زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

# r-clean rings

Research Project

Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of BSc. in (Mathematics)


***By***:

***Shahen Star Khidr***

***Supervised by***:

***Dr. Neshtiman Nooraldeen Sulaiman***


**April – 2024**

# Certification of the supervisors

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:

Supervisor : **Dr . Neshtiman N. Sulaiman**

Scientific grade: Assist. Professor

Date: / / 2024

In view of the available recommendations, I forward this work for debate by the examining committee.

Signature:

Name: **Dr. Rashad Rashid Haji**

Scientific grad: Assist. Professor

Chairman of the Mathematics Department

Date: / / 2024

# ACKNOWLEDGEMENT

I would like to thank my supervisor, ***Dr. Neshtiman Sulaiman***, for bringing the weight of his considerable experience and knowledge to this project. His high standards have made me better at what I do.

I sincerely thank *Dr. Rashad Rashid,* the head of the Department of Mathematics, College of Education, for all the facilities provided to us in the pursuit of this project.

I acknowledge our deep sense of gratitude to my loving parents for being a constant source of inspiration and motivation. I also thank my friends who have played significant roles throughout my life.

# ABSTRACT

An element of a ring $R$ is called clean if it is the sum of an idempotent and a unit. A ring $R$ is called clean if each of its element is clean. An element $r \in R$ called regular if $r = ryr$ for some $y \in R$. The ring R is regular if each of its element is regular. In this paper we define a ring is r-clean if each of its elements is the sum of a regular and an idempotent element. We give some relations between r-clean and clean rings. Finally we investigate some properties of r-clean rings.

# TABLE OF CONTENTS

**Certification of the supervisors** ..... ii
**ACKNOWLEDGEMENT** ..... iii
**ABSTRACT** ..... iv
**TABLE OF CONTENTS** ..... v
**INTRODUCTION** ..... 1
**CHAPTER ONE** ..... 2
**1.1. Background** ..... 2
**CHAPTER TWO** ..... 4
**Clean ring and r-Clean ring** ..... 4
**2.1. Some properties of r-clean elements** ..... 4
**2.2. Some properties of r-clean rings** ..... 6
**References** ..... 12
**خلاصه** ..... a
**پوخته** ..... a

# INTRODUCTION

Let R be an associative ring with unity. An element $a \in R$ $i$s said to be clean if $a = e + u,$ where e is an idempotent and u is a unit in R. If every element of R is clean, then R is called a clean ring. Clean rings first were introduced by Nicholson in (Nicholson, 1977), He proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring. Several peoples worked on this subject and investigate properties of clean rings, for example see (Ashrifi & Nasibi, 2013) (Chen, 2008) (Chen & Abdolyousefi, 2022) and (Camillo & Yu, 1994). In 1936, von Neumann defined that an element $r \in R$ is regular if $r = ryr$ for some y ∈ R, the ring R is regular if each of its element is regular. A ring R is called unit regular if, for each a ∈ R, there exists a unit u ∈ R such that $aua = a$. In (Camillo & Yu, 1994) proved that every unit regular ring is clean. Let $Reg(R) = \{a \in R : a$ is regular$\}$. We call an element x of a ring R is r-clean if $x = r + e,$ where $r \in Reg(R)$ $a$nd $e \in Id(R)$. A ring R is r-clean if each of its elements are r-clean.

This project consists two chapter. In chapter One, we give some necessary definition and theorems in a ring theory we needed in the project. The second chapter, consists two section, in the first section, we introduce the definition of r-clean element and r-clean ring and we give some properties of r-clean elements. The second section, we give some properties of r- clean ring and what are the relations with clean rings.

# CHAPTER ONE

## 1.1. Background

**Definition 1.1.1**: (Fraleigh, 1982)

A nonempty set R with two binary operation + and $\cdot$ called addition and multiplication (also called product) respectively, such that $(R, +)$ is an additive abelian group.

(i) $(R, +)$ is an additive abelian group.

(ii) $(R, \cdot)$ is a multiplicative semigroup.

(iii) Multiplication is distributive (on both side ) over addition ; that is for all $a, b, c \epsilon R$,

$$a \cdot (b + c) = a \cdot b \ + \ a \cdot c \ , \qquad (a + b) \cdot c = a \cdot c + b \cdot c$$

**Example:** $(Z , + ,\cdot)$ , $(R , + ,\cdot)$

**Definition 1.1.2:** (Fraleigh, 1982)

A commutative ring is a ring R in which the multiplication is commutative that is

$a \cdot b = b \cdot a$ for all $a , b \epsilon R$ .

**Definition 1.1.3:** (Fraleigh, 1982)

A ring R with a multiplicative identity 1 such that $1. x = x. 1 = x$ for all $x \in R$ is a ring with unity A multiplicative identity in a ring is unity.

**Definition 1.1.4:** (Bhattacharya, et al., 1994)

An element a in a ring R is called idempotent $if\ a^2 = a$ clearly 0 and 1 (if R has unity) are idempotent elements.

**Example:** $(Z_{6\ ,+6\ ,\cdot 6})$

{0,1,3,4} are called idempotent in $Z_6$.

**Definition 1.1.5:** (Bhattacharya, et al., 1994)

An element a in a ring R is called nilpotent, if there exists a positive integer n such that $a^n = 0$.

**Example:** ( $Z_{4\ ,+4,\cdot 4}$)

{0,2} are nilpotent element.

**Definition1.1.6:** (Fraleigh, 1982)

If a and b are two nonzero elements of a ring R such that ab=0, then a and b are divisors of 0 (or 0 divisors) in particular, a is a left divisors of 0 and is a right divisors of 0.

**Definition1.1.7:** (Ashrafi & Nasibi, 2011; Chen & Abdolyousefi, 2022)

An element r ϵ R called regular if r = ryr for some y ϵ R. The ring R is regular if each of its element is regular.

**Definition 1.1.8:** A ring is called abelian if all its idempotents are central.

# CHAPTER TWO

# Clean ring and r-Clean ring

## 2.1. Some properties of r-clean elements

**Definition 2.1.1:** (Ashrafi & Nasibi, 2011)

An element $a \in R$ $i$s said to be clean if $a = e + u$, where e is an idempotent and $u$ is a unit $in$ R. If every element of R is clean then R is called a clean ring.

**Definition 2.1.2:** (Ashrifi & Nasibi, 2013)

An element $a \in R$ is said to be r-clean $if$ $a = e + r$, where $e$ is an idempotent and $r$ is a regular (von Neumann) element in R. If every element of R is r-clean, then R is called an r-clean ring.

**Lemma 2.1.3:** Every clean and regular rings are r-clean ring**.**

The converse of above is not true.

**Example:** $\mathrm{Z}_4$, the ring of integers modulo 4.
Since 2 ∈ $\mathrm{Z}_4$ but is not regular element, hence is not regular ring and
It is easy to check that $\mathrm{Z}_4$ is r-clean.

**Lemma:** every elements in $Z_n, n$ is prime are r-clean

**Definition2.1.4:** (Chen, 2008)

Let $R$ be a ring and $x \in R$. Then $x$ is strongly clean if and only if there exist $e \in Id(R), u \in U(R)$ such that $e = xue, e - 1 = (x - 1)u(e - 1)$ and $eu = ue$. Hence $R$ is strongly clean if and only if every $x$ of R satisfies the above conditions.

**Lemma 2.1.5:**

Let $R$ be a ring and $e \in Id(R)$. if $a \in eRe$ is strongly clean in $eRe$, then $a$ is strongly clean in $R$.

**Proof:** $eRe$ suppose $a = f + v$ where $fv = vf, f^2 = f \in eRe, v \in eRe$ and $vw = e = wv$ for $w \in eRe$. Then $u = v + (1 - e)$ is a unit in $R$( with $u^{-1} = w + (1 - e)$) and $a - u = f + (1 - e)$ is idempotent because $f$ and $1 - e$ are orthogonal.

**Lemma 2.1.6:**

Let $R$ be an abelian ring. Let $a \in R$ be a clean element in $R$ and $e \in Id(R)$. Then

(1) The element $ae$ is clean.

(2) If $-a$ is clean, then $a + e$ is also clean.

**Proof:** (**1**) Since $a$ is clean in R , $a = u + f$ with $u \in U(R)$ and $f \in Id(R)$. So $ae = ue + fe$. Clearly $ue \in U(e\,Re)$ and $fe \in Id(e\,R\,e)$ since $R$ is Abelian. And $R$ is Abelian implies $ae$ is strongly in $eRe$. By Lemma 2.1.5, $ae$ is strongly clean in $R$ and hence $ae$ is clean.

(**2**) It is known and easy to prove that $a$ is clean if and only if $1 - a$ is clean for $a \in R$. Since $a$ and $-a$ are clean, so are $a$ and $1 + a$. Let $a = u + f$ and $1 + a = v$ + g where $u, v \in U(R)$ and $f, g \in Id(R)$. Then $a + e = a\,e + a(1 - e) + e = (1 + a)e + a(1 - e) = (v + g)e + (u + f)(1 - e) = v\,e + u(1 - e) + ge +$

$f(1-e)$. Note that $R$ is Abelian, it is easy to check that $v\,e+u(1-e)\in U(R)$ with$(ve+u(1-e))^{-1} = v^{-1}e + u^{-1}(1-e)$ and $ge+f(1-e)\in Id(R)$. Hence $a+e$ is clean in $R$.

**Theorem 2.1.7:**

Let $R$ be a ring, then $x\in R$ is r-clean if and only if $1-x$ is r-clean.

**Proof:** Let $x\in R$ be r-clean. Then write $x=r+e$, where $r\in Reg(R)$ and $e\in Id(R)$. Thus $1-x = -r+(1-e)$. But there exists $y\in R$ such that $ryr=r$. Hence $(-r)(-y)(-r)=-(ryr)=-r$ and since $-r\in Reg(R)$ and $1-e \in Id(R)$, so $1-x$ is r-clean.

Conversely, if $1-x$ is r-clean, write $1-x=r+e$, where $r\in Reg(R)$ $a$nd $e\in Id(R)$. Thus $x=-r+(1-e)$, like previous part, $-r\in Reg(R)$ and $1-e\in Id(R)$. Therefore $x$ is r-clean.

## 2.2. Some properties of r-clean rings

**Theorem 2.2.1:**

Let R be an abelian ring. Then R is r-clean if and only if R is clean.

**Proof:** One direction is trivial. Conversely, let R be r-clean $and$ $x\in R$. Then $x=e'+r$, where e′∈ Id(R) $and$ $r\in Reg(R)$. So there exists $y\in R$ such that $ryr= r$. Clearly, $e=ry$ and $y$ rare idempotents and $\left(re+(1-e)\right).(ye+(1-e))=1$ Also since R is abelian, we have

$$\left(ye+(1-e)\right)\left(re+(1-e)\right)=yre+1e=eyr+1-e=$$

$$ry(yr)+1-e=r(y\,r)y+1-e=e+1-e=1.$$

So $u = re + (1-e)$ is a unit. Furthermore, $r = eu$. Now, set $f = 1 - e$. Then $eu + f$ and hence, $-(eu + f)$ is a unit. Since f is an idempotent, so $-r = f + (-(eu + f))$ is clean. Then since $r \in Reg(R)$, it follows by Lemma 1.7(2) that $x$ is clean, as required.

**Theorem 2.2.2:**

Every factor ring of an r-clean ring is r-clean. In particular a homomorphic image of an r-clean ring is r-clean.

**Proof:** Let R be r-clean and I⊲ R. Also let $\bar{x} = x + I \in \frac{R}{I}$. Since R is r-clean so we have $x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. Thus $\bar{x} = \bar{r} + \bar{e}$. But there exists y ∈ R such that $ryr = r$. Therefore $\overline{ryr} = \bar{r}$. So $\bar{r}$ ∈ Reg(R) and since $\bar{e}$ ∈ Id($\frac{R}{I}$), it follows that $\frac{R}{I}$ is r-clean.

**Remark 2.2.3:**

In general, inverse of above theorem may not be correct. For example, if $p$ be a prime number, then $\frac{Z}{pZ} \cong Z_p$ is r-clean, but Z is not r-clean.

**Theorem 2.2.4:**

A direct product R = $\prod_{i \in I} R_i$ of rings $\{R_i\}_{i \in I}$ is r-clean if and only if so is each $\{R_i\}_{i \in I}$ .

**Proof**. One direction immediately follows from Theorem 2.2.2 . Conversely, let $R_i$ be r-clean for $each\ i \in I$ Set $x = (x_i)_{i \in I} \in \prod_{i \in I} R_i$ . For each $i$, write $x_i = r_i + e_i$ , where $r_i \in Reg(R_i)$ and $e_i \in Id(R_i)$. Since $r_i \in Reg(R_i)$, there exists $y_i \in R_i$ such that $r_i y_i r_i = r_i$ . Thus $x = (r_i)_{i\epsilon I} + (e_i)_{i\epsilon I}$, where $(r_i)_{i\epsilon I} \in Reg(\prod_{i\epsilon I} R_i)$ and $(e_i)_{i\epsilon I} \in Id(\prod_{i\epsilon I} R_i)$. Therefore $\prod_{i\epsilon I} R_i$ is r-clean.

**Lemma 2.2.5:**

Let R be a commutative ring $and\ f = \sum_{i=0}^{n} a_i\, x^i \in R[x]$ be regular. Then $a_0$is regular and $a_i$is nilpotent for each $i$.

**Proof:** Since $f$ is regular, thus there exists $g = \sum_{i=0}^{m} b_i x^i \in R[x]$ such that $fgf = f$. So $a_0 b_0 a_0 = a_0$ Therefore $a_0$ is regular. Now to end the proof, it is enough to show that for each prime ideal $P$ of $R$; every $a_i \in P$. Since $P$ is prime, thus $\frac{R}{P}[x]$ is an integral domain. Define $\varphi : R[x] \rightarrow \frac{R}{P}[x]$ by $\varphi(\sum_{i=0}^{k} a_i x^i) = \sum_{i=0}^{k} (a_i + p)\, x^i$. Clearly $\varphi$ is an epimorphism. $But\ \varphi(f)\varphi(g)\varphi(f) = \varphi(f),$ $so\ deg(\varphi(f)\varphi(g)\varphi(f)) = deg(\varphi(f))$. Thus $deg(\varphi(f)) + deg(\varphi(g)) + deg(\varphi(f)) = deg(\varphi(f)). Therefore\ deg(\varphi(f)) + deg(\varphi(g)) = 0$. So $deg(\varphi(f)) = 0$. Thus $a_1 + P = \ldots = a_n + P = P, as$ required.

**Theorem 2.2.6:**

If $R$ is a commutative ring, then $R[x]$ is not r-clean.

**Proof.** We show that $x$ is not r-clean $in\ R[x]$. Suppose $that\ x = r + e$, where $r \in Reg(R[x])\ and\ e \in Id(R[x]). Since\ Id(R) = Id(R[x])\ and\ x = r + e, so\ x - e\ is$ regular. Hence by pervious Lemma, 1 should be nilpotent, which is a contradiction.

**Lemma 2.2.7:**

Even If $R$ is a field, then $R[x]$ is not r-clean.

**Corollary 2.2.8:**

If $R$ is a commutative ring, then $R[x]$ is neither clean nor regular.

**Theorem 2.2.9:** Let $R$ be a ring. Then the ring $R[[x]]$ is r-clean if and only if so is $R$.

**Proof:** $If$ $R[[x]]$ is r-clean, then by Theorem 2.2.2, R≅ $\frac{R[[x]]}{(x)}$ is r-clean.

Conversely, suppose $that$ $R$ is r-clean. We know that R[[$x$]] ≅ {$(a_i)$ : $a_i$ ∈ R, for each $i \geq 0$} = $\prod_{i\geq 0} R$.So the result is clear by Theorem 2.2.4.

**Theorem 2.2.10:**

For every ring $R$, we have the following statements.

**(1)** If $e$ is an central idempotent element of $R$ and $eRe$ $and$ $(1 - e)R(1 - e)$ are both r-clean, then so is $R$.

**(2)** If R is r-clean, then so is the matrix ring $M_n(R)$ for any $n \geq 1$.

**Proof:** We use $\bar{e}$ to denote $1 - e$ and apply the Pierce decomposition for the ring $R$, i.e.,

$$R = eRe \oplus eR\bar{e} \oplus \bar{e}Re \oplus \bar{e}R\bar{e}.$$

But idempotents in $R$ are central, $so$ $R = eRe \oplus \bar{e}R\bar{e}$ = $\begin{pmatrix} eRe & 0 \\ 0 & \bar{e}R\bar{e} \end{pmatrix}$ .

For each $A \in R$, $write$ $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a$ , $b$ belong to $eRe$ and $\bar{e}$R$\bar{e}$ respectively. By our hypothesis $a, b$ are r-clean. Thus $a = r_1 + e_1$, $b = r_2 + e_2$, where $r_{1,} r_2 \in Reg(R)$ and $e_1 e_2$ ∈ Id(R). So

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1 + e_1 & 0 \\ 0 & r_2 + e_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} .$$

But there exists $y_1, y_2 \in R$ such that $r_1 y_1 r_1 = r_1$ , $r_2 y_2 r_2 = r_2$ . Therefore

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1 y_1 r_1 & 0 \\ 0 & r_2 y_2 r_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

So $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \epsilon\ Reg(R)$, since $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in Id(R)$, it follows $that\ R$ is r-clean .

**Proposition 2.2.11:**

Let $R$ be a r-clean ring and $e$ be a central idempotent in $R$. Then $eRe$ is also r-clean.

**Proof:** Since $e$ is central, it follows that $eRe$ is homomorphic image of $R$. Hence the result follows from Theorem2.2.2.

**Theorem 2.2.12:**

Let $R$ be a ring in which 2 is invertible. Then R is r-clean if and only if every element of R is the sum of a regular and a square root of 1**.**

**Proof:** Suppose that R is r-clean and $x \in R$, then $\frac{x+1}{2} \in R$. Write$\frac{x+1}{2} = r + e$, where $r \in Reg(R)$ and e ∈ Id(R). So $x = (2e - 1) + 2r$. But there exists y ∈ $R$ such that $ryr = r$ . Thus $(r + r)\ \frac{y}{2}\ (r + r) = \frac{ryr}{2} + \frac{ryr}{2} + \frac{ryr}{2} + \frac{ryr}{2} = \frac{1}{2}$(r + r + r + r) = 2r. Thus $2r \in Reg(R)$ and since $(2e - 1)^2 = 1$, so $x$ is a sum of a regular and a square root of 1.

Conversely, if $x \in R$ , then $2x - 1 = t + r$, where $t^2 = 1$ and $r \in Reg\ (R)$. Thus $x = \frac{(t+1)}{2} + \frac{r}{2}$ $i$t is easy to check $that$ $\frac{t+1}{2} \in Id(R)$. Now since $\frac{r}{2}(y + y)\frac{r}{2} = \frac{ryr}{4} + \frac{ryr}{4} = \frac{r}{2}$, it follows that $\frac{r}{2} \in Reg(R)$. Hence $x$ is r-clean, which shows that $R$ is r $-$clean.

**Lemma 2.2.13:** Let $R$ be a ring with no zero divisors. Then $R$ is clean if $R$ is r-clean.

**Proof:** For every $x \in R$, we write $x = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. Then there exists $y \in R$ such that $ryr = r$. Now, if $r = 0$, then $x = e = (2e - 1) + (1 - e)$ is clean. But if r $\neq$0, then since R is a ring with no zero divisors and $ryr = r$, so $r \in U(R)$. Hence $x$ again is clean.

**Theorem 2.2.14:** Let R be a ring. Then R is r-clean if and only if every element $x \in$ R can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$.

**Proof:** Let R be r-clean and $x \in R$. Then as R is r-clean, so $-x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (-r) - e$, where $-e \in Reg(R)$ and $e \in Id(R)$.

Conversely, suppose that every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. So for every element $x \in R$, we can write $-x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (-r) + e$, where $-r \in Reg(R)$ and $e \in Id(R)$.

## References


Ashrafi, N. & Nasibi, E., 2011. *R-clean rings*, Iran: Mathematical Reports.

Ashrifi, N. & Nasibi, E., 2013. Rings in which elements are the sum of an idempotent and a regular element. *Bulletin of the Iranian Mathematical Society*, July, 39(3), pp. 579-588.

Bhattacharya, P. B., Jian, S. K. & Nagpaul, S. R., 1994. *Basic abstract algebra.* Second edition ed. s.l.:Cambridge university press.

Camillo, V. P. & Yu, H. P., 1994. Exchange rings, units and idempotents. *Communications in Algebra*, 22(12), pp. 4737-4749.

Chen , H. & Abdolyousefi, M. S., 2022. *Theory of Clean Rings and Matrices.* s.l.:World Scientific Publishing Co. Pte. Ltd..

Chen, W., 2008. Notes on clean rings and clean elements,. *Southeast Asian Bulletin of Mathematics*, 32(5).

Fraleigh, J. B., 1982. *A first course in abstract algebra ,.* third edition ed. Simuiltaneously in canada,: Addison-Wesley,.

Nicholson, W. K., 1977. Lifting Idempotents and Exchange Rings. *Transactions of the American Mathematical Society,* May, Volume 229, pp. 269-278.

# خلاصه

العنصر فى الحلقة تسمى نظيفة اذا تكونت من مجموع عنصر عديمة القوى وعنصر الوحدة, وتسمى الحلقة بحلقة نظيفة اذا كانت جميع عناصرها نظيفة.العنصر $r \in R$ فى الحلقة تسمى منتظم اذا وجد $y \in R$ بحيث ان $r = ryr$. وتسمى الحلقة بحلقة منتظمة اذا كانت جميع عناصرها منتظمة. في هذا العمل نعرف حلقة **r-clean** اذا كانت جميع عناصرها هو مجموع عنصر عديمة القوى وعنصر المنتظم. اعطينا بعض العلاقات بين الحلقة **clean** والحلقة **r-clean**. واخيرا عرضنا بعض الصفات في الحلقة **r-clean**.


# پوخته