Metric Space:

Open sets: Let (X,d) be a metric space . A subset G of X is said to be d-open iff to each $x \in G$ there exist r >o such that $S(x,r) \subset G$.

Def^n: Let (X,d) be a metric space, and let $x_o \in X$ if $r \in R^+$ then the set $\{x \in X;\ d(x,x_o) < r\}$ is called an open sphere (or open ball).the point x_o is called the center and r the radius of the sphere. and we denoted by $S(x_o,r)$ or by $B(x_o,r)$: i.e. $S(x_o,r)=\{ x_o \in X;\ d(x,x_o) < r\}$

Closed set is define and denoted by $S[x_o,r]=\{ x_o \in X;\ d(x,x_o) \leq r\}$.

Ex: Let $x \in R$ then a subset N of R is U nbd of x iff there exist a u-open set G such that $x \in G \subset N$, but G is U-open and $x \in G$ implies that there exist an $\delta>0$ such that $(x-\delta,x+\delta) \subset G$. Thus N is a U-nbd of x if N contains an open interval $(x-\delta,x+\delta)$ for some $\delta>0$. In particular every open interval containing x is a nbd of x.

Ex: Consider the set R of all real numbers with usual metric space $d(x,y) = |x-y|$ and find whether or not the following sets are open.

A= (0,1) ,B=[0,1), C= (0,1] ,D=[0,1], E= $(0,1)\cup(2,3)$, F={1}, G={1,2,3} .

Sol^n : A is open set Let x be appoint in A, we take r=min{x-0,1-x}, then it is evident that $(x-r, x+r) \subset A$

For example consider $\frac{1}{4} \in (0,1)$, then $r=\min\{\frac{1}{4}-0,1-\frac{1}{4}\}=\min\{\frac{1}{4},\frac{3}{4}\}=\frac{1}{4}$

$(\frac{1}{4}-\frac{1}{4},\frac{1}{4}+\frac{1}{4}) =(0, \frac{1}{2}) \subset (0,1)=A$.

B is no open set, since however small we choose a positive number r, the open interval (0-r,0+r) = (-r,r) is not contained in B. Thus there exists no open ball with 0 as centre and contained in B .

Theorem 1: In a metric space the intersection of a finite number of open sets is open.

Proof: Let (X,d) be a metric space and let $\{G_i ; i=1,2,3,\ldots,n\}$ be a finite collection of open subsets of X, to show that $H=\cap\{G_i; i=1,2,3,\ldots n\}$ is also open. let $x\in G_i$ for every $i=1,2,3,\ldots n$, since each G_i is open there exist $r_i>0$ such that $S(x,r_i)\subset G_i$ $i=1,2,3,\ldots n$. let $r=\min\{r_1,r_2,r_3,\ldots,r_n\}$, then $S(x,r)\subset S(x,r_i)$ for all $i=1,2,3,\ldots n$, it follows that $S(x,r)\subset G_i$, for all $i=1,2,3,\ldots,n$, this implies that $S(x,r)\subset\cap\{G_i, i=1,2,3,\ldots,n\}=H$, thus it is shown that to each x in H there exist $r>0$, such that $S(x,r)\subset H$. Hence H is open.

Theorem 2: In a metric space the union of an arbitrary collection of open set is open.

Proof: let (X,d) be a metric space and let $\{G_\lambda ; \lambda\in\Delta\}$ be an arbitrary collection of open subset of X, to show that $G=\cup\{G_\lambda : \lambda\in\Delta\}$ is open, let $x\in G$, then by def^n of union $x\in G_\lambda$ for some $\lambda\in\Delta$, since G_λ is open there exists $r>0$ such that $S(x,r)\subset G,\ but\ G_\lambda\subset G,\ hence\ S(x,r)\subset G$, thus we have shown that to each $x\in G$, there exists a positive numbers r such that $S(x,r)\subset G$,hence G is open

Theorem 3: A subset of a metric space is open iff it is the union of family of open ball.

Proof: Let (X,d) be a metric space and $A\subset X$, let A be open ,if $A=\phi$, then it is The union of empty family of ball , now let $A\neq\phi$, and $x\in A$, since A is

open, there exist an open ball B(x,r), r>0 such that B(x,r) $\subset$ A, it follows that A $\subset$ {B(x,r), x in A} $\subset$ A. Hence A=$\bigcup\{B(x,r), x \in A\}$

So A is the union of a family of open ball.

Conversely if A is the union of a family of open ball then A is open by Theorem 2.

Ex: Show that in a discrete metric space every set is open.

Sol^n: Let A be a subset of discrete metric space if A=ϕ, then A is open, if A$\neq\phi$, let x$\in$A, since S(x,$\frac{1}{2}$)={x}, we have S(x,$\frac{1}{2}$) $\subset$ A. Hence A is open.

Ex: Show that in a metric space, the complement of every singleton set is Open . More generally the complement of a finite set is open.

Sol^n: H.W

Ex: Give an example to show that the intersection of an infinite number of open sets is not open.

Sol^n: Consider the collection {(-$\frac{1}{n}$,$\frac{1}{n}$), $n \in N$} of open intervals in R with usual metric d(x,y)=$|x-y|$, then$\bigcap$ {(-$\frac{1}{n}$,$\frac{1}{n}$), $n \in N$}={0}, which is not open since there exist not r>0 such that (-r,r) $\subset$ {0}.

Closed sets:

Def^n: Let (X,d) be a metric space , a subset A of X is said to be closed iff the complement of A is open.

Ex: Show that every singleton set in R is closed for the usual metric d for R.

Sol^n: Let a$\in$R, to show that {a} is closed. Now R-{a}=(-∞,a)$\cup$(a,∞), but (-∞,a) and (a,∞) are open sets, hence their union is also open.

Theorem 4: Let (X,d) be a metric space and let $\{H_\lambda; \lambda \in \Delta\}$ be an arbitrary

collection of closed subsets of X. then $\bigcap\{H_\lambda;\lambda\in\Delta\}$ is also a closed set. In other words, the intersection of an arbitrary family of closed sets is closed.

Proof: H_λ *is closed*, $\forall\lambda\in\Delta$,

then X-H_λ *is open*, $\forall\lambda\in\Delta$,

then $\cup$ {X-H_λ, $\forall\lambda\in\Delta$ }is open by theorem

then X-$\bigcap$ { H_λ, $\forall\lambda\in\Delta$ } is open De-Morgan

then $\bigcap$ { H_λ , $\forall\lambda\in\Delta$ } is closed.

Topologies:

Def^n: Let X be anon empty set and let π be a collection of subsets of X satisfying the following three condition:

T_1: $\phi\in\pi,\quad X\in\pi$.

T_2: if $G_1\in\pi$ *and* $G_2\in\pi$ *then* $G_1\bigcap G_2\in\pi$.

T_3 : *If* $G_\lambda\in\pi$ *for every* $\lambda\in\Delta$ *where* Δ *is arbitrary set then* $\bigcup\{G_\lambda;\lambda\in\Delta\}$

Then π is called a topology for X, the members of π are called π-open sets and the pair (X,π) is called a topological space.

Ex: Show that the union of empty collection of sets is empty i.e. $\bigcup\{A_\lambda,\lambda\in\phi\}=\phi$ and the intersection of empty collection of subsets of X is X itself i.e. $\bigcap\{A_\lambda,\lambda\in\phi\}=X$

Ex: Let X={a,b,c}, and consider the following collections of the subset of X:

$1-\pi_1=\{\phi,X\}$

$2-\pi_2=\{\phi,\{a\},\{b,c\},X\}$

$3-\pi_3=\{\phi,\{a\},\{b\},X\}$

$4-\pi_4=\{\phi,\{a\},X\}$

$5-\pi_5=\{\phi,\{a\},\{b\},\{a,b\}X\}$

$6-\pi_6=\{\{b\},\{a,c\},X\}$

$7-\pi_7=\{\phi,\{a,b\},\{b,c\},X\}$

$8-\pi_8=\{\phi,\{b\},\{b,c\},X\}$

Let we verify these axioms for π_8,

$T_1:\phi\in\pi_8\ ,\ X\in\pi_8$

$T_2:\phi\cap\{b\}=\phi\cap\{a,b\}=\phi\cap X=\phi\in\pi_8$

$\{b\}\cap\{a,b\}=\{b\}\cap X=\{b\}\in\pi_8$

$\{a,b\}\cap X=\{a,b\}\in\pi_8$

$T_3:\phi\cup\{b\}=\{b\}\ ,\ \phi\cup\{a,b\}=\{a,b\}\ \ \phi\cup X=X\ \ \{b\}\cup\{a,b\}=\{a,b\}$
$\{a,b\}\cup X=\{a,b\}\qquad\{b\}\cup\{a,b\}\cup X=X$ } All are in π_8 .

So π_8 is a topology on X.

Theorem 5: Every metric space is a topological space, but the converse is not true .

Proof: Let (X,d) be any metric space to prove that X, ϕ is open set.

$Let\ x\in X\ then\ \exists B_r(x)\ such\ that\ B_r(x)\subseteq X\ so\ X\ is\ open$

$If\ x\in\phi\rightarrow\exists B_r(x)\ such\ that\ B_r(x)\subset\phi\rightarrow\phi\ is\ open$

Let A,B be an open sets, to prove that A∩B is open,

$Let\ x\in A\cap B\rightarrow x\in A\ and\ x\in\mathbf{B}\rightarrow\exists B_r(x)\subset A\ and\ B_s(x)\subset B\ Let\ i=\min\{r,s\}\ so$
$B_i(x)\subset B_r(x)\cap B_s(x)\subset A\cap B\quad so\ A\cap B\ is\ open$

$Let\ \{A_i\ ;i\in I)\ be\ a\ faimly\ of\ open\ set\ to\ prove\ that\ \bigcup_{i\in I}A_i\ is\ open$

$Let\ x\in\bigcup_{i\in I}A_i\ then\ \exists i\in I\ such\ that\ x\in A_i\rightarrow\exists B_{r_i}(x)\subset A_i\rightarrow B_{r_i}(x)\subset A_i\subset\bigcup_{i\in I}A_i$

$\therefore\bigcup_{i\in I}A_i\ \ is\ open\ set.$

But the converse is not true for example let X={a,b,c} and

π={ϕ,{a},X}, suppose hat d is a metric of X , ρ=d(a,b) but $B_\rho(b)=\{b\}$

Which is not open.

Ex: Let X be any set. Then the collection I= { ϕ,X) consisting of empty set and the whole space. Is always a topology for X called the indiscrete or (trivial) topology, the pair (X,I) is called an indiscrete topological space.

Ex: Let D be the collection of all subsets of X, then D is a topology for X called the discrete topology.

Sol^n: Since $\phi \subset X, X \subset X, we\, have \phi \in D, and\ X \in D$ so that T_1 satisfied.

T_2 : Also holds since the intersection of two subset of X is a gain a subset of X.

T_3: Is satisfied since the union of any collection of subset of X is again a subset of X.

Ex : Let R be the set of all real numbers and let S consist of subsets of R defined as follows:

i- $\phi \in S$ ii- A non-empty subset G of R belong to S iff to each $p \in G, \exists$ a right half open interval [a,b) where a,b are in R, a<b such that $p \in [a,b) \subset G$ show hat S is a topology for R called the lower limit topology or in short RHO topology for R.

Sol^n ; T_1: $\phi \in S\ also\ R \in S$ since to each p∈R there exists aright half-open interval [p,p+ε), $\varepsilon > 0,\ such\, that\ p \in [p, p+\varepsilon) \subset R$

T_2: Let $G_1, G_2 \in S, and\ Let\ p \in G_1 \cap G_2, then\ p \in G_1\ and\ p \in G_2$ so there exists a right half-open intervals H_1 and H_2 such that $p \in H_1 \subset G_1\ and\ p \in H_2 \subset G_2$, it follows that $p \in H_1 \cap H_2 \in G_1 \cap G_2, \sin ce\, H_1 \cap H_2 \neq \phi$ so its clear that $H_1 \cap H_2$ is a right half-open intervals, thus to each $p \in G_1 \cap G_2$, there exist a right half-open interval $H_1 \cap H_2, such\, that\ p \in H_1 \cap H_2 \subset G_1 \cap G_2$, hence $G_1 \cap G_2 \in S$.

T_3: Let $G_\lambda \in S,\ \forall \lambda \in \Delta$ where Δ is an arbitrary set, let $p \in \bigcup\{G_\lambda ; \lambda \in \Delta\}$. Then there exist $\lambda_p \in \Delta\, such\, that\, p \in G_{\lambda_p}.since\ G_{\lambda_p}$ is S-open, there is a right half-open intervals H such that $p \in H \subset G_{\lambda_p}$. it follows that $p \in H \subset \bigcup\{G_\lambda ; \lambda \in \Delta\}$.

Hence $\bigcup\{G_\lambda ; \lambda \in \Delta\} \in S$. Thus S is a topology for R.

Similarly the upper limit topology for R consist of ϕ and all those subset G of R having the property that to each $p \in G$ there exist a left half- open interval (a,b] such that $p \in (a,b] \subset G$.

Ex: let π be the collection of subsets of N consisting of empty set ϕ and all subset of N of the form G_m ={m,m+1,m+2, …}, m in N show that π is a topology for N, what are the open sets containing 5.

Sol^n : $T_1 ; \phi \in \pi\ and\ A_1 = \{1,2,3,...\} = N \in \pi$

$$T_2 : Let\, G_m \in \pi\, and\, G_n \in \pi\ , m,n \in N, then\, G_m \cap G_n = \begin{cases} G_n & as\ m > n \\ G_m & as\, n < m \end{cases}\ hence\, G_m \cap G_n \in \pi$$

$T_3 : G_\lambda \in \pi\, \forall \lambda \in \Delta$ where Δ is arbitrary subset of N, since N is a well ordered Set (prove that) Δ contains a smallest positive integer m_0 so that $\bigcup\{G_\lambda : \lambda \in \Delta\} = \{m_0, m_0+1, m_0+2,...\} = G_{m_0} \in \pi$, hence π is a topology for N.

G_1 =N ={1,2,3,…}, G_2={2,3,4,…}, G_3 ={3,4,5,6,…} G_4={4,5,6,…} G_5={5,6,7,8,…}

Note: A partially ordered set X is said to be well ordered if every subset of X contains a first element.

Partial ordered set the pair(x,≤) is called p.o. set if x≤y for x,y in X If $a \in X$ be such that $a \le x\, \forall x \in,\ then\, a\, is\, a\, first\, element\, of\ X$.

Ex: List all possible topologies for the set X={a,b,c}.

Ex: Let U consist of ϕ and all those subsets G of R having the property that to each $x \in G$ there exist $\varepsilon > 0 \, such \, that \, (x-\varepsilon, x+\varepsilon) \subset G$ to show that U is a topology for R called the usual topology.

Sol^n : T_1 - $\phi \subset U \, by \, definition \, also \, R \in U, \sin ce \, to \, each \, x \in R \, (x-1, x+1) \subset R, In \, fact \, for \, any \, \varepsilon > 0$

$(x-\varepsilon, x+\varepsilon) \subset R$

T_2 : Let $G_1, G_2 \in U$, if $G_1 \cap G_2 = \phi$ there is nothing to prove if $G_1 \cap G_2 \neq \phi$, let

$x \in G_1 \cap G_2$ then $x \in G_1 \, and \, x \in G_2, hence \, \exists \varepsilon_1 > o, \varepsilon_2 > o \, such \, that \, (x-\varepsilon, x+\varepsilon) \subset G_1$

$(x-\varepsilon, x+\varepsilon) \subset G_2 \;\; take \, \varepsilon = \min\{\varepsilon_1, \varepsilon_2\}, then \, \varepsilon > 0 \, and \, (x-\varepsilon, x+\varepsilon) \subset G_1 \cap G_2, hence \, G_1 \cap G_2 \subset U.$

T_3: Let $\{G_\lambda ; \lambda \in \Delta\}$ be an arbitrary collection of members of U an let

$x \in \bigcup\{G_\lambda ; \lambda \in \Delta\}, then \, x \in G_\lambda \;\; for \, some \, \lambda \in \Delta, \sin ce \, G_\lambda \in U \, \exists \varepsilon > 0 \, such \, that \, (x-\varepsilon, x+\varepsilon) \subset G_\lambda$

But $(x-\varepsilon, x+\varepsilon) \subset \bigcup\{G_\lambda : \lambda \in \Delta\}$, therefore $\bigcup\{G_\lambda ; \lambda \in \Delta\} \in U$, so U is a topology for R.

Comparison of topology:

Def^n : Let π_1 and π_2 be two topologies for a set X , we say that π_1 is weaker or (smaller) than π_2 or that π_2 is stronger or (Larger) than π_1 iff $\pi_1 \subset \pi_2$ that is iff ev ery π_1 –open is π_2-open, if either $\pi_1 \subset \pi_2 \; or \, \pi_2 \subset \pi_1$ we say that the topologies π_1 and π_2 are comparable. If $\pi_1 \not\subset \pi_2$ and $\pi_2 \subseteq \pi_1$, then we say that π_1 and π_2 are not comparable.

For any set X , (X.I) is weaker topology and (X,D) is stronger topology.

Ex : Find three mutually non comparable topologies for the set X={a,b,c}

Sol^n : Let $\pi_1 = \{\phi, \{a\}, X\}$ $\pi_2 = \{\phi, \{b\}, X\}$, $\pi_3 = \{\phi, \{c\}, X\}$ Also from the following topology $\pi_1 = \{\phi.\{a\}.X\}$, $\pi_2 = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$, $\pi_3 = \{\phi, \{b\}, \{b,c\}, X\}$, we see that π_1 and π_3 are not comparable since $\pi_1 \not\subset \pi_3 \, and \, \pi_3 \not\subset \pi_1$ but π_1 and π_2 are comparable.

Intersection and union of topologies:

The union of two topology need not be a topology for example Let X={a,b,c}, consider two topology defined on X as follows $\pi_1 = \{\phi, \{a\}, X\}$, $\pi_2 = \{\phi, \{b\}, X\}$, then which is not topology for X

Theorem 6: Let $\{\pi_\lambda ; \lambda \in \Delta\}$ where λ is an arbitrary set be a collection of topologies for X then the intersection $\cap\{\pi_\lambda ; \lambda \in \Delta\}$ is also a topology for X.

Proof: Let $\{\pi_\lambda : \lambda \in \Delta\}$be a collection of topologies for X, we have to show that $\cap\{\pi_\lambda : \lambda \in \Delta\}$ is also a topology for X , if $\Delta = \phi$, then $\cap\{\pi_\lambda : \lambda \in \Delta\}$=P(X). Thus in this case the intersection of topologies is the discrete topology. Now let $\Delta \neq \phi$,

T_1 : since $\pi_\lambda : \forall \lambda \in \Delta$ is a topology, it follows that $\phi, X \in \pi_\lambda ; \forall \lambda \in \Delta$, but

$\phi \in \pi_\lambda , \forall \lambda \in \Delta, then\ \phi \in \cap\{\pi_{\lambda,}, \lambda \in \Delta\}$ and $X \in \pi_\lambda\ \forall \lambda \in \Delta\ then\ X \in \cap\{\pi_\lambda ; \lambda \in \Delta\}$

T_2 : Let $G_1, G_2 \in \cap\{\pi_\lambda ; \lambda \in \Delta\}$ then $G_1, G_2 \in \pi_\lambda ; \forall \lambda \in \Delta, since\ \pi_\lambda\ is\ a\ topology\ for\ X\ \forall \lambda \in \Delta$

It follows that $G_1 \cap G_2 \in \pi_{\lambda,} ; \forall \lambda \in \Delta$, hence $G_1 \cap G_2 \in \cap\{\pi_\lambda ; \lambda \in \Delta\}$.

T_3: Let $G_\alpha \in \cap\{\pi_\lambda ; \lambda \in \Delta\}$, $\forall \lambda \in \Delta$ where Δ is an arbitrary set, then $G_\alpha \in \pi_\lambda ; \forall \lambda \in \Delta, and\ \forall \alpha \in \Delta$, since for each π_λ is a topology for X, it follows that $\cup\{G_\alpha ; \alpha \in \Delta\} \in \pi_\lambda ; \forall \lambda \Delta$. Hence $\cup\{G_\alpha ; \alpha \in \Delta\} \in \cap\{\pi_\lambda ; \lambda \in \Delta\}\ thus \cap\{\pi_\lambda ; \lambda \in \Delta\}$ is a topology for X.

Closed sets:

Def^n : Let (X, π) be a topological space, a subset F of X is said to be π-closed Iff its complement F^c is open.

Ex: Let X={a,b,c}, and let π={ϕ,{a},{b,c},X} since $\{a\}^c$ ={b,c}, $\{b,c\}^c$ ={a}

It follows that the closed sets are ϕ, {a},{b,c}, and X.

Def^n : A topological space (X,π) is said to be a door space iff every subset of X is either open or closed. For example let X={a,b,c} and

$\pi=\{\phi,\{b\},\{a,b\},\{b,c\},X\}$ then the closed sets are $X,\{a,c\},\{c\},\{a\},\phi$.

Hence all the subsets of X are either open or closed and consequently (X,π) is a door space.

Ex: If $a \in R$ show that {a} is closed set in the usual topology for R.

Sol^n : $\{a\}^c = (-\infty, a) \cup (a, \infty)$ but $(-\infty,a)$ and (a,∞) are open sets hence their union is also open, it follows that $\{a\}^c$ is open , therefore {a} is closed.

Intersection and union of closed sets:

Theorem7 : If $\{F_\lambda ; \lambda \in \Delta\}$ is any collection of closed subsets of a topological space X, then $\cap\{F_\lambda ; \lambda \in \Delta\}$ is closed set.

Proof : F_λ *is closed* $\forall \lambda \in \Delta$ *then* $F^c{}_{\lambda\lambda}$ *is open* $\forall \lambda \in \Delta$ *then* $\bigcup\{F^c{}_\lambda : \lambda \in \Delta\}$ is open By T_3

$[\bigcap\{F_\lambda ; \lambda \in \Delta\}]^c$ *is open De – Morgan Law*

then $\bigcap\{F_\lambda ; \lambda \in \Delta\}$ *is closed by Def* ^{n} *of closed set.*

Theorem 8: if F_1 and F_2 b any two closed subsets of a topological space X Then $F_1 \cup F_2$ is a closed set .

Proof : F_1, F_2 *are closed* $\Rightarrow F_1{}^c, F_2{}^c$ *are open* $\Rightarrow F_1{}^c \cap F_2{}^c$ *is open by* T_2 *of Def* ^{n}

$(F_1 \cup F_2)^c$ *is open By De – Morgan law* $\Rightarrow F_1 \cup F_2$ *is closed.*

Note: $F_1,F_2,F_3,\ldots F_n$ be a finite number of closed subsets of X, then their union will also be a closed subset of X.

Ex : Give an example to show that the union of an infinite collection of closed sets in a topological space is not necessarily closed.

Sol^n : Let (R,U) be the usual topological space. And let $F_n=[1/n,1]$, $n \in N$. So that F_n is closed interval on R, then $[\frac{1}{n},1]^c = \{x \in R, x < \frac{1}{n} \text{ or } x > 1\} = (-\infty, \frac{1}{n}) \cup (1, \infty)$ which is open hence $[1/n,1]=F_n$ is closed set, Now

$\bigcup\{F_n, n \in N\} = \{1\} \bigcup [\frac{1}{2},1] \bigcup [\frac{1}{3},1] \bigcup \ldots = (0,1]$ since (0,1] is not closed it follows that the union of an infinite collection of closed sets is not necessarily closed.

Characterization of a topological space in terms of closed sets:

Theorem 9: Let X be anon-empty set $F_1, F_2 \in F \Rightarrow F_1 \bigcup F_2 \in F$

$$F_3 : F_\lambda \in F \quad \forall \lambda \in \Delta \Rightarrow \bigcap\{F_\lambda ; \lambda \in \Delta\} \in F$$

Then there exist a unique topology on X such that the π-closed subsets of X are precisely the members of F.

Proof: Let π consist of the complements of the members of F, then π is a topology for X.

T_1: $X \in F \Rightarrow X^c \in \pi \Rightarrow \phi \in \pi$ and $\phi \in F \Rightarrow \phi^c \in \pi \Rightarrow X \in \pi$

$$\begin{aligned} T_2 : \; & G_1, G_2 \in \pi \Rightarrow G_1^c, G_2^c \in F \\ & \Rightarrow G_1^c, G_2^c \in F \quad by\, F_2 \\ & \Rightarrow (G_1 \bigcap G_2)^c \in F \qquad by\, De-Morgan \\ & \Rightarrow G_1 \bigcap G_2 \in F \qquad by\, Def^n \end{aligned}$$

$$\begin{aligned} T_3 : \quad & G_\lambda \in \pi \quad \forall \lambda \in \Delta \\ & \Rightarrow G_\lambda^c \in F \; \forall \lambda \in \Delta \\ & \Rightarrow \bigcap\{G_\lambda^c ; \lambda \in \Delta\} \in F \quad by\, F_3 \\ & \Rightarrow [\bigcup\{G_\lambda ; \lambda \in \Delta\}]^c \in F \; De-Morgan \\ & so \bigcup\{G_\lambda : \lambda \in \Delta\} \in \pi \end{aligned}$$

Hence π is a topology for X.

further a subset F for X is closed iff $F^c \in \pi$, that is iff $F \in F$. to show the uniqueness of topology, let π and π⁻ be two topologies have the same system of closed sets.

$$\begin{aligned} &\text{then } G \in \pi \Leftrightarrow G \text{ is } \pi - \text{open} \\ &\Leftrightarrow G^c \text{ is } \pi - \text{closed} \\ &\Leftrightarrow G^c \text{ is } \pi^- - \text{closed } [\text{since } \pi \text{ and } \pi^- \text{ have the same system of closed sets}] \\ &\Leftrightarrow G \text{ is } \pi^- - \text{open} \\ &\Leftrightarrow G \in \pi^- \quad \text{hence } \pi = \pi^- \end{aligned}$$

Neighbourhoods:

Def^n : Let (X,π) be a topological space and let $x \in X$. A subset N of X is said to be a π-neighbourhood of x iff there exist a π-open set G such that $x \in G \subset N$. Similarly N is called a π-nbd of A subset of X iff there exist an open set G such that $A \subset G \subset N$. The collection of al nbd of in X is called the neighbourhood system at x and denoted by N(x).

EX : Let X={1,2,3,4,5} and let π={ϕ,{1},{1,2},{1,2,5},{1,3,4},{1,2,3,4}X}

then π-nbd of 1 are

{1},{1,2},{1,3},{1,4},{1,5},{1,2,3},{1,2,4},{1,2,5},{1,3,4},{1,3,5},{1,4,5},{1,2,3,4},{1,2,4,5},{1,3,4,5},{1,2,3,5},and X

Not that {1,3} is not an open set but it is a π-nbd of 1 since is a π-open set such that $1 \in \{1\} \subset \{1,3\}$

Ex: Which of the following subsets of R are nbd of 1?

(0,2),(0,2][1,2], [0,2]-1.5 , R

Theorem 10: A subset of a topological space are open iff it's a nbd of each its points.

Proof: Let a subset G of a topological space be open. Then for every x∈G, $x \in G \subset G$ and therefore G is a nbd of each its points.

Conversely let G be a nbd of its point, if G=ϕ, then there is nothing to prove, if x≠ϕ, then to each x∈G there exist an open set G_x such that $x \in G_x \subset G$. It

follows that $G = \bigcup\{G_x, x \in G\}$, hence G is open.

Ex: Let X be a t.s. If F is closed subset of X, and $x \in A^c$, prove that there is a nbd N of x such that $N \cap F = \phi$.

Sol^n: Since F is closed then F^c is open and so by above theorem F^c contains a nbd of each its points. Hence there exist a nbd N of x such that

$N \subset F^c$ *i.e.* $N \cap F = \phi$

Theorem 11: Let X be a topological space, and for any $x \in X$, Let $N_{(x)}$ be the collection of all nbds of x then:

1- $\forall x \in X,\ N(x) \neq \phi$, i.e. Every point x has at least one nbd.

2- $N \in N(x) \text{ then } x \in N(x)$, i.e. Every nbd of x contains x.

3- $N \in N(x), N \subset M \text{ then } M \in N(x)$ i.e. Every set containing a nbd of x is a nbd of x.

4- $N \in N(x), M \in N(x) \text{ then } N \cap M \in N(x)$, i.e. the intersection of two nbd of x is nbd of x.

5- $N \in N(x) \text{ then there exist } M \in N(x) \text{ such that } M \subset N \text{ and } M \in N(y)$. i.e. If N is a nbd of x, then there exist a nbd M of x which is a subset of N such that M is a nbd of each of its points.

Proof:1-Since X is an open set it is a nbd of every $x \in X$. Hence there exist at least one nbd (namely X)for each $x \in X$. Hence $N_{(x)} \neq \phi$ for all $x \in X$.

2-If $N \in N_{(x)}$, then N is a nbd of x, so by Def^n of nbd $x \in X$.

3- If $N \in N(x)$, there exist an open set G such that $x \in G \subset N$, since $N \subset M$, $x \in G \subset M$, and so M is a nbd of x, hence $M \in N(x)$.

4- Let $N \in N_{(x)}$ and $M \in N_{(x)}$, the by Def^n of nbd , there exist an open sets G_1 and G_2 such that $x \in G_1 \subset N \text{ and } x \in G_2 \subset M$ hence $x \in G_1 \cap G_2 \subset N \cap M$, since

$G_1 \cap G_2$, is an open set, it follows from (1) that $N \cap M$ is a nbd of x, hence $N \cap M \in N(x)$.

5-If N$\in$N$_{(x)}$, then there exist an open set M such that $x \in N \subset M$. Since M is open set it is a nbd of each of its point therefore $M \in N(y)\ \forall\, y \in M.$

Base for the neighbouhood system of a point ; Base for a topology

Local Base at a point.

Def^n: Let (X,π) be a topological space, a non-empty collection **B**(x) of π-neighborhoods of x is called a base for π-nbd system of x iff for every π-nbd N of x there is B$\in$ **B** (x) such that B$\subset$ N, we say that **B** (x) is a local base at x or a fundamental system of nbds of x. If **B**(x) is local base at x, then the members of **B**(x) are called basic π-nbds of x.

Ex: Let X= {a,b,c,d,e} and let $\pi = \{\phi, \{a\}, \{a,b\}, \{a,b,e\}, \{a,c,d\}, \{a,b,c,d\}, X\}$

Then the local base at each point a,b,c,d,e is given by **B**(a)={{a}}, **B**(b)={{a,b}}, **B**(c)={{a,c,d}}, **B**(d)={{a,,c,d}},**B**(e)={{a,b,e}}.

Ex : Let (X,π) be any topological space, and let x$\in X$, show that the collection B(x) of all π-open subset of X containing x is a local base.

Sol^n : Let N be any nbd of x. then there exist an open set G such that $x \in G \subset N$. since G is an open set containing x, $G \in \beta(x)$, this show that $\beta(x)$ is a local base at x.

Properties of local base:

Theorem 12: Let X be a topological space and let β(x) be a local base at any point x of X, then β(x) has the following properties.

B$_0$: β(x)$\neq \phi$ for every x in X.

B_1: If $B \in \beta(x)$ then $x \in B$

B_2: If $A \in \beta(x)$ and $B \in \beta(x)$ then $\exists$ s a $C \in \beta(x)$ such that $C \subset A \bigcap B$

B_3: If $A \in \beta(x)$ then $\exists$ s a set B such that $x \in B \bigcap \subset A$, and such that for every $y \in B$, $\exists$ s a set $C \in \beta(y)$ satisfying $C \subset B$

Proof: B_0- Since X is open , it is a nbd of its points, since β (x) is a local base at any point x of X, and X is a nbd of X, it follows that there must exist a $B \in \beta(x)$ such that $B \subset X$. Hence $\beta(x) \neq \phi \ \forall x \in X$..

B_1: If $B \in \beta(x)$, then B is a nbd of x, so by Def^n of nbd $x \in B$.

B_2:If $A \in \beta(x)$ then A is a nbd of x, similarly B is a nbd of x it follows that $A \bigcap B$ is a nbd of x, since $\beta(x)$ is a local base at x, it follow that there exist $C \in \beta(x)$ such that $C \in A \bigcap B$.

B_3: Since $A \in \beta(x)$, A is a nbd of x, hence there exist an open set B Such that $x \in B \subset A$, since B is an open set it's a nbd of every $y \in B$ Again since $\beta(y)$ is a local base at y and B is a nbd of every $y \in B$ It follows that for every $y \in B$ $\exists s, C \in \beta(y)$ such that $C \subset B$.

Ex : Consider the usual topology U for R and any point x$\in$R. then the collection $\beta(x) = \{(x-\varepsilon, x+\varepsilon); 0 < \varepsilon \in R\}$ constitutes a base for the U-nebd system for x, to prove this, let N be any nbd of x, then there exist U-nbd set G such that $x \in G \subset N$, since G is U-open there exist $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset G \subset N$, thus to each nbd N of x, there exist a member $(x-\varepsilon, x+\varepsilon) \in \beta(x)$ such that $(x-\varepsilon, x+\varepsilon) \subset N$

H.W/ Also show that $\beta(x) = \{(x-\frac{1}{n}, x+\frac{1}{n}), n \in N\}$ is anther local base for U-nbd

First countable space:

Def^n : A topological space (x,π) is said to satisfy the first axiom of count-

ability if each points of X possesses a countable locale base, such a topology is said to be a first countable space.

Ex: A discrete space (X,D) is a first countable, for in a discrete space every subset of X is open, in particular each singleton {x}, x$\in$ X is open and so is a nbd of x. Also every nbd N (i.e. open set containing x in this case) of x must be a superset of {x}.

hence the collection {{x}} consisting of the single nebd {x} of x, constitutes member is countable. Hence there exists a countable base at each point of X.

Ex : Show that the topological space (R,U) is first countable.

Sol^n : Let $x \in R$ then the collection $\{(x-\frac{1}{n}, x+\frac{1}{n}); n \in N\}$ is a countable base at x and so (R,U) is first countable.

Base for a topology:

Def^n: Let (X,π) be a topological space, a collection β of subsets of X is said to form a base for π iff:

$1-\beta \subset \pi \quad 2-For\,each\,Point\; x \in X\; and\; each\, nebd\; N\; of\; x\, \exists\, some\; B \in \beta\; such\, that\; x \in B \subset N$

Ex : Let X={a,b,c,d} and let $\pi = \{\phi, \{a\}, \{b\}, \{c,d\}\{a,b\}, \{a,c,d\}, \{b,c,d\}, X\}$, then the collection $\beta = \{\{a\}, \{b\}, \{c,d\}\}$ is a base for π since β$\subset$π and for each nbd of a contains {a} which is a member of β containing a. Similarly each nbd of b contains {b}$\in$β, and each of c or d contains {c,d}$\in$ β.

Ex : Consider the discrete space (X,D), then the collection β={{x}, x$\in$X} Consisting of all singleton subset of X is abase for D, since each singleton set is D-open so that $B \subset D$, also for each $x \in X$ and each nbd N of x, $\{x\} \in \beta,$ is such that $x \in \{x\} \subset N$

Def^n : Let (X,π) be a topological space the space X is said to be second countable (or to satisfy the second axiom of count-ability) if there exist a countable base for π.

Ex: The space (R.U) is second countable since the set of all open intervals (r,s) where r,s are rational numbers forms a countable base for U. This Follows from the fact that between any two real numbers there exists infinitely many rational numbers. thus to each point x in R and each nbd N of x $\exists r,s \in Q \, such \, that \, x \in (r,s) \subset$ N

Theorem 13:Let (x,π) be a topological space, a collection β of π is abase for π iff every π-open set can be expressed as the union of members of β.

Proof: Let β be a base for π and let G∈π, since G is π-open, it is a π-nbd of each of its point, hence by def^n of base to each x∈G there exist a member $B \in \beta \, such \, that \, x \in B \subset G$ it follows that $G = \bigcup\{B; B \in \beta \; and \; B \subset G\}$.

Conversely, Let $\beta \subset \pi$ and every open set G be the union of members of β , we have to show that β is abase for π, we have

i- $\beta \subset \pi \;\; given$

$i - Let \, x \in X \, and \, let \, N \, be \, any \, nebd \, of \; x, then \, \exists s \, an \, open \, set \, G \, such \, that \, x \in G \subset N$

But G is the union of members of β, hence there exists

$B \in \beta \, such \, that \, x \in B \subset G \subset N, thus \, \beta \, is \, a \, base \; for \, \pi$.

Ex: Let π and π* be topologies for X, which have a common base β then π=π*.

Sol^n; Let G∈π, and x∈G, since G is π-open, it is π-nbd of x ,, and since β is a base for π, there exists B∈β such that $z \in B \subset G$β. Since β is a base forπ* and B∈β, it follows that B∈π*. Hence G is π*-nbd of x, since x is arbitrary G∈π*

$Thus\ \pi \subset \pi^*,\ similarly\ we\ can\ prove\ \pi^* \subset \pi$, hence $\pi = \pi^*$

Properties of a base for a topology:

Theorem14: let (X,π) be a topological space and let β be a base for π, then B has the following properties:

[B^*_1] For every $x \in X$ there exists a $B \in \beta$ such that $x \in \beta$, i.e. $X = \bigcup\{B; B \in \beta\}$.

[B^*_2] For every $B_1 \in \beta$, $B_2 \in \beta$ and a point $x \in B_1 \cap B_2$ there exists a $B \in \beta$ such That $x \in B \subset B_1 \cap B_2$, that is the intersection of any two members of β is a union of members of β .

Proof: [B^*_1] since X is a π-open set it is a nbd of each of its points hence by def^n of base, for every $x \in X$, there exists some $B \in \beta$ such that

$x \in B \subset X,\ \ in\ other\ words\ X = \bigcup\{B, B \in \beta\}$

[B^*_2] If $B_1 \in \beta$ and $B_2 \in \beta$, then B_1 and B_2 are π-open, hence their intersection $B_1 \cap B_2$ is also π-open, and therefore $B_1 \cap B_2$ is a nbd of each of its points and so by def^n of base to each $x \in B_1 \cap B_2$ there exists $B \in \beta$ such that $x \in B \subset B_1 \cap B_2$, that is $B_1 \cap B_2$ is the union of members of β.

Limit points :

Def: Let (X, π) be a topological space , and let A be a subset of X , a point $x \in X$ is called a limit point (or a cluster point or an accumulation point) of A iff every nbd of x contains a point of A other than x.

i.e. x will be a limit point of A iff every nbd of x meets A in a point different from x , that is $N \setminus \{x\} \cap A \neq \phi$ for all N is and of x or we say that x is a limit point of A iff every open set G containing x , $G \setminus \{x\} \cap A \neq \phi$, also we say that x will not be a limit point of A if there exists a nbd N of x Such that $N \cap A = \phi$ or $N \cap A = \{x\}$.

Def: Let A be a subset of a topological space X, and let $x \in X$, the x is called an adherent point (or contact point) of A iff every nbd of x contains a point of A and denoted by d(A).

The set of all limit point of A is called derived set and denoted by D(A).

Def: A point x is said to be an isolated point of a subset A of a topological space X , if $x \in X$ but x is not a limit point of A . A closed set which has no isolated point is said to be perfect.

Ex: let (X.D) be descried topological space, and let A be any subset of X Is A has a limit point?

Sol: let $x \in X$, if $G\backslash\{x\} \cap A \neq \phi$ $N\backslash\{x\} \cap A \neq \phi$for every open set G containing x But we have $\{x\}\backslash\{x\} \cap A=\phi$, therefore x is not a limit point of A . Hence A has not a limit point in a descried topology.

Ex: let X={a,b,c} and let $\pi=\{\phi,X,\{a\},\{b\}\{a,b\}\}$ find all limit point of the set A ={a,c}.

Sol: we have three points in X

1- $a \in X$, the open set which contain a are {a}, {a,b} X so since $\{a,b\}\cap\{a\}\backslash\{a\}= \phi$, a is not a limit point of A.

2- $b \in X$, the open set which contain b are {b},{a,b}, X and $\{a,c\}\cap\{b\}\backslash\{b\}=\phi$ b is not a limit point of A.

3- $c \in X$, and the open set which contain c is X only, and $X\backslash\{c\}\cap A=\{c\}\neq\phi$, so c is a limit point of A, the isolated point of A is a, since a is in A and not a limit point , and D(A)={c}

Ex: let X={a,b,c,d,e} and let $\pi=\{\phi,X,\{b\},\{d,e\},\{b,d,e\},\{a,c,d,e\}\}$ then π is a topology on X. Consider the subset A={b,c,d} ,the point c is a limit

point of A since the π-open nbds of c are {a,c,d,e}, X each contains a point of A other than c. But b is not a limit point of A since {b} is nbd of b which contains no point of A other than b similarly a,e are limit point of A so D(A)={a,c,e}. The isolated points of A are b and d since b,d are belong to A but are not limit points of A. then an adherent point of A are a,b,c,d,e.

Theorem 15: Let X be a topological space, and let A be a subset of X then A is closed iff $D(A)\subset A$.

Proof: Let A be closed, then A^c is open and so to each $x\in A^c$ there exist a nbd N of x such that $N\subset A^c$. Since $A\cap A^c=\phi$, the nbd N contains no point of A and so x is not a limit point of A. Thus no point of A can be a limit point of A ,that is A contains all its limit points. Hence $D(A)\subset A$. Conversely let $D(A)\subset A$ and let $x\in A^c$, then $x\notin A$. since $D(A)\subset A$, $x\notin D(A)$ hence there exist a nbd of x such that $N\cap A=\phi$ so that $N\subset A^c$, thus A^c contains a nbd of each of its points and so A^c is open, that is A is closed.

Closure:

Def: Let X be a topological Space and let $A\subset X$. the intersection of all π-closed supersets of A is called the closure of A and denoted by A or c(A) or ClA. When confusion is possible as to what space the closure is to be take in, we shall Cl (A).

Theorem 16: Let A be a subset of a topological space , then

1- ClA is the smallest closed set containing A.

2- A is closed iff ClA=A

Proof: 1- this follows from definition.

2- If A closed, then A itself is the smallest closed set containing A and hence ClA=A. Conversely if ClA=A by 1 ClA is closed and so A is also closed.

Theorem 17: prove that ClA $=A \cup D(A)$.

Proof: We first prove that $A\cup D(A)$ is closed i.e.$[A \cup D(A)]^c = A^c \cap D(A)^c$ is open , let $x\in A^c \cap D(A)^c$, then $x\in A^c$ and $x\in D(A)^c$ so that $x\notin A$ and $x\notin D(A)$. This means that x is not a limit point of A, and hence there exist an open nbd N of x which contains no point of A, it follows that $N \subset A^c$. Now no point $y\in N$ can be a limit point of A, since N is a nbd of y which contains no point of A. hence $N\subset D(A)^c$. since $N\subset A^c$ and $N\subset D(A)^c$, So $N\subset A^c\cap D(A)^c$. thus $A^c\cap D(A)^c$ contains a nbd of each of its point and consequently $A^c\cap D(A)^c$ is open. We now show that ClA= $A\cup D(A)$,since $A\cup D(A)$ is closed set containing A and ClA is the smallest closed set containing A, we have $ClA\subset A\cup D(A)$. Again since ClA is closed, it contains all its limit points, and thus in particular, all limit points of A, so that $D(A)\subset ClA$ also $A\subset ClA$. Hence $A\cup D(A)\subset ClA$, it follows that $ClA=A\cup D(A)$.

Corollary: Prove that ClA=adh(A)={x; each nbd of x intersect A}

Proof: $x\in adh(A)$ iff every nbd of x intersects A

Iff $x\in A$ or every nbd of x contains a point of A other than x

Iff $x\in A$ or $x\in D(A)$

Iff $x\in A\cup D(A)$

Iff $x \in ClA$.

An adherent point is also some times called a closure point.

Ex: Let X={a,b,c,d} and let π={ϕ,X,{a},{b,c},{a,d},{a,b,c}}

Closed subsets are X, {b,c,d}, {a,d},{b,c},{d},then Cl{b}={b,c}, ϕ, since {b,c} is the intersection of all closed subsets of X which contain b . Again Cl{a,b}=X, since X is the only closed set containing {a,b}. similarly we have Cl{b,c,d}={b,c, d}.

Ex: Let X={a,b,c} and let $\pi = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ .Find the limit point of the sets A={b,c.} , B={a,c}

Properties of closure: "Kuratiwski theorem "

Let X be a topological space, and let A,B be any subset of X, then

$i - Cl\phi = \phi, \quad ii - A \subset ClA \quad iii - if\ A \subset B\ ,\ then\ ClA \subset ClB$

$iv - Cl(A \cup B) = ClA \cup ClB \quad v - Cl(A \cap B) \subset ClA \cap ClB \quad vi - Cl(ClA) = ClA$

Proof: i-Since ϕ is closed, we have $Cl\phi = \phi$.

ii- By theorem ClA is the smallest closed set containing A, so $A \subset ClA$

iii- By (ii) $B \subset ClB,\ \sin ce\ A \subset B$ we have $A \subset ClB$,but ClB is a closed set. Thus ClB is closed set containing A. Since ClA is the smallest closed set containing A, we have $ClA \subset ClB$.

$iv - Since\ A \subset A \cup B\ and\ B \subset A \cup B, we\ have\ ClA \subset Cl(A \cup B)\ and\ ClB \subset Cl(A \cup B)$
$by\ iii\ we\ have\ ClA \cup ClB \subset Cl(A \cup B)$ ..... (1)

Since ClA and ClB are closed sets, then $ClA \cup ClB$ is also closed, also $A \subset ClA\ and\ B \subset ClB$ implies that $A \cup B \subset ClA \cup ClB\ thus\ ClA \cup ClB$ is closed set containing $A \cup B$,since $Cl(A \cup B)$ is the smallest closed set Containing $Cl(A \cup B) \subset ClA \cup ClB$ .....2 , from 1and 2 we get

$Cl(A \bigcup B) = ClA \bigcup ClB$.

v- $A \bigcap B \subset B$ $then Cl(A \bigcap B) \subset ClB$ and $A \bigcap B \subset A then Cl(A \bigcap B) \subset ClB$. Hence

$Cl(A \bigcap B) \subset ClA \bigcap ClB$

vi-Since ClA is closed, we have Cl(Cl(A)).

Theorem 18: Let X be a topological space, and let A be a subset of X then the following statements are equivalent:

i- A is closed ii- ClA=A iii-A contains all its limit point.

Ex: Consider the usual topological space and find the closure of the following subsets of R.

i-A={ $\frac{1}{n}, n \in N$} ii- B=The set of all integer numbers ,

iii-C= The set of all rational number, iv- D=$\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...\}$

Interior point and interior set:

Def^n : Let X be a topological space and let $A \subset X$, a point x in X is said to be an interior point of A iff A is a nbd of X, that is iff there exists an open set G such that $x \in G \subset A$, the set of all interior point of A is called the interior of A and is denoted by A^0 or IntA

Theorem 19: $A^\circ = \{G : G \, is \, open, G \subset A\}$

Proof:

$x \in A^\circ$ iff $A is a nbd of$ x

iff $ther exsit an open set G suh that x \in G \subset A$

iff $x \in \bigcup\{G; G is open, G \subset A\}$

$Hence A^\circ = \bigcup\{G; G is open, G \subset A\}$

Theorem 20: Let X be a topological space. And let A be a subset of X, then

i- IntA is an open set.

ii- IntA is the largest open set contained in A.

iii- A is open if IntA=A.

Proof: i- Let x be an arbitrary point of IntA, Then x is an interior point of A Hence by Def^{n} , A is a nebd of x, then there exist an open set G such that $x \in G \subset A.$ Since G is open, it is a nbd of each of its points and so A is also a nbd of each of G. It follows that every pint of G is an interior point of A so that $G \subset IntA$, thus it is shown that to each point $x \in IntA$ there exist an open set G such that $x \in G \subset IntA$, hence IntA is a nbd of each of its points and consequently IntA is open.

ii-Let G be any open subset of A and let $x \in G, \; so\, that\, x \in G \subset A$ since G is open, A is a nbd of x and consequently x is an interior point of A, hence $x \in IntA$, thus we have shown that $x \in G \Rightarrow x \in IntA, and\, so\, G \subset IntA \subset A$. Hence IntA contains every open subset of A and it is therefore the largest open subset of A.

iii-Let A=IntA By(i) IntA is an open set and therefore A is also open. Consequently let A be open. Then A is usually identical with the largest open subset of A. but by (ii) IntA is the largest open subset of A. Hence A=IntA

Ex: Let (X,D) be s discrete topological space and let A be any subset of X. Since A is open, we have IntA=A, thus in a discrete space every subset of X coincides with its interior.

Theorem 21: Let X be a topological space and let A be a subset of X. Then IntA equals the set all those points of A which are not limit pints of A^{c}

Proof: Let x be a point of A, which is not a limit point of $A^{c.}$ Then there exists a nbd N of x which contains no point of A^{c} , and so $N \subset A$ this implies that A is also a nbd of x and so $x \in IntA$. Conversely let

$x \in IntA$, since IntA is open, it is a nbd of x, also IntA contains no point of A^c , it follows that x is not a limit point of A^c , thus no point of IntA is a limit point of A^c , hence IntA consists of precisely those point of A which are not limit point of A^c.

Theorem 22 : Let X be a topological space, and let A,B be any subset of X, then:

$$i - IntX = X, Int\phi = \phi \qquad ii - IntA \subset A \quad iii - A \subset B \Rightarrow IntA \subset IntB$$
$$iv - Int(A \cap B) = IntA \cap IntB \quad v - IntA \cup IntB \subset Int(A \cup B) \quad vi - Int(IntA) = IntA$$

Proof : i- Since X and ϕ are open set, we have by iii Theorem IntX =X, Intϕ=ϕ .

ii- $x \in IntA \Rightarrow x \text{ is an interior point of } A \Rightarrow A \text{ is a nebd of } x \Rightarrow x \in A, \text{hence } A = IntA$

iii-Let $x \in IntA$,then x is an interior point of A, and so A is a nbd of x, since $A \subset B$, B is also a nbd of x, this implies that $x \in IntB$ thus we shown that $x \in IntA \Rightarrow x \in IntB,\ IntA \subset IntB$

iv-Since $A \cap B \subset A \text{ and } A \cap B \subset B \text{ we have by iii } Int(A \cap B) \subset IntA \text{ and } Int(A \cap B) \subset IntB$

this implies that $Int(A \cap B) \subset IntA \cap IntB$ …….(1)

a gain let $x \in IntA \cap IntB. \text{Then } x \in IntA \text{ and } x \in IntB$, hence x is an interior point of each of the sets A and B, it follows that A and B era nebds of x so that their intersection $A \cap B$ is also a nebd of x, hence $x \in Int(A \cap B)$ thus $x \in IntA \cap IntB \Rightarrow x \in Int(A \cap B)$ so

$IntA \cap IntB \subset Int(A \cap B)$ …….(2)

From 1 and 2 we get $Int(A \cap B) = IntA \cap IntB$

$v -$ By(iii) $A \subset A \cup B \Rightarrow IntA \subset Int(A \cup B)$

$B \subset A \cup B \Rightarrow IntB \subset Int(A \cup B)$

$\text{hence } IntA \cup IntB \subset Int(A \cup B)$

Not that in general $IntA \cup IntB \neq Int(A \cup B)$

For example Let A=[0,1) and [1,2) then IntA=(0,1) and IntB=(1,2)

$IntA \cup IntB = (0,1) \cup (1,2) = (0,2) \setminus \{1\}$ also $A \cup B$=[0,1) $\cup$[1,2)=[1,2]

So Int($A \cup B$)=(1,2)

Thus in this case Int$A \cup$ IntB is a proper subset of Int($A \cup B$), and

Int$A \cup$ Int$B \neq$ Int($A \cup B$)

vi-Now by i of Theorem 20 IntA is an open set, hence by iii of the same theorem Int(IntA)=IntA

Exterior point and the exterior of a set:

Def^n : Let A be a subset of a topological space X, A point $x \in X$ is said to be an exterior point of A iff it is an interior point of A^c, that is there exist an open set G such that $x \in G \subset A^c$ or equivalently $x \in G \, and \, G \cap A = \phi$. The set of all exterior points of A is called the exterior of A and is denoted by extA or e(A). thus extA=Int(A^c), it follows that $ext(A^c) = [A^{c^{c^c}}]^0 = A^0$ also we have $A \cap extA = \phi$, that is no point of A can be exterior point of A.

Remark: Since extA is the interior of A^c, it follows from Theorem 20 that extA is open and is the largest open set contained in A^c .

Theorem 23: Let (X,π) be a topological space and let A be a subset of X then

$$extA = \bigcup\{G \in \pi, G \subset A^c\}$$

Proof: By Def^n , extA=Int(A^c), but by Theorem 19

$$IntA^c = \bigcup\{G \in \pi; G \in A^c\} \quad hence \; extA = \bigcup\{G \in \pi; G \subset A^c\}$$

Theorem 24: Let A be a subset of a topological space X, then a point x in X is an exterior point of A iff x is not an adherent point of A, that is iff $x \in ClA^c$.

Proof : let x b an exterior point of A, then x is an interior point of A^c, so A^c

is a nbd of x containing no point of A, it follows that x is not an adherent point of A, that is $x \in ClA^c$.

Conversely, suppose that x is not an adherent point of A, then there exist a nbd N of x which contains no points of A. This implies that $x \in N \subset A^c$. It follows that A^c is a nbd of x and consequently x is an interior point of A^c, that is x is an exterior point of A.

Theorem 25: Let X be a topological space and let A and B be subsets of X. Then:

$$i - extX = \phi,\ exr\phi = X \quad ii - extA \subset A^c \quad iii - extA \subset ext[(extA)^c] \quad iv - A \subset B \Rightarrow extB \subset extA$$

$$v - IntA \subset ext(extA) \quad iv - ext(A \bigcup B) = extA \bigcap extB$$

Proof: $i - extX = IntX^c = Int\phi = \phi \qquad ext\phi = Int\phi^c = IntX = X$

$$ii - extA = IntA^c \subset A^c \qquad by\ ii\ Theorem\ I_4$$

$$\begin{aligned} iii - ext[ext(A^c)] &= ext[IntA^c]^c = ext(IntA^c)^c = Int\{[IntA^c]^c\}^c \\ &= Int(IntA^c) \qquad \{by\ A^{c^c} = A\} \\ &= IntA^c \qquad \{by\ Int(IntA) = IntA\} \\ &= extA \end{aligned}$$

$$iv - A \subset B \Rightarrow B^c \subset A^c \Rightarrow IntA^c \subset IntB^c \Rightarrow extB \subset extA$$

$$v - By\ ii\ we\ have\ extA \subset A^c\ then\ iv\ gives\ extA^c \subset ext(extA),\ But\ IntA = extA^c$$
$$hence\ IntA \subset ext(extA)$$

$$\begin{aligned} vi - ext(A \bigcup B) &= Int[(A \bigcup B)^c] \\ &= Int(A^c \bigcap B^c) \qquad By\ Demorganlaw \\ &= IntA^c \bigcap IntB^c \qquad \mathbf{B}y\ iv\ Theorem\ I_4 \\ &= extA \bigcap extB \end{aligned}$$

Frontier point and the frontier of a set.

Def^n :A point x of a topological space is said to be a frontier point (or boundary point) of a subset A of X iff it is neither an interior nor an exterior point of A. the set of all frontier points of A is called the frontier of A and shall be denoted by FrA.

FrA=[IntA $\cup$ extA]^c

Theorem 26: Lt X be a topological space and let A be a subset of X. then a Point x in X is a frontier point of A iff every nbd of x intersections both A and A^c .

$$\begin{aligned}\text{Proof: We have } x \in FrA &\Leftrightarrow x \notin IntA \text{ and } x \notin extA = IntA^c \\ &\Leftrightarrow \text{neither } A \text{ nor } A^c \text{ is a nebd of } x \\ &\Leftrightarrow \text{no nebd of } x \text{ can be containd in } A \text{ or in } A^c \quad \text{why?} \\ &\Leftrightarrow \text{every nebd of } x \text{ intersects both } A \text{ and } A^c\end{aligned}$$

Corollary: FrA=FrA^c . for we have

$$\begin{aligned}x \in FrA &\Leftrightarrow \text{every nebd of } x \text{ intersects both } A \text{ and } A^c \\ &\Leftrightarrow \text{every nebd of } x \text{ intersects both } A^c \text{ and } A^{c^c} \\ &\Leftrightarrow x \in FrA^c \qquad \text{since } A^{c^c} = A\end{aligned}$$

Theorem 27 : LetA be any subset of a topological space X. then IntA, extA and FrA are disjoint and $X = IntA \cup extA \cup FrA$ Further FrA is a closed set.

Proof: By Def^n extA=IntA^c , also $IntA \subset A \text{ and } IntA^c \subset A^c, \text{since } A \cap A^c = \phi, \text{it follows that}$

$IntA \cap extA = IntA \cap IntA^c = \phi$ a gain by Def^n of frontier, we have

$$\begin{aligned}x \in FrA &\Leftrightarrow x \notin IntA \text{ and } x \notin extA \\ &\Leftrightarrow x \notin \{IntA \cup extA\} \\ &\Leftrightarrow x \in [IntA \cup extA]^c\end{aligned}$$

$$\text{Thus } FrA = [IntA \cup extA]^c \quad \ldots\ldots\ldots(1)$$

$$\text{It follows that } FrA \cap IntA = \phi \text{ and } FrA \cap extA = \phi \text{ and } X = IntA \cup extA \cup FrA$$

Since IntA and extA are open, we see from 1 that FrA is closed.

Dense and non-dense sets:

Def^n : Let X be a topological space and let A,B be subset of X. then

i- A is said to be dense in B iff $B \subset ClA$

ii- A is said to be dense in X or every where dense iff ClA=X it follows that A is every where dense iff every point of X is an adherent point of A,

iii- A is said to be nowhere dense or non-dense in X iff Int(ClA)=ϕ, that is, iff interior of the closure of A is empty.

iv- A is said to be dense in itself iff $A \subset D(A)$.

It follows from Def^n (a closed set which has no isolated points is said to be perfect) and iv of a above definition that a set A is perfect iff A is dense in itself and closed. This implies that A is perfect iff A=D(A)

For A is perfect iff A is closed and A has no isolated points

iff A is closed and every point of A is a limit point of A

iff $D(A) \subset A$ and $A \subset D(A)$

iff A=D(A).

Separable space:

Def^n: A topological space is said to be separable iff X contains a countable Dense subset, that is, iff there exist a countable subset A of X such That ClA=X.

For example the usual topological space (R,U) is separable since the set Q of all rational numbers is countable dense subset of R.

Ex: Let X={a,b.c.d,e} and let π={ϕ,{b},{c,d},{b,c,d},{a,c,d},{a,b,c,d},X}.

Find interior, exterior and frontier of the following subset of X.

A={c} B={a,b} C={a,c,d} D={b,c,d}

Sol^n :1- i- since A is not a nebd of c, so $c \notin IntA$, hence IntA=ϕ

ii- Now A^c ={a,b,d,e} it is easy to see that b is an interior point of A^c , since A^c is a nebd of b , but a,d,e are not interior points of A^c , hence extA =b

iii- Since IntA=ϕ and extA=b it follows that FrA={a,c,d,e}.

2- i-Here b is an interior point of B , but a is not. IntB={b}.

ii- Now B^c ={c,d,e}, since c,d$\in \{c,d\} \subset B^c$, it follows that B^c is a nbd of c,d hence c,d are interior points of B^c. that is c,d are exterior points of B. that is extB={c,d}

iii-Since IntB={b}, and extB= {c,d} then FrB={a,e}

3- here C is open then IntC=C={a,c,d}, and extC= $IntC^c$=Int{b,e}={b} also FrC={e}.

4- Also D is open set so that it is a nbd of each of its points and consequently every point of D is its interior point, hence IntD=D={b,c,d}, D^c ={a,e}. Since thee exists no open set G such that $a \in G \subset D^c$, D^c is not a nbd of a hence $a \notin IntD^c$, similarly $e \notin IntD^c$. therefore extD=$IntD^c$= ϕ also FrD={a,e}.

Ex: If A is open and closed then FrA=ϕ

Sol^n: Since A is open then IntA=A and also since A is closed A^c is open and Ext A = $IntA^c$=A^c but $FrA = \{IntA \cup extA\}^c = \{A \cup A^c\}^c = X^c = \phi$

Ex: consider the usual topology U on R and find interior, exterior and frontier Of the following subset of R. A=(0,1) B=[0.1) C=[0,1] D=$\{\frac{1}{n}; n \in N\}$,N , Q

Sol^n:1- Since A is open, it is a nbd of each of its points and so every point of A is its interior point. Hence IntA=(0.1)

Now A^c=$(-\infty, o)\cup(1,\infty)$, here A^c is a nebd of each of its point except 0 and 1, hence extA =IntA^c=$(-\infty, o)\cup(1,\infty)$.

Also $FrA = \{IntA \cup extA\}^c = \{0,1\}$.

2- proceeding as in 1 we have IntB=(0,1) $extB = IntB^c = (-\infty,0)\cup(1,\infty)$ and

$FrB = \{IntB \cup extB\}^c = \{0,1\}$.

4- Here D cannot be a nbd of any points of its points 1/n , n=1,2,3,...

Since there exists no $\varepsilon > 0 \, such \, that \; (\frac{1}{n}-\varepsilon, \frac{1}{n}+\varepsilon) \subset D$, hence no point of D can be its interior point so that IntD=ϕ .

It is easy to see that D^c is a nbd of each of its points except 0, hence extD=IntD^c $[D\cup\{0\}]^c$

$FrA = [IntD \cup extD]^c = [D \cup \{0\}]$

Theorem 28 : Let X be a topological space and let A be a subset of X

$$ClA = IntA \cup FrA$$

Proof: By Def^n of ClA, we have $ClA = \cap\{F; F \, is \, closed \; A \subset F\}$,

then by De-Morgan law $[ClA]^c = \cup\{F^c; F^c \, is \, open \, and \; F^c \subset A\} = extA$, taking complements, we get $[(ClA)^c]^c = [extA]^c = IntA \cup FrA \, so \, that \, ClA = IntA \cup FrA$

Corollary: $ClA = A \cup FrA$

$Proof: Since \; A \subset ClA \; and \; FrA \subset ClA \; so \, that \; A \cup FrA \subset ClA$ .....(1)

$Also \, FrA = [IntA \cup extA]^c = [IntA]^c \cap [extA]^c \quad again \sin ce \; IntA \subset A \, and$
$ClA = IntA \cup FrA it \; follows that \; ClA \subset A \cup FrA$ .....(2)
$from 1 \, and \, 2 \, we \, get \; ClA = A \cup FrA$

Subspace:

Def^{n} : Let X be a topological space and let $Y \subset X$. The π-relative topology for Y is the collection π_Y given by $\pi_Y = \{G \cap Y ; G \in \pi\}$.

The topological space (Y, π_Y) is called a subspace of (X,π), the topology π_Y on Y is said to be induced by π.

Theorem29: Let (X,π) be a topological space and let $Y \subset X$, then the collection $\pi_Y = \{G \cap Y; G \in \pi\}$ is a topology on Y.

Proof : T_1: $Since \phi \in \pi \; and \; \phi \cap Y = \phi \Rightarrow \phi \in \pi_Y \; a\,gain Y \cap X = Y \sin ce Y \subset X \; and \sin ce X \in \pi$,

we have $Y \in \pi_Y$.

T_2: Let $H_1, H_2 \in \pi_Y$, $Then\, H_1 = G_1 \cap Y \; and \; H_2 = G_2 \cap Y \; for\; same\; G_1, G_2 \in \pi$.

Now $H_1 \cap H_2 = G_1 \cap Y \cap G_2 \cap Y = (G_1 \cap G_2) \cap Y \in \pi_Y$ $[Since G_1, G_2 \in \pi]$.

T_3: Let $H_\lambda \in \pi_Y; \forall \lambda \in \Delta, \; then \; \exists \; open\, set \; G_\lambda \; such\, that \; H_\lambda = G_\lambda \cap Y \;\; \forall \lambda \in \Delta$, now

$\bigcup\{H_\lambda; \lambda \in \Delta\} = \bigcup\{G_\lambda \cap Y; \lambda \in \Delta\} = \bigcup\{G_\lambda; \lambda \in \Delta\} \cap Y \in \pi_Y \;\; \sin ce \;\; \bigcup\{G_\lambda, \lambda \in \Delta\} \in \pi$

Hence π_Y is a topology for Y.

Ex: Let X={a,b,c,d,e}, π={ϕ,{a},{b},{a,b},{a,c},{a,b,c},{a,b,e}{a,b,d,e},X}

Y={b,c,e} then

$\pi_Y = \{\phi, \{b\}, \{c\}, \{b,c\}, \{b,e\}\}$

Def^n : Hereditarily property :

A property of a topological space is said to be hereditary if every subspace of the space has that property.

Ex: Consider the usual topology U of R and the subset [0,1] of R, then the set [0,1/2) is open in the U-relative topology of [0,1], since

$[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0,1] \, and \, (-1, \frac{1}{2})$ is U-open, similarly (3/4,1] is open in the U-relative Topology for [0,1], since $(\frac{3}{4}, 1] = (\frac{3}{4}, \frac{3}{2}) \cap [0,1] \; and \; (\frac{3}{4}, \frac{3}{2})$ is U-open.

Ex; Let U be the usual topology for R describe the relativization of U to the Set N of natural numbers.

Theorem 30: Let (Y,π_Y) be a sub-space of (X,π); then:

i- A subset A of Y is closed in Y iff there exists a set F closed in

X such that A=F $\cap$ Y.

ii- For every $A \subseteq Y$, $cl_Y A = cl_X A \cap Y$.

iii- A subset M of Y is π_Y-nbd of a point $y \in Y$ iff $M = N \cap Y$ for some π-nbd N of Y.

iv- A point y in Y is π_Y- limit point of $A \subseteq Y$ iff y is a π -limit point of A , further $D_Y(A) = D(A) \cap Y$.

v- For every $A \subset Y$, $Int_Y A \supset Int_X A$

vi- For every A in Y $Fr_Y(A) \subset Fr_X(A)$.

Proof: i- A closed in Y iff Y/A is open in Y.

If f Y/A=G $\cap$ Y for some open set G of X.

If f A=Y/(G$\cap$Y)=(Y/G)$\cup$(Y/Y)

If f A=Y/G [since Y/Y=ϕ] De-Morgan law

If f A=Y$\cap G^c$ "The complement of G in X"

If f A=Y$\cap$F where $F=G^c$ is closed in X.

ii- By def^n $Cl_Y A = \bigcap\{K;\ K\ is\ closed\ in\ Y, and\ A \subset K\}$

$$\begin{aligned} Cl_Y A &= \bigcap\{F \cap Y : F\ is\ closed\ in\ X\ and\ A \subset F \cap Y \\ &= \bigcap\{F \cap Y; F\ is\ closed\ and\ A \subset F\} \\ &= [\bigcap\{F; F\ is\ closed\ and\ A \subset F\}] \cap Y \\ &= Cl_X(A) \cap Y \end{aligned}$$

iii- Let M be a π_Y-nbd of y, then there exists a π_Y-open set H such that

$y \in H \subset M \Rightarrow \exists a\ \pi - open\ set\ G\ such\ that\ y \in H = G \cap Y \subset M$. Let $N = M \cup G$.

Then N is a π-nbd of y since G is a π-open set such that $y \in G \subset N$.

$$\begin{aligned} Further\quad N \cap Y = (M \cup G) \cap Y = (M \cap Y) \cup (G \cap Y) &= M \cup (G \cap Y) \qquad Since\ M \subset Y \\ &= M \qquad since\ G \cap Y \subset M \end{aligned}$$

Conversely Let M=N $\cap$ Y for some π-nbd N of y, then there exists

A π-open set G such that $y \in G \subset N,\ \ which implies that\ y \in G \bigcap Y \subset N \bigcap Y = M$

since G∩Y is π_Y –open set , M is π_Y-nbd of y,

vi- y is a π_Y-limit point of A if f [M/{y}∩A] ≠Φ for all π_Y –nbds M of y.

if f [N∩Y/{y}∩A ≠ Φ for all- nbds N of y

if f [N/{y}∩A] ≠Φ for all nbds N of y

if f y is a π-limit point of A.

$$v - x \in IntA \Rightarrow x \text{ interior point of } A \Rightarrow A \text{ is a } \pi - nbd \text{ of } x$$
$$\Rightarrow A \bigcap Y \text{ is } \pi_Y nbd \text{ of } x$$
$$\Rightarrow A \text{ is a } \pi_Y nbd \text{ of } x \quad [\text{since } A \subset Y \Rightarrow A \bigcap Y = A$$
$$\Rightarrow x \in Int_Y A$$
$$Hence\ Int_X A \subset Int_Y A.$$

$$iv - y \in Fr_Y A \Rightarrow y \text{ is } \pi_Y - \text{fronier point of } A \text{ and } Y / A$$
$$\Rightarrow every\ \pi_Y - nbd of y \text{ intersects both } A and Y - A$$
$$\Rightarrow N \bigcap Y \text{ intersection both } A and Y / A \quad \forall \pi - nbd\ N \text{ of } y$$
$$\Rightarrow every\ \pi - nbd\ N \text{ of } y \text{ intersection both } A and\ X - A$$
$$\Rightarrow y \text{ is } \pi - Frontier of\ A$$
$$\Rightarrow y \in Fr_X A$$
$$Hence\ Fr_Y A \subset Fr_X A.$$

Theorem 31: let (Y,π_Y) be a subspace of a topological space of (X,π) and let B be a base for π, then $\beta_y = \{\beta \bigcap Y; B \subset \beta\}$ is a base for π_Y

Proof: Let H be a π_Y open subset of Y and let x in H , then there exists a π- open subset G of X such that H=G∩Y. since β is a base for the topologyπ

$$\exists s\ B \in \beta \text{ such that } x \in B \subset G, \text{since } H \subset Y, it\ follows that\ x \in Y\ and\ x \in B \bigcap Y \subset G \bigcap Y = H$$

$$hence \exists s\ a\ set\ B \bigcap Y \in \beta_Y, Such that\ x \in B \bigcap Y \subset H.$$
$$Thus\ to\ each\ x \in H, there\ exists\ a\ member\ \mathbf{B} \bigcap Y \text{ of } B_Y \text{ such that } x \in \mathbf{B} \bigcap Y \subset H,$$
$$that\ is\ H = \bigcup\{B \bigcap Y; B \bigcap Y \in \beta_y\ and\ B \bigcap Y \subset H\}$$

Hence β_y is a base for π_Y.

Ex: X={a,b,c.d,e} and Y={a,c,e} $\pi_X = \{\phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}, X\}$

$\pi_Y = \{\phi, \{a\}, \{a,c\}, \{a,e\}, Y\}$ *let* $A = \{a,e\} \subset Y$ $Int_Y A = \{a,e\}$ *and* $Int_X A = \{a\}$

Separated Set

Definition: Let (X, τ) be a t.s. two non-empty subset A & B of X are said to be τ -separated iff $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

Or equivalent we say $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \emptyset$.

Note : Every separated set are disjoint but the converse not true in general

Example: Let A= (-∞,0) and B=[0,∞) of R . A & B are disjoint which is not separated .

$\overline{A}$ =(-∞,0] and $\overline{A} \cap B$=(-∞,0] ∩[0,∞)={0}≠Ø

Theorem(1) : Let (Y,τ_Y) be a subspace pf a t.s. (X, τ) and Let A , B be two subset of Y , then A , B are τ –separated iff τ_Y-separated .

Proof: since $CL_Y A = CL_X A \cap Y$ and $CL_Y B = CL_x B \cap Y$

Now $(CL_Y A \cap B) \cup (CL_Y B \cap A)$=

$= (CL_X A \cap Y) \cap B] \cup [(CL_X B \cap Y) \cap A]$

$=(CL_X A \cap B) \cup (CL_X B \cap A)$ [since $A,B \subset Y$]

Hence [$(CL_Y A \cap B) \cup (CL_Y B \cap A) = \emptyset$ iff $(CL_X A \cap B) \cup (CL_X B \cap A) = \emptyset$] .

It follows that A,B are τ *–separated iff* τ_Y*-separated*

Theorem(2) *:* Two closed (open)subset A,B of a t.s (X, τ) are separated iff subset are disjoint

Proof: Since any two separated sets are disjoint , we need only to prove that two disjoint closed (open) sets are separated if A& B are both disjoint and closed , than A∩B=Ø

$A = \overline{A}$ and $B = \overline{B}$ so that

$A \cap B = A \cap \overline{B} = \emptyset$ and $A \cap B = \overline{A} \cap B = \emptyset$

Showing that A&B are separated

If A and B are both disjoint and open then A^c *and* B^c are both closed so that

$clA^c = A^c$ *and* $clB^c = B^c$. Also

$$A \bigcap B = \phi \Rightarrow A \subset B^c \quad and \quad B \subset A^c$$
$$\Rightarrow clA \subset clB^c = B^c \quad and \quad clB \subset clA^c = A^c$$
$$\Rightarrow clA \bigcap B = \phi \quad and \quad clB \bigcap A = \phi$$
$$\Rightarrow A \, and \, B \, are \, separated.$$

Connected and disconnected sets

Definition: Let (X, τ) be t.s A subset A of X is said to be τ -disconnected iff it is the union of two non-empty τ -separated sets iff there exist two non-empty sets C and D .such that C∩D= and C∩D= , A=C U D , A is τ - connected if is not τ -disconnected .

Note: two points a and b of a t.s X are said to be connected iff they are contained in a connected subsets of X.

Theorem(3): *At.s X is disconnected iff* $\exists$*s a non empty proper subset which is both open and closed.*

Proof: let A be a non empty proper subset we have to prove that X is disconnected

Let B=A^c , then B is a non empty set moreover X=A$\cup$B and A$\cap$B=ϕ

Since A is both open and closed , hence $\overline{A}$=A and $\overline{B}$=B , it follows that A$\cap$ $\overline{B}$=ϕ and $\overline{A}$ $\cap$B=ϕ , thus X can be expressed as the union of two non-empty separated sets so X is disconnected

Conversely: let X be a disconnected set then $\exists$s a non empty subset A and B of X such that A$\cap$ $\overline{B}$=ϕ , $\overline{A}$ $\cap$B=ϕ , and X=A$\cup$B.

Since A$\subset$ $\overline{A}$, $\overline{A}$ $\cap$B=ϕ $\Rightarrow$A$\cap$B=ϕ , hence A=B^c and B is non –empty

A is proper subset of X

Now A$\cup\overline{B}$=X ,[A$\cup$B=X and B$\subset$ $\overline{B}$, so A$\cup\overline{B}$ $\supset$ X and A$\cup\overline{B}$ $\subset$ X] always

Also A$\cap$ $\overline{B}$=ϕ $\Rightarrow$ A=($\overline{B}$)^c and simillery B=($\overline{A}$) ^{c}

Since $\overline{A}$ and $\overline{B}$ are closed so A&B are open , since A= B^c therefore A is closed thus A is a non-empty proper subset of X

Which both open and closed

Continuity in a topological space

Let (X,τ) and (Y,τ) be a topological space . A function $f(X,\tau)\rightarrow(Y,\tau)$ is said to be continuous iff for every μ-nbd M of f(x) $\exists$s a τ-nbd N of x s.t f(N)3M.

Also f is said to be continuous or (τ-μ continuous) iff it is continuous at each point of X.

It follows that from the definition that f is continuous at x_0 iff for every μ-open set H containing $f(x_0)$ $\exists$s an τ-open set G containing x_0 s.t $f(G)\subset H$.

Ex: X={a,b,c,d} and Y={1,2,3,4} ι={ϕ,X,{a},{b,a},{a,b,c}} μ={ϕ,Y,{1,2,3},{1,2}}

And f :X→Y defined by f(a)=4 , f(d)=1,f(b)=2 , f(c0=3 . discuss the continuity X.

Solution : since $a\in X$ and f(a)=4 f(a)=4$\in$Y , H$\in$Y is μ-open . {a}=G , f({a})={4}$\subset$Y f(G)$\in$H

$\therefore$ f is continuous at a .

Since $b\in X$ f(b)=2

The μ-open set containing 2 are {1,2},{1,2,3} and Y.

The τ-open set containing b are {a,b},{a,b,c],X .

F(b)=2$\in${1,2} b$\in${a,b} f({a,b})={2,4}$\not\subset${1,2} b$\in${a,b,c}

F({a,b,c})={2,4,3$\not\subset${1,2} f is not continuous at b .

$c\in X$,f(c)=3 the μ-open set containing f(c)=3 are {1,2,3} and Y .

The τ-open set containing c are {a,b,c}and X.

F({a,b,c})={1,2,3}$\not\subset${1,2,3} , f(X)=Y$\not\subset${1,2,3} f is not τ-μ continuous.

$\therefore$f is not continuous at c . f is not continuous at X .

A$\in$, f(d)=1 , μ-open set ={1,2},{1,2,3},Y f :Y→X , τ-open set = X .

F(X)=Y$\not\subset${1,2} f is not continuous at d .

Theorem(4) *: let X and Y be a topological space A function f :X→Y is continuous iff the inverse image under f of every open set in Y is open in x.*

Proof : let f be continuous , and let H be an μ-open set.

We have to prove that $f^{-1}(H)$ is open .

if $f^{-1}(H)=\phi$ there is nothing to prove

if $f^{-1}(H)\neq\phi$ and let $x\in f^{-1}(H)$ so that $f(x)\in H$.

by continuity of f , $\exists$ an open set G containing x in X and $f(G)\subset H$ that is $x\in G\subset f^{-1}(H)$, $f^{-1}(H)$ is an open .

conversely : suppose that v is an open set for every open set H in Y

we shall show that f is continuous

let H be an open set Y containing f(x) , $x\in f^{-1}(H)$ but $f^{-1}(H)$ is an open set by hypothesis .

there for $f^{-1}(H)$ is an open set in X containing x.

put $G = f^{-1}(H)\rightarrow f(G)=f(f^{-1}(H))\subset H$

$\therefore$ $f(G)\subset H$, f is continuous (by def) .

Theorem(5) : *let X and Y be a topological space A function f :X→Y is continuous iff the inverse image under f of every closed set Y is closed in* X .

Proof : let f be a function and $F\subset Y$ is closed . $f^{-1}(F)$ is closed

Since F is closed in Y then Y\F is open in Y

By theorem $f^{-1}(Y\backslash F)=X\backslash f^{-1}(F)$ is open in X

$\therefore$ $f^{-1}(F)$ is closed in X

Conversely : to show that f is continuous , let $f^{-1}(F)$ be any closed subset in X for every $F\subset Y$ is closed . let G be any open set in Y …….

Theorem(6*): let X and Y be any t.s then a function f :X→Y is continuous iff the inverse image of every sub base for Y is open in X* .

Proof : suppose f is continuous , and B* be a sub base for Y , since each member of B* is open in Y it follows from ((theorem 1)) that $f^{-1}(D)$ is open in X for every $D\in B^*$

Conversely : let $f^{-1}(D)$ be an open set in X for every $D\in B^*$ to show that f is continuous , let H be any open set for Y . let B , so that B is abase for Y ,

If $B \in B$ then $\exists$ $D_1, D_2, D_3, \ldots, D_n$ (n finite) in B* s.t $B = D_1 \cap D_2 \cap \ldots \cap D_n$

$f^{-1}(D) = f^{-1}\{D_1 \cap D_2 \cap \ldots \cap D_n\} = f^{-1}(D_1) \cap f^{-1}(D_2) \cap f^{-1}(D_3) \cap \ldots \cap f^{-1}(D_n)$ by hypothesis

each of $f^{-1}(D_i)$ i=1,2,…,n are open set in X , and there for $f^{-1}(B)$ is an open set in X .

since B is abase for Y , $H \subset \cup \{B; B \subset G \subset B\}$, $f^{-1}(H) \subset f^{-1}(\cup \{B; B \in B\} = 99\{ f^{-1}(B); B \in B\}$

$\therefore$ $f^{-1}(H_1)$ is an open set in X , so by (theorem 1.) f is continuous.

Theorem(7):*let X and Y be an t .s and f :X→Y is continuous iff the inverse image of every member base for Y is an open set in X .*

Theorem(8):*A function f from a space X in the another space Y is continuous iff f(clA)⊂clfA), 00⊂X.*

Proof: let f be a continuous function and let $A \subset X$, $\overline{f(A)}$ is closed set in Y

$\therefore$ $f^{-1}(clf(A))$ is closed in X . by theorem 2 , and there for $clf^{-1}(clf(A)) = f^{-1}(clf(A))$---(*)

Now $f(A) \subset clf(A)$[$\therefore$ $A \subset \overline{A}$]

$A \subset f^{-1}(f(A)) \subset f^{-1}(clf(A))$

$\therefore clA \subset f^{-1}(clf(A))$

$A \subset f^{-1}(clf(A))$

$\therefore clA \subset f^{-1}(clf(A))$

$F(clA) \subset f(f^{-1}(clf(A)) \subset clf(A)$

$\therefore$ $f(clA) \subset clf(A)$.

Conversely : suppose that $f(clA) \subset clf(A)$ $00A \subset X$, to show that f is continuous

Let F be any closed subset of Y , that is clF=F .

$f^{-1}(F)$ subset X so that by hypotheses $f^{-1}(clf(F)) \subset cl\ f\ f^{-1}(F) \subset clF = F$

there for $fclf^{-1}(F) \subset F$.

$clf^{-1}(F) \subset f^{-1}(F)$----(1)

but $f^{-1}(F) \subset clf^{-1}(F)$----(2) always by $[A \subset clA]$

from 1 and 2 we get $f^{-1}(F = cl\ f^{-1}(F)$, it follows that $f^{-1}(F)$ is closed subset of X

hence f is continuous by theorem 2

theorem(9):*A function f from a space X in the another space Y is continuous iff* $cl\ f^{-1}(B) \subset f^{-1}(clB)\ \forall B \in Y$.

proof: let f be a continuous function and let $B \subset Y$, since clB is a closed subset of Y , then $f^{-1}(clB)$ is a closed subset in X (bythe2) $cl\ f^{-1}(clB) = f^{-1}(clB)$---(1)

now $B \subset clB \rightarrow f^{-1}(B) \subset f^{-1}(clB)$

$\therefore\ cl\ f^{-1}(B) \subset cl\ f^{-1}(clB) = f^{-1}(clB)$.

$cl\ f^{-1}(B) \subset f^{-1}(clB)$

conversely : let the condition hold let F be any closed subset in Y . so that clF=F . by hypothesis $cl\ f^{-1}(F) \subset f^{-1}(clF) = f^{-1}(F)$

$f^{-1}(F) \subset cl\ f^{-1}(F)$ always

$\therefore\ f^{-1}(F) = cl\ f^{-1}(F)$

$\therefore\ f^{-1}(F)$ is closed in X .

Ex: let τ and μ be two topology for R . find whether the function f: R →R , define by $f(x)=1\ \forall\ x \in R$ is τ - μ continuous

Solution : let H be any μ-open set , if $1 \in H$ then $f^{-1}(H)=R$ and if $1 \notin H$ then $f^{-1}(H)=\Phi$

Since each of R and Φ, are open set in τ , so f is continuous

Example: let f and g be a function from R to R defined as follows:

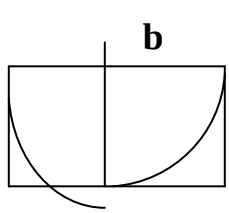
(a) $f(x)=x^2$, $\forall x \in R$ **(b)** $g(x)=|x|$, $\forall x \in R$

Find whether each of these function is :

i- μ - μ continuous . ii-S- μ continuous

iii-I - μ continuous iv- D- μ continuous

solution : since the set of all interval (a,b) with a<b form a base for μ it is enough to see whether $f^{-1}((a,b))$, $g^{-1}(a,b)$ are open w.r.t the given topology for R



$-\sqrt{b}$ a $\sqrt{b}$

$f^{-1}(G)= (-\sqrt{b}, \sqrt{b})$

$$f^{-1}(G)=\begin{cases} \Phi & \text{if } a<b\leq 0 \\ (-\sqrt{b}, \sqrt{b}) & \text{if } a<0<b \\ (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) & \text{if } 0< a < b \end{cases}$$

i- as show above the inverse image of every interval (a,b) is μ-open .

$\therefore$ f is μ-μ continuous .

ii- since S is finer then μ [that is every μ-open is S-open] so that f is S-U-continuous

iii- If we take (a,b)=(1,2) then $f^{-1}(1,2)= (-\sqrt{2}, -1) \cup (1, \sqrt{2})$ which is not I-open so f is not I-U continuous .

iv- since the inverse image of every open interval is D-open hence the space is D-U continuous .

Q1: let f be a function of R into R defined as f(x)= $|x|$, $\forall x \in$ R . find whether f is

I-U continuous U-U continuous D-U continuous S-U continuous

Example: let f be a function of R in to R defined by

$$F(x)=\begin{cases} 1/x & x\neq 0 \\ 0 & x=0 \end{cases}$$

find whether f is U-U , I-U , S-U and D-U continuous .

solution :consider the open interval (-1,1) where $f^{-1}(-1,1)= f^{-1}\{(-1,0) \cup \{0\} \cup (0,1)\}$

$= f^{-1}(-1,0) \cup f^{-1}\{0\} \cup f^{-1}(-1,0)$

$=(-\infty,-1) \cup \{0\} \cup (1,\infty)$

Homomorphism

Definition : let (X,τ) and (Y,μ) be two topological spaces and let f be a function from X in to Y .. then

i-f is open function(interior function) iff f(G) is μ-open for every τ-open set G.

ii- f is closed function iff f(F) is μ-closed for every τ-closed set F.

iii- f is bicontinuous iff f is continuous and open function .

iff [f and f^{-1} is continuous]

iv- f is homomorphism iff

1- f is bijective [1-1 and onto]

2-f is continuous

3- f is open [or f is closed or f^{-1} is continuous]

Definition : A space X is said to be homomorphism to another space Y if $\exists$ a homomorphism from X in to Y . and Y is said to be homeomorphic image of X we write $(X,\tau) \approx (Y,\mu)$.

Definition : A property of a topological space X is said to be a topological property if each homeomorphism of X has that property whenever X has that property .

[The image of every open set is open]

[The image of every closed set is closed]

Example: consider $\tau=\{\varphi,\{a\},\{a,b\},X\}$, X={a,b,c} , Y={r,p,q},

$\mu=\{\varphi,\{r\},\{p,q\},Y\}$

F(a)=f(b)=f(c)=r , find whether f is continuous , open , closed , continuous and homomorphism .

Solution : since $f^{-1}(\varphi)=\varphi$, $f^{-1}(\{x\})=X$, $f^{-1}(\{p,q\})=\varphi$, $f^{-1}(Y)=X$

Are τ-open hence f is continuous also since $f(\varphi)$, f({a})={r} , f({a,b})={r} , f(x)={r}

Which μ–open so f is open .

Since every τ-open (and μ–open) sets are τ-closed and μ–closed function .

F is continuous and open so f is continuous .

F is bijective so f isn't homomorphism .

Example : show that the function f:R→R defined by

$$F(x)=\begin{cases} X & \text{where } x<1 \\ 1 & \text{where } x\in[1,2] \\ X^2/4 & \text{where } x>2 \end{cases}$$

Discuses the continuity and opens of f .

Solution : let (a,b) be any open interval then $f^{-1}[(a,b])=\begin{cases} (a,b) & \text{if } a<b<1 \\ (a,2\sqrt{b}) & \text{if } a<1<b \\ (2\sqrt{a},2\sqrt{b}) & \text{if } 1<a<b \end{cases}$

Since the inverse image of every μ-open set is μ–open hence the function f is continuous.

open:let G be any open set containing x , let G=(1.5,1.9) , f(G)={1} which is not open

theorem(10):***let (X,τ) and (Y,μ) be two t.s the mapping f:X→Y is open iff f(IntA)$\subset$Int(f(A),***

proof : let f be an open function and let A$\subset$X , IntA is an open set in X , f(IntA) is μ-open since f is open , since IntA$\subset$A " always"

f(IntA)$\subset$f(A) ,

again since f(IntA) is μ-open there for f is an open function , then Int f(IntA)=f(IntA)---1

also f(IntA)$\subset$f(A) , Int f(IntA)=f(IntA)$\subset$Int f(A)

hence f(IntA)$\subset$Int f(A) .

conversely:

suppose that the hypothesis hold , to show that f is open , let G be an τ-open set so Int G=G

f(G)=f(IntG)$\subset$Intf(G) by hypothesis

$\therefore$ f(G)$\subset$Int f(G) , but Int f(G)$\subset$f(G) always

$\therefore$ Int f(G)=f(G) which implies that f(G) is open .

Definition : A property of a topological space is said to be hereditary if every subspace of the space has that property .

Separation Axioms

T_0-space (KOLOMOGORV)

Def: the space (X,τ) is said to be a T_0-space iff for every two distinct point of X $\exists$ an open set G which contain one of them but not other **.**

Ex: the (X,I) is not T_0-space , (X,D) is T_0-space **.**

Theorem(11) : ***A t.s (X,τ) is T_0-space iff for all $x,y\in X$, $x\neq y$ then $\{\bar{x}\}\neq\{\bar{y}\}$.***

Proof : suppose that (X,τ) is T_0-space and , Let , $x\neq y$ we wont to show that $\{\bar{x}\}\neq\{\bar{y}\}$

$\therefore (X,\tau)$ is a T_0-space , then $\forall$ $x\neq y$, $\exists$ an open set G containing x but not y . i.e $x\in G$ but $y\notin G$.

$\therefore y\in G^c$, then $\{\bar{y}\}\subset G^c$

Since $x\in G$, $x\notin G^c$, that $x\notin\{\bar{y}\}$, but $x\in\{\bar{x}\}$, hence $\{\bar{x}\}\neq\{\bar{y}\}$.

Conversely :Let $x\neq y$ and $\{\bar{x}\}\neq\{\bar{y}\}$, we have to show that (X,τ) is T_0-space

Since $\{\bar{x}\}\neq\{\bar{y}\}$, $\exists$ an element $z\in X$ s.t $z\notin\{\bar{y}\}$ but $z\in\{\bar{x}\}$.

Suppose that $x\in\{\bar{y}\}$then$\{\bar{x}\}\subset\{\bar{\bar{y}}\}=\{\bar{y}\}$which implies that $z\in\{\bar{y}\}$which is contradiction

$\therefore x\notin\{\bar{y}\}\Rightarrow (x\in\{\bar{y}\})^c=X\backslash\{\bar{y}\}$

$\therefore\{\bar{y}\}^c$ is open set containing x but not containing y since $y\in\{\bar{y}\}$

$\therefore (X,\tau)$ is T_0.

Theorem(12):. ***Every subspace of a T_0-space is a T_0-space. And hence the property is hereditary*****.**

Proof :.let(X,τ) be a T_0-space and let (y,τ_y) be any subspace of (X,τ) .we have td show that (y,τ_y) is a T_0-space.

let y_1, y_2 be any two distinct point of Y , since $Y \subset X$, so y_1, y_2 are two distinct point in X . but (X, τ) is a T_0-space , so an open set G . s.t containing one of them (say) y_1 but not y_2 then $G \cap Y$ is an open set in Y

therefore $G \cap Y$ is a τ_y–open set containing y_1 but not y_2 it follows that (y, τ_y) is a T_0-space.

Theorem(13)*: the property of space being a T_0-space is preserved under 1-1 , onto open function and hence is a topological property .*

Proof : let (X, τ) a T_0-space and let f be a 1-1 , onto open function from(X, τ) to another topological space (Y, μ) we have to show that (Y, μ) is a T_0-space

Let y_1, y_2 be any two distinct point in Y .

Since f is 1-1 , onto function , $\exists x_1, x_2 \in X$, s.t $f(x_1)=y_1$ and $f(x_2)=y_2$, $x_1 \neq x_2$.

Since $(X, \tau)$is a T_0-space, $\exists$ a τ -open set G containing one of them(say) x_1 but not x_2

Since f is open function , so f(G) is μ-open set containing $f(x_1)=y_1$,but not $f(x_2)=y_2$.

Hence$(y, \mu)$is a T_0-space.

T_1-space :"Frechet space "

Definition :A t.s. $(X, \tau)$is said to be aT_1-space iff for every two distinct points x and y of x. $\exists$two open set. G and H s.t. $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

Note: $T_1 \subset T_0$; that is every T_1-space is a to- space but the converse may not be true in general .

For example: let x be any set and $a \in x$, a is an arbitrary element : Z={ φ,every subset containing a}

$(X, \tau)$is a T_0-space , but$(X, \tau)$is not T_1-space **.**

Since every open set containing b contains a also :where $a \neq b$ **.**

Example : IS (R,U) is a T_1- space **.**

Solu: let x,ybe any two distinct real numbers . and let y >x , let y-x=k then

G={(x-k/4,x=x+k/4)}and H={(y-k/4,y+k/4)}are μ–open , s.t.$x \in G$ but $x \notin H$ and $y \in H$ but $y \notin G$. hence (R,u)is T_1-space

Theorem(14): the *space (X, τ) is T_1-space iff every singleton on subset of x is closed .*

Proof: suppose that every singleton subset of x is closed ,to show that$(X,\tau)$is aT$_1$-space

Let x, y$\in$X and x $\neq$ y ,{x}and {y}are closed set .

y$\notin$ {x} $\Rightarrow$then y$\in$ {x}^{c}

$\therefore$ {x}^{c}is an open set containing y but not x. and {y}^{c} is an open set containing x but not y

$\therefore (X,\tau)$ is a T$_1$-space .

Conversely: Let (X,τ) be a T$_1$-space and let x $\in$X ,we have two show that {x}is closed ,

Since $(X,\tau)$is a T$_1$- space

$\therefore \forall$ y$\in$X , and x $\neq$ y.

$\exists$an open set G containing y but not x.

x$\notin$G$_y$ $\subseteq$ {x}

$\therefore$ {x}^{c} is the union of all open set containing y . { x}^{c} is open ,{x}is closed

Theorem(15): ***the property of a space being a T$_1$- space preserved under 1-1 ,on to open function and hence is a topological property .***

Proof : let (X,τ) be a T$_1$-space and let f be 1-1 ,open function of (X,τ) on to another t.s. $(y, \mu)$is we shall show that (y, μ) is a T$_1$- space .

Let y$_1$,y$_2$ be any two distinct points of y, since f is 1-1 and on to,$\exists$a distances points x$_1$, x$_2$ $\in$X, s.t. y$_1$=f (x$_1$)and y$_2$=f (x$_2$)

since (X,τ) is a T$_1$-space , $\exists$T$_1$-open set G and H s.t x$_1\in$G,x$_1\notin$H and x$_2\in$H but x$_2\notin$G

since f is an open function . f(G)and f(H)are μ-open subset in y .such that y$_1$=f(x$_1$)$\in$f(G) but y$_2$=f(x$_2$)$\notin$f(G) . and y$_1$=f (x$_1$)$\in$f(H) but y$_2$ =f(x$_2$)$\notin$f(H).

hence (y, μ) is a T$_1$-space .

EXersises:

1- show that every finite T$_1$-space is discreet .
2- show that a t.s (X,τ) is T$_1$-space iff τ –contains a co-finite topology on X
3- show that every topology finer than T$_1$-topology on any set X is a T$_1$-topology .
4- prove that for any set X ,$\exists$s a unique smallest topology τ –set (X, τ) is a T$_1$-space

5- prove that a finite subset of a T_1-space has no a accumulation points.

T2-space : Hausdoff space

Definition : a t.s (X,τ) is said to be a T_2-space iff for every two disjoint points x_1,x_2 , $\exists$ disjoint open set G_1,G_2 s.t , $x_1\in G_1$ and $x_2\in G_2$, that is $\forall x_1,x_2\in X$, $x_1\neq x_2$, $\exists$ two open set G_1,G_2 , $G_1\cap G_2=\phi$, and $x_1\in G_1$, $x_2\in G_2$.

Example: show that (R,U) and (R,S) are T_2-space **.**

Solution: let a,b be any tow distinct points in R , and a>b so $|a-b|=\zeta$ then

(a- $\zeta/4$, $a+\zeta/4$)=G and ($b-\zeta/4$,$b+\zeta/4$) =H are tow W-open set containing a &b respectively and $G\cap H=\phi$, so the space is T_2-space **.**

Example: Consider the co-finite topology on an infinite set X , show that it is not T_2-space .

Solution: For this topology no two open set can be disjoint , suppose if possible that G,H are tow disjoint open subsets of X so that $G\cap H=\phi$.

Then $(G\cap H)^c=\phi^c$

$G^c\cup H^c=\phi^c=X$ (De Morgan)

$G^c\cup H^c=X$

But G^c and H^c are finite [by definition of co finite then $G^c\cup H^c$ is finite also which is contradiction .

Theorem(16): ***let (X,τ) be a t.s and let (Y,μ) be a housdorff space , let $f:X\to Y$ be a 1-1 , onto and continuous function then X is also housdorff .***

Proof: let x_1,x_2 be any tow distinct point of X , since f is 1-1 , and $x_1\neq x_2$ then $f(x_1)\neq f(x_2)$.

Let $y_1=f(x_1)$, $y_2=f(x_2)$ so that $x_1=f^{-1}(y_1)$, $x_2=f^{-1}(y_2)$.

Then $y_1,y_2\in Y$ s.t $y_1\neq y_2$

Since (Y,μ) is ahousdorff space , s a μ-open set G and H s.t $y_1\in G_1$, $y_2\in G_2$ and $G\cap H=\phi$, Since f is continuous , $f^{-1}(G)$ and $f^{-1}(H)$ are τ–open set

Now $f^{-1}(G)\cap f^{-1}(H)=f^{-1}(G\cap H)=f^{-1}(\phi)=\phi$

And $y_1\in G\Rightarrow f^{-1}(y_1)\in f^{-1}(G)\Rightarrow x_1\in f^{-1}(G)$

$Y_2 \in H \Rightarrow f^{-1}(y_2) \in f^{-1}(H) \Rightarrow x_2 \in f^{-1}(H)$

Hence the space is housdorff .

Theorem(17): ***every subspace of T_2-space is a T_2-space .***

Proof: let (X, τ) be a T_2-space and let(Y, μ) be any subspace of X ,

Let y_1, y_2 be any tow distinct points of y ,

Since $Y \subset X$, then y_1, y_2 are tow distinct point in X but (X, τ) is T_2-space , so tow open set H,G s.t $y_1 \in G$, $y_2 \in H$ and $G \cap H = \phi$

But by def , $G \cap Y$ and $Y \cap H$ are τ_y–open sets and

$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi \cap Y = \phi$

Thus $G \cap Y$, $H \cap Y$are tow disjoint τ_y–open sets , Hence the subspace (Y_1, τ_y) is T_2-space.

Theorem(18): ***Each singleton subset of a T_2-space is closed .***

Proof : Let X be a housdorff space , Let $x \in X$

To show that {x} is closed , Let y be an arbitrary point of X distinct from x . Since the space is T_2-space , $\exists$ an open set G containing y , $x \notin G$ it follows that y is not an accumulation points of {x} , so $D(\{x\}) = \phi$.

Hence $\{\bar{x}\} = \{x\}$ it follows that {x} is closed set .

Theorem(19): ***Every T_2-space is a T_1-space but the converse is not true in general***

Proof: let(X, τ) be a T_2-space and let y_1, y_2 be any two distinct point of X , since the space X is a T_2-space so , tow open set G , H s.t $y_1 \in G$, $y_2 \in H$ and $G \cap H = \phi$ this implies that $y_1 \in G$ but $y_1 \notin H$ and $y_2 \notin G$ but $y_2 \in H$.

Hence the space is a T_2-space .

But the converse in above example of co-finite topology on an infinite set X , is not T_2-space , but it is T_1-space since for if x is an arbitrary point of , then by Def of τ X/{x} is open {be any the finite set } and consequently {x} is closed

The every singleton subset of X is closed and hence the space is T_1-space .

Example: Let (X,τ) be a t.s and Let(Y,μ) be a housdorff space . if f and g are continuous function from X in to Y , show that the set A={x$\in$X; f(x)=g(x)} is closed

Solution: we shall show that X\A is open set .

Now X\A={x$\in$X; f(x)$\neq$g(x)}------(1) , Let p be an arbitrary point of X\A .

Put y_1=f(p) and y_2=g(p),we have $y_1 \neq y_2$, thus y_1,y_2 are tow distinct point in a housdorff space,$\exists$two μ-open sets G and H s.t y_1=f(p)$\in$G,y_2=g(p)$\in$H and G$\cap$H=ϕ

p$\in$ f^{-1} (G), p$\in$ g^{-1} (H), p$\in$ f^{-1} (G) $\cap$ g^{-1} (H)=V ,

since f , g are continuous function

$\therefore$ f^{-1} (G) , g^{-1} (H) are open set, Hence is open set We have to show that V$\subset$X\A

Let y$\in$V= f^{-1} (G) $\cap$ g^{-1} then y$\in$ f^{-1} (G)and y$\in$ g^{-1} (H)

f(y) $\in$G and g(y)$\in$H , since G$\cap$H=ϕ it follows that f(y) $\neq$g(y) and by(1)

y$\in$X\A , thus we shown that to each arbitrary point y$\in$V, also y$\in$X\A ,

hence V$\subset$X\A

X\A is an open set

There for A is closed

Regular and T_3-space

Def:A t.s (X,τ) is said to be a regular space iff for every closed set F and every point p$\in$F, $\exists$Tow open sets G and H s.t p$\in$G,F$\subset$G and G$\cap$H=ϕ

The regular space which is also T_1-space is called a T_3-space

Example: Let X={a,b,c} , and Let τ ={ϕ,{a},{b,c},X}

τ^c ={,X{b,c},{a},ϕ}

Example: show that (R,U) is a T_3-space .

Solution: let F be a U-closed subset and let x$\in$R, s.t x$\notin$F…….

Theorem(20): ***A t.s X is regular iff for every point x$\in$X and every nbd N of x 9 a nbd M of x such that $\overline{M}$ $\subset$N .***

Proof :"The only if part" let N be any nbd of x .then $\exists$an open set G such that $x \in G \subset N$.

Since G^c is closed and $x \notin G^c$,

But the space is regular $\exists$two disjoint open set L&M such that $G^c \subset L$ and $x \in M$.

So that $M \subset L^c$ it follows that

$\overline{M} \subset \overline{Lc} = L^c$--------- (*)

But $G^c \subset L \rightarrow L^c \subset G \subset N$-----(**)

From (*) and (**) we get $\overline{M} \subset N$.

The" if part" let the condition hold .

Let f be any closed subset of x .and $x \notin F$, then $x \in F^c$,

Since F^c is an open set containing , so by hypothesis $\exists$an open set M such that $x \in M$

and $\overline{M} \subset F^c \rightarrow F \subset (\overline{M})^c$ then $(\overline{M})^c$ is an open set , containing F also

$M \cap M^c = \emptyset, M \cap (\overline{M})^c = \emptyset$

:. The space is regular

Example: Every T_3-space is a T_3-space

Solu :let (X, τ)be a T_9-space , and let x,y be any two distinct point.

Now by definition of X , the space is R T_1 and so {x} is a closed set also $y \notin \{x\}$.

Since X is regular . $\exists$two open set G&H such that $y \in G$,$\{x\} \subset H$ & $G \cap H = \emptyset$,but $x \in \{x\}$ $\subset H$, hence the space is T_2.

Theorem*(21): Every compact housdorf space is a T_3-space*

Proof //let (X,τ)be compact housdorff space

To show that (X,τ)is a T_3-space

since X is housdorff , so X is a T_1-space , it suffices to show that (X,τ)is a regular , let F be a closed subset of X and let $p \in X$ such that $p \notin F$

so $p \in X \backslash F$, since (X, τ) is a housdorff space so for every $x \in F$,there must exist two open sets $G(x) \cap H(x) = \emptyset$…(*)

The collection C={H(x) ; $x \in F$ } is open cover of F.

Since F is a closed subset of a compact space X, so that F is compact (by theorem)

Hence $\exists$s a finite numbers of points $x_1,x_2,..,x_n$ in F such that $F\subset\{H(x_i),i=1,2,...,n\}$,let $H=U\{H(x_i),i=1,2,...,n\}$

And $G=\cap\{G(x_i), i=1,...,n\}$

Then $p\in G$, since $p\in G(x_i)$ for each x_i also $G\cap H=\emptyset$,

[other wise $G(x_k)\cap H(x_k)\neq\emptyset$ for some $x_k\in F$ this contradict(*)]

hence the space is regular .

Normal $+T_3=T_4$

Normal space and T_4 –space

Definition : At.s.(X, τ) is said to be normal iff for every pair of disjoint τ-closed subset L and M of x , $\exists$s τ- open sets G and H such that $L\subset G$, $M\subset H$ and $G\cap H=\emptyset$.

A normal space which T_1 –space is called a T_4–space

Example :lets X={a,b,c}, T={Ø,X,{a},{b,c}}since the only disjoint closed subsets are {a} ,{b,c} which is also are τ-open sets.

The space is normal.

But τ is not a T_1-space .

Since $b\neq c$,there does not exist an open set containing one of them but not the other .

Theorem(22);*A t.s (X,τ) is normal iff for any closed set F , and open set G^* containing F,$\exists$ an open set V such that $F\subset H^*$ and $\overline{H}^*\subset G^*$*

Proof // the "only if part "let X be a normal space , and let F be any closed set and G be an open set containing F.

G is open $\Rightarrow G^c$ is closed , and $F\cap G^c=\emptyset$, since the space is normal $\exists$two disjoint open set H^* and G^* such that $F\subset H^*$, $G^c\subset G^*$ and $H^*\cap G^*=\emptyset$ so that $H^*\subset G^*$

But $H^*\subset G^{*c}\Rightarrow\overline{H*}\subset\overline{G*c}=G^{*c}$ ……. 1

Also $G^c\subset G^*\rightarrow G^{*c}\subset G$ …… 2

From 1 and 2 we get $\overline{H*}\subset G$

The "if part "suppose the hypothesis is hold and to show that the space (X, τ) is normal .

Let L and M be any two disjoint closed subset of X . that is $L\cap M=\emptyset$ then $L\subset M^c$,[L is closed , M^c is an open set containing by hypothesis $\exists$an open set H^*such that $L\subset H^*$, and $\overline{H*}\subset M^c$ which implies that also $H^*\cap(\overline{H*})^c=\emptyset$ thus the space is normal

Theorem(23): ***normality is topological property***

Theorem(24)*: every closed subset of a normal space is normal space is normal* .

Proof :let(X,τ) be a normal space , and let (Y,τ y) be any closed subspace of X we have to show that (Y,τ y) is normal

Let L^*, M^*be any two disjoint closed subset of Y, then $\exists$a subset L,M of X such that $L^*=L\cap Y$,$M^*=M\cap Y$ since Y is closed it follows that $L^*$and $M^*$are τ -closed subset in X.

Since X is normal , $\exists$two τ -open set G and H such that $L^*\subset H$,

$M^*\subset G$ and $H\cap G=\emptyset$.

So $L^*\subset H$ and $L^*\subset Y\rightarrow L^*\subset H\cap Y$

$M^*\subset G$ and $M^*\subset Y\rightarrow M^*\subset G\cap Y$

And $(H\cap Y)\cap(G\cap Y)=(H\cap G)\cap Y=\emptyset\cap Y=\emptyset$

$L^*\subset H\cap Y$, $M^*\subset G\cap Y$ and $(H\cap Y)\cap(G\cap Y)=\emptyset$, hence the space is normal .

Example: ***show that if the space is normal***.

Let L,M be any U-closed subset of R s.t $L\cap M=\emptyset$

Let $r\in L$ then $r\notin M$ and so $r\in R\backslash M$ since R\M is U – open ,$\exists$ $\zeta>0$ such that $(r-\zeta , r+\zeta)\subset R\backslash M$, therefore $(r-\zeta,r+\zeta)\cap M=\emptyset$

Let G=U{ $(r-\zeta/3, r+\zeta/3)$; $r\in L$ then $L\subset G$. similarly it can be shown that for each $m\in M$, $\exists_s\delta>0$ such that $(m-\delta,m+\delta)\cap L=\emptyset$, and let H=U{$(m-\delta/3, m+\delta/3)$; $m\in M$} therefore $m\subset H$,thus G,H are two open set such that $L\subset G$,$M\subset H$

we have two show that $G\cap H=\emptyset$.

Suppose is possible that $x\in G\cap H$ so $x\in G$ and $x\in H$. then $x\in$ $(r-\zeta/3 ,\zeta/3)$for some $r\in L$ and $x\in(m-\zeta/3 , m+\zeta)$ for some $m\in M$ we then have $/r-x/<\zeta/3$ and $/m-x/<\zeta/3$ hence $/r-m/=/r-x+x-m/\leq/r-x/+/m-x/<\zeta/3+\zeta/3$ if $\zeta<\delta$ then $/r-m/<\zeta$ and so $r\in$ $(m-\zeta/3,m+\zeta)$ which is C!

if $\delta<\zeta$ then \r-m\ $<\zeta$, and $m\in$ $(r-\zeta/3 , r+\zeta/3)$which is contradiction

it follows that $G \cap H = \emptyset$ hence the space is normal

Urysohn's lemma

let F_1, F_2 be any pair of disjoint closed set in a normal space X, $\exists$ a continuous function $F: X \to [o,1]$ s.t $f(x) = o$ for $x \in F_1$, and $f(x)=1$ for $x \in F_2$

Completely regular space and tychonoff space .

Def: A topological space X is said to be completely regular iff for every closed subset F of X and every point $x \in X \backslash F$, $\exists$ a continuous function f of X in to the subspace [0,1] of R . s.t f(x)=0 and f(F)=1

A tychonoff space (or T_3-1/2space) is completely regular and T_1-space .

Theorem(25): ***A t.s(X, τ)is completely regular iff for every $x \in X$ and every open set G containing x $\exists_s$ a continuous function f of X in to [0,1] such that f(x)=0 and f(y)=1 $\forall$ $y \in X \backslash G$***

Proof: Let (Y, τ) be a completely regular space and G be an open set containing x , such that $x \notin X \backslash G$ then $X \backslash G$ is a closed set which dose not containing x .

By definition of completely regular $\exists$ a continuous function f from (X, τ) in to a subset [0,1] such that f(x)=0 , f(y)=1 for all $y \in X \backslash G$.

Conversely : Let the condition is hold

Let F be any closed subset of X and x be a point of X such that $x \notin F$. then $x \in X \backslash F$ and since F is closed so $X \backslash F$ is an open set containing x

By hypothesis $\exists$ s a continuous function f from (X, τ) into a subset [0,1] s.t f(x)=0 , f(y)= 1 for all $y \in X\{X \backslash F\}=F$

Hence the space is C.R

Theorem(26)*: Every completely regular space is regular . Hence every tychonoff space is a T_3-space .*

Proof: Let X be a completely regular, Let F be a closed subset of X , and let x be a point of X such that $x \notin F$ since the space is completely regular . $\exists$ a continuous function f from(X, τ) into subset [0,1] such that f(x)=0, f(F)={1} .

Also we can see that the space [0,1] with the relative usual topology is a T_2-space

Hence $\exists$ open sets G and H of [0,1] s.t $0 \in G$ and $1 \in H$ and $G \cap H = \emptyset$ since f is a continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are open set in (X, τ) s.t

$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset$

Further $f(x)=0 \in G \rightarrow x \in f^{-1}(G)$ and $f(F)=\{1\} \subset H \rightarrow F \subset f^{-1}(H)$

Hence the space is regular

Theorem(27): ***Every T_4-space is a tychonoff space***.

Proof: Let (X, τ) be a T_4-space by definition T_4=normal+T_1

To show that the space is tychonoff space it suffices to show that the space is C.R,

So Let F be a closed subset of X , and let x be a point of X s.t $x \notin F$,

since the space $(X, \tau)$is a T_1- so {x} is closed subset of X ,

thus {x} and F are two disjoint closed subset of a normal space

So by ((Urshon's Lemma)) $\exists$ a continuous function f from (X, τ) in to the set [0,1] s.t

f({x})=0 i.e f(x)=0 and f(F)={1}

it follows that the space is C.R .