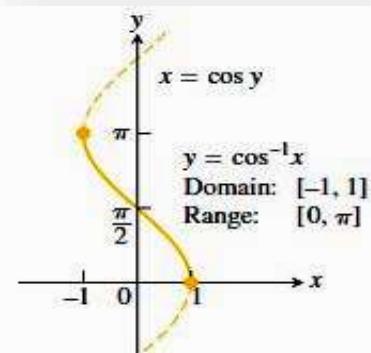


Figure 2.11

2. The inverse of cosine function

It's denoted by \cos^{-1} , and defined to be the inverse of the cosine function for $0 \leq x \leq \pi$,

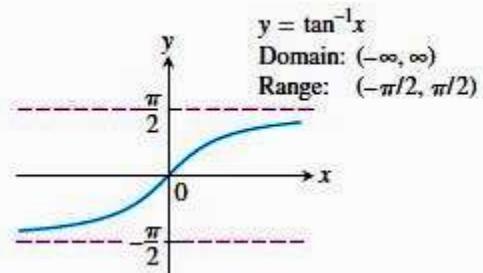
$$f(x) = \cos^{-1} x, \cos^{-1}: [-1, 1] \rightarrow [0, \pi]$$



3. The inverse of tangent function

It's denoted by \tan^{-1} , and defined to be the inverse of the tangent function

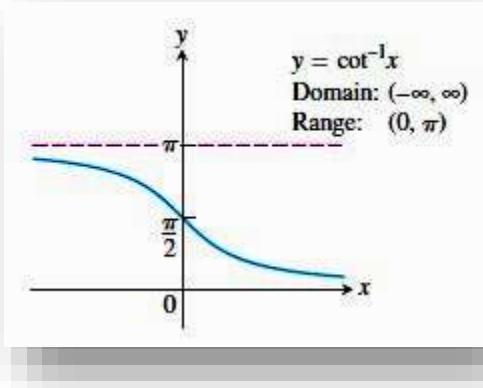
$$f(x) = \tan^{-1} x, \tan^{-1}: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$$



4. The inverse of cotangent function

It's denoted by \cot^{-1} , and defined to be the inverse of the cotangent function

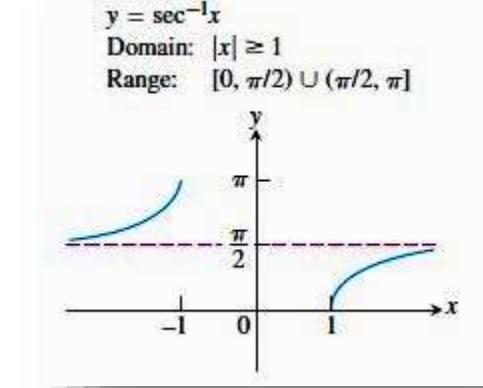
$$f(x) = \cot^{-1} x, \cot^{-1}: \mathbb{R} \rightarrow (0, \pi)$$



5. The inverse of secant function

It's denoted by \sec^{-1} , and defined to be the inverse of the secant function

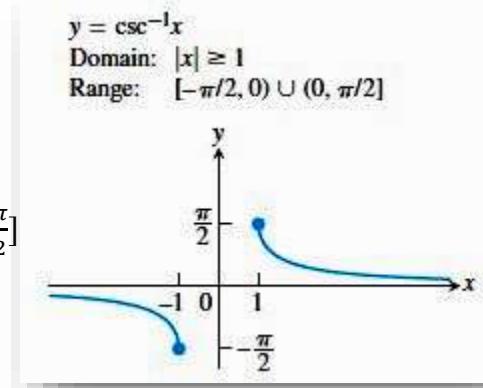
$$f(x) = \sec^{-1} x, \sec^{-1}: (-\infty, -1] \cup [1, \infty) \rightarrow [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$$



6. The inverse of cosecant function

It's denoted by \csc^{-1} , and defined to be the inverse of the cosecant function

$$f(x) = \csc^{-1} x, \csc^{-1}: (-\infty, -1] \cup [1, \infty) \rightarrow [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$$



Logarithmic Functions

1. The General Logarithmic Functions

Let $a > 0$, $a \neq 1$ be any real number, the general logarithmic from $\text{Log}_a: \mathbb{R}^+ \rightarrow \mathbb{R}$ where a is the base of logarithmic function.

Some Properties

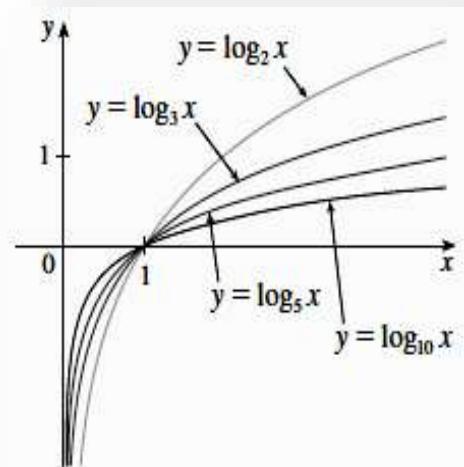
If x and y are positive numbers, then

1. $\text{Log}_a(xy) = \text{Log}_a x + \text{Log}_a y$
2. $\text{Log}_a\left(\frac{x}{y}\right) = \text{Log}_a x - \text{Log}_a y$
3. $\text{Log}_a x^r = r \text{Log}_a x$ (where r is any real number)
4. $\text{Log}_a 1 = 0$
5. $\text{Log}_a a = 1$
6. For $0 < x < 1$, $\text{Log}_a x < 0$
7. For $x \geq 1$, $\text{Log}_a x \geq 0$
8. $\lim_{x \rightarrow 0^+} \text{Log}_a x = -\infty$, $\lim_{x \rightarrow \infty} \text{Log}_a x = \infty$

9. It is one to one and onto function, so it is bijective function.

-If $a = 10$, then we denote this function by $f(x) = \text{Log } x$.

-If $a = e$ (e is the Euler's number and $e = 2.718281828 \dots$), we denote this function by $f(x) = \ln x$ and it is called the natural logarithmic function.



2. The Natural Logarithmic Functions

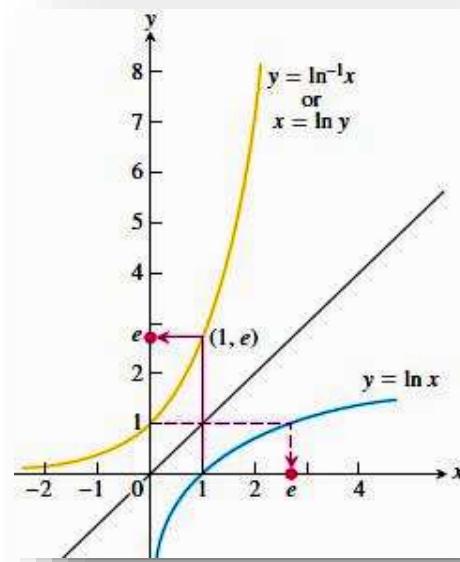
It is the logarithmic function with the base $a = e$.

i.e. $f(x) = \text{Log}_e x = \ln x$, $\ln: \mathbb{R}^+ \rightarrow \mathbb{R}$.

Some Properties

If x and y are positive numbers, then

1. $\ln(xy) = \ln x + \ln y$
2. $\frac{\ln x}{\ln y} = \ln x - \ln y$
3. $\ln 1 = 0$, $\ln e = 1$
4. $\ln x^r = r \ln x$



5. $\ln \frac{1}{a} = \ln a^{-1} = -\ln a$
6. For $0 < x < 1$, $\ln x < 0$
7. For $x \geq 1$, $\ln x \geq 0$
8. $\lim_{x \rightarrow 0^+} \ln x = -\infty$, $\lim_{x \rightarrow \infty} \ln x = \infty$
9. It is one to one and onto function, so it is bijective function.

Exponential Functions

1. The Natural Exponential Functions

Since the natural logarithmic function is a bijective function, so it has an inverse, which's the natural exponential, hence $\exp: \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = e^x, \forall x \in \mathbb{R}$.

Some Properties

The natural exponential e^x obeys the following laws:

$$1. e^x e^y = e^{x+y} \quad \forall x, y \in \mathbb{R}$$

$$2. e^{-x} = \frac{1}{e^x}$$

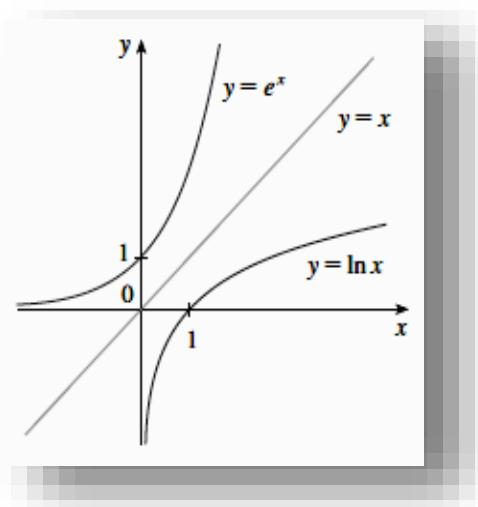
$$3. \frac{e^x}{e^y} = e^{x-y}$$

$$4. (e^x)^y = e^{xy} = (e^y)^x$$

$$5. e^0 = 1$$

$$6. e^{\ln x} = x$$

$$7. \ln e^x = x$$



2. The General Exponential Functions

It is defined $f: \mathbb{R} \rightarrow \mathbb{R}^+$ by

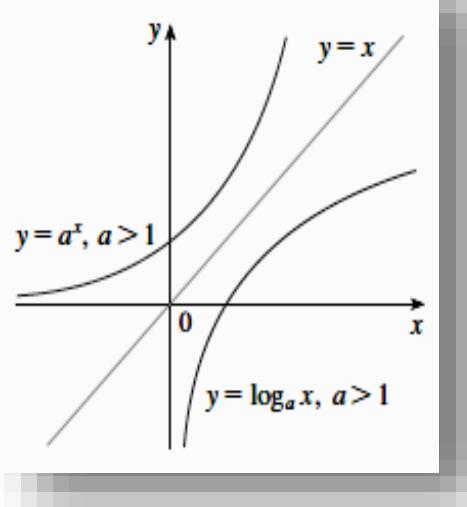
$$f(x) = a^x, \forall x \in \mathbb{R}, a > 0, a \neq 1$$

It is the inverse function of the logarithmic function.

Since $e^{\ln x} = x \Rightarrow e^{\ln a} = a$

$$\Rightarrow (e^{\ln a})^x = a^x$$

$$\Rightarrow a^x = e^{x \ln a}$$



$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a}$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \quad (x > 0)$$

$$\log_a(a^x) = x \quad (\text{all } x)$$

Some Examples**Example :**

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \text{ since } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\sin^{-1}(-1) = -\frac{\pi}{2}, \text{ since } \sin\left(-\frac{\pi}{2}\right) = -1$$

$$\sin \sin^{-1}\left(\frac{2}{5}\right) = \frac{2}{5}$$

But

$$\sin^{-1} \sin(2\pi) = 0, \text{ not } 2\pi$$

$\sin^{-1} \sin(2\pi)$ can't be 2π because \sin^{-1} always returns an angle in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Recall:-

The inverse cosine by inverting $\cos x$, restricted to $[0, \pi]$.

The inverse tangent by inverting $\tan x$, restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The inverse secant by inverting $\sec x$, restricted to $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

Example : Find $\sin^{-1}\left(\sin\left(\frac{\pi}{16}\right)\right)$

Solution: Since $\frac{\pi}{16} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$\sin^{-1}\left(\sin\left(\frac{\pi}{16}\right)\right) = \frac{\pi}{16}$$

Exponential and logarithmic functions:

Example :

(1) $f(x) = 3^x$ is exponential function with base 3 .

(2) $f(x) = \left(\frac{1}{5}\right)^x$ is exponential function with base $\frac{1}{5}$.

(3) $f(x) = \pi^x$ is an exponential function with base π .

Examples:-

$\log_{10} 100 = 2$	because $100 = 10^2$	$\log_{10}\left(\frac{1}{1000}\right) = -3$	because $\frac{1}{1000} = 10^{-3}$
$\log_2 16 = 4$	because $16 = 2^4$	$\log_b 1 = 0$	because $1 = b^0$
$\log_b b = 1$	because $b^1 = b$	$\log_{3/2} \frac{27}{8} = 3$	because $\frac{27}{8} = \left(\frac{3}{2}\right)^3$
$\log_{1/6} 36 = -2$	because $36 = \left(\frac{1}{6}\right)^{-2}$	$\log_2 \sqrt[7]{32} = \frac{5}{7}$	because $\sqrt[7]{32} = (2)^{5/7}$
$\log 1000 = 3$	because $1000 = 10^3$	$\ln 1 = 0$	because $1 = e^0$
$\ln e = 1$	because $e = e^1$	$\ln \frac{1}{e} = -1$	because $\frac{1}{e} = e^{-1}$
$\ln e^2 = 2$	because $e^2 = e^2$	$\ln \sqrt[3]{e} = \frac{1}{3}$	because $\sqrt[3]{e} = e^{1/3}$

Example : Find the domain of the function $f(x) = \ln(9 - 4x^2)$.

Solution: We must have $9 - 4x^2 > 0$, so that

$$4x^2 < 9 \Rightarrow x^2 < \frac{9}{4} \Rightarrow \sqrt{x^2} < \sqrt{\frac{9}{4}} \Rightarrow |x| < \frac{3}{2} \Rightarrow x \in \left(-\frac{3}{2}, \frac{3}{2}\right).$$

Thus $\text{Dom}(f) = \left(-\frac{3}{2}, \frac{3}{2}\right)$.

Example : If $f(x) = 4e^{bx}$, find the value of b for which $f(2) = 2$.

Solution:

$$f(2) = 2 \Rightarrow 4e^{2b} = 2 \Rightarrow e^{2b} = \frac{1}{2} \Rightarrow \ln(e^{2b}) = \ln\left(\frac{1}{2}\right) \Rightarrow 2b = \ln\left(\frac{1}{2}\right) \Rightarrow b = \frac{1}{2} \ln\left(\frac{1}{2}\right) = -\frac{1}{2} \ln 2 = -\ln \sqrt{2}$$

Example : If $f(x) = e^x + 3e^{-x}$, find $f(\ln 2)$.

Solution

$$f(\ln 2) = e^{\ln 2} + 3e^{-\ln 2} = e^{\ln 2} + 3e^{\ln\left(\frac{1}{2}\right)} = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$$

Example : Solve $\log_{10} x = \sqrt{2}$.

Solution

$$\log_{10} x = \sqrt{2} \Rightarrow x = 10^{\sqrt{2}}$$

Example : Solve $\ln(x+1) = 5$.

Solution : $\ln(x+1) = 5 \Rightarrow x+1 = e^5 \Rightarrow x = e^5 - 1$

Example : Solve $5^x = 7$.

Solution

$$5^x = 7 \Rightarrow \ln(5^x) = \ln(7) \Rightarrow x \ln 5 = \ln 7 \Rightarrow x = \frac{\ln 7}{\ln 5}$$

H.W. : Solve $\log_{10} \sqrt{x} = -1$.

H.W.: Solve $\ln\left(\frac{1}{x}\right) = -2$.

H.W.: Solve $\log_5 5^{2x} = 8$.

H.W.: Solve $\ln\left(\frac{1}{x}\right) + \ln(2x^3) = \ln(3)$.