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Boolean Zero Square Ring

Research Project

Submitted to the department of (Mathematics) in partial fulfillment of the requirements for the degree of BSc. in (Mathematic)

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Certification of the Supervisor

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ABSTRACT

In this work we study Boolean-zero square ring (BZS), We will say that R BZR if every element of R is either idempotent or nilpotent of index 2. We introduce some proprieties of these rings.

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INTRODUCTION

Let R be an associative ring, not necessarily commutative and not necessarily with identity. We will say that R is Boolean zero square or BZS if every non-zero element of R is either idempotent or nilpotent of index 2.

BZS rings generalize both the well-known class of Boolean rings. See (Stanley, 1969) and (Stone, 1936) for more information about Boolean rings and zero square rings.

A BZS ring which is neither Boolean nor zero square is called properly BZS. In recent times, there have been many authors are studies BZS as (Farag & Tucci, 2021), you investigation of the structure of finite BZS rings and (Pinto, 2021) We introduce a new class of semigroups BZS.
This work consists of three chapters. In chapter one, we give a background about a fundamental of some definition and theorems in rings we will need in other chapter. Chapter two, ion, in the first section we find BZS in rings of integer modulo n, and in the second section we give some properties the structure of BZS. The last chapter, consist two sections, section one is about BZS rings with cyclic additive groups, and in the final section we study ideals of properly BZS rings. We note that many of the examples and results in this paper derive from (Farag & Tucci, 2019).

Throughout we denote the set of idempotent elements of the BZS ring R by Id and the set of nilpotent elements of R by N. We also let e, f denote idempotent elements, and x, y denote nilpotent elements.

CHAPTER ONE

1.1. Background

Definition 1.1. (Fraleigh, 1982) A *ring* $(R, +, \ .)$ is a set R together with two binary operations $+$ and ., such that the following axioms are satisfied:

a) $(R, +, \ .)$ Is an abelian group
b) Multiplication is associative
c) For all $a, b, c \ \epsilon R$ the left distributive law $a(b + c) = (ab) + (ac)$ and the right distributive law $(a + b)c + (ac) + (bc)$ hold.

Definition 1.2. (Fraleigh, 1982) A ring in which the multiplication is commutative is called *commutative ring*.

Definition 1.3. (Fraleigh, 1982)An element a of a ring R is *idempotent* if $a^2 = a$.

Definition 1.4. (Fraleigh, 1982)An element a of a ring R is *nilpotent* if $a^n = 0$ for some $n \in Z^+$

Definition 1.5. (Fraleigh, 1982)A ring R is a *Boolean ring* if $a^2 = a$ for all $a \in R$.

Definition 1.6. (Fraleigh, 1982)A group G is *cyclic* if there is some element a in G that generates G.

Definition 1.7. (Fraleigh, 1982) An *identity element* for a binary operation $*$ on a set S is any element e satisfying $e * x = x * e = x \quad \forall x \in S$.

Definition 1.8. [ ] A map ϕ of a group G into a group G' is a *homomorphism* if $(ab)\phi = (a\phi)(b\phi)$For all elements a and b in G.

Definition 1.9. (Fraleigh, 1982)Two sets are isomorphic if such an *isomorphism* between them exists..

Definition 1.10. (Fraleigh, 1982)The *center* of a group G is the set of all $a \in G$ such that $ax = xa \quad \forall x \in G$, that is the set of all elements of G that commute with every element of G.

Definition 1.11. (Fraleigh, 1982) An additive subgroup $\langle H, +\rangle$ of a ring R satisfying $rH \subseteq H \ and \ Hr \subseteq H \ for \ all \ r \in R$ is an *ideal* or(two-sided ideal) of R

Definition 1.12. (Fraleigh, 1982) An ideal $I \neq R$ in a commutative ring R is a *prime ideal* if $ab \in I$ implies that either $a \in I$ or $b \in I$ for all $a, b \in R$.

Definition 1.13. (Fraleigh, 1982) A *maximal ideal* of a ring R is an ideal M different from R such that there is no proper ideal I of R properly containing M.

Definition 1.14. (Fraleigh, 1982) A set R together with two binary operations + (called *addition*) and · (called *multiplication*) is called a (right) *near-ring* if:

- R is a group (not necessarily abelian) under addition;
- multiplication is associative (so R is a semigroup under multiplication); and
- multiplication *on the right* distributes over addition: for any x, y, z in R, it holds that $(x + y)\cdot z = (x\cdot z) + (y\cdot z)$.

Definition 1.15. (Fraleigh, 1982) A ring R is *nil ring* if every element of R is nilpotent.

CHAPTER TWO

2.1. The BZS in ring of integers modulo n

Definition 2.1.1. (Stanley, 1969)A ring R for which $x^2 = 0$ for all $x \in R$ is called a *zero square ring.*

Definition 2.1.2. R is *Boolean zero square* ring if every non zero element of R is either idempotent or nilpotent of index 2.

Now, we find BZS elements in the rings of integer modulo n.

Lemma 2.1.3. In Z_n, the ring of integer modulo n. If $n = p, p$ is prime has no nontrivial BZS.

Example: In Z_5

Idempotent and nilpotent elements of Z_5 are $\{0,1\}$ then

$Z_5 = \{0,1\}$ is BZS ring just have trivial BZS.

Lemma 2.1.4. In Z_{p^n}, the ring of integer modulo p^n, p is prime. The BZS are

$$\begin{cases} m.p^{\frac{n}{2}} & if\ n\ if\ even \\ m.p^{\frac{n+1}{2}} & if\ n\ is\ odd \end{cases}$$

Example: In Z_{25}

Nontrivial BZS elements are $\{5,10,15,20\}$

Lemma 2.1.5. In Z_{pq}, the ring of integer modulo $p.q,\ p\ and\ q$ are primes, $p > q,\ q \neq 5$, has nontrivial BZS are q and $nq + 1, n = 2,3,4, \ldots$

Example: In Z_{15}

Nontrivial BZS elements are $\{6,10\}$

Lemma 2.1.6. In Z_{2p}, the ring of integer modulo $2p, p$ is prime then has only two nontrivial BZS are p and $p+1$.

Example: $In\ Z_{22}$

Nontrivial BZS elements are $\{11,12\}$

Lemma 2.1.7. In $Z_{p^n q}$, the ring of integer modulo $p^n q,\ p\ and\ q$ are prime $p < q,\ q \neq 5$, has nontrivial BZS are are $q+1$ and $nq, n = 2,3,4, \ldots$

Example: $In\ Z_{24}$

Nontrivial BZS elements are $\{9,12,16\}$

2.2. Some Properties the Structure of BZS

Lemma 2.2.1. Every Boolean ring is Boolean zero square.

Proof: The prove is clearly by definition of Boolean ring.

But the converse is not true

Example: let $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, A is Boolean zero square ring but not Boolean ring.

Lemma 2.2.2. Every zero square is Boolean zero square

Proof: It is clearly by definition of zero square ring.

But the converse is not true

Example: $Z_2 = \{0,1\}$

$$0^2 = 0\ ,\ \ 1^2 = 1$$

Therefore, Z_2 is Boolean zero square but not zero square.

Note: In general in the set of all integers only nZ_2 becomes Boolean zero square ring.

Proposition 2.2.3.

Let R be a ring with identity 1. Let x be a non-zero nilpotent element of index 2. Then $1 + x$ is neither idempotent nor nilpotent of index 2.

Proof:

Note that $(1 + x)^2 = 1 + 2x$.

If $1 + x$ is idempotent then $1 + 2x = 1 + x \ and \ x = 0$, Contradiction !

If $1 + x$ is nilpotent of index 2 then $1 + 2x = 0$.

Multiply this equation by x to get $x = 0$, Contradiction !

Lemma 2.2.4. If R is a BZS ring and if $e \in Id(R)$, then $2e = 0$.

Proof:

If $e = 0$, the result is obvious.

Otherwise, we have $e \in Id(R) \Rightarrow e^2 = e$

Since R is BZS, then $a \in R, \ a^2 = a \ \ or \ \ a^2 = 0$

$(-e)^2 = e^2 = e \neq 0$.

Thus $(-e)^2 = -e \ \ and \ so \ e = -e \Rightarrow e + e = 0 \Rightarrow 2e = 0$

Lemma 2.2.5. Let R be a BZS ring, let $e \neq 0$ be idempotent, and let x be nilpotent. Then the element $e + x$ is idempotent.

Proof:

If $x = 0$ then the result is trivial.

Assume that $x \neq 0$. The proof is by contradiction.

Suppose that $e + x$ is nilpotent of index 2

Note that $(e + x)^2 = 0$

$e^2 + ex + xe + x^2 = 0$

$e + ex + xe = 0$

Multiply on the left by x to get

$xe + xex + x^2e = 0 \Rightarrow xe = -xex$

Multiply on the right by x to get

$ex + ex^2 + xex = 0 \Rightarrow ex = -xex$

Thus $ex = xe$

Then the equation $e + ex + xe = 0$ becomes

$e + 2ex = 0$

By Lemma 2,2,4 we have that $2ex = 0$

Hence we have $e = 0$ contradiction !

Lemma 2.2.6. If R is a BZS ring, then

a) $exe = 0$
b) $xex = 0$
c) $if\ e \neq 0\ then\ ex + xe = 0$

Proof:

If $x = 0$ then the results are trivial.

If $e = 0$ then a) and b) are trivial.

Otherwise, $e + x$ is idempotent by Lemma 2.2.5 , and we get

$e + ex + xe = e + x \Rightarrow ex + xe = x$ … (1)

Multiply eq(1) by e on the right and simplify to get $exe = 0$

Multiply eq(1) by x on the left to get $xex = 0$.

Lemma 2.2.7. If R is a properly BZS ring and $x \in N$ $then$ $2x = 0$

Proof:

By Lemma 2.2.6 c) and Lemma 2.2.5 we have $2x = 2ex + 2xe = 0$.

Lemm2.2.8. if R is a properly BZS ring, then the center of R is trivial.

Proof:

Let e be a non-zero idempotent element of R and let x be a non-zero nilpotent element of R.

If $ex = xe$ then by Lemma 2.2.6 c), $2ex = x \neq 0$,

Contradiction!

CHAPTER THREE

3.1. BZS Rings with Cyclic Additive Groups

Proposition 3.1.1.

Let $(R,+,.)$ be a BZS ring such that $(R,+)$ is a cyclic group of order $n \geq 2$, then either R is isomorphic to the ring of integers modulo 2 or it is a ring with identically zero multiplication.

Proof:

Suppose that g generates R additively

$R =< g >= \{g^n | n \in Z\}$

Since R is BZS either $g^2 = 0$ or $g^2 = g$

If $g^2 = 0$ then $g.g = 0$

Then for all integers $0 \leq \alpha, \beta < n - 1$

We have $(\alpha g).(\beta g) = 0$ since R is distributive

$$(\alpha.\beta)g^2 = 0$$

$$\alpha = \beta = 0$$

Then we get R is a ring with identity zero multiplication.

If $g^2 = g$ then $g.g = g$

$$(n-1)g.(n-1)g = (n-1)^2 g.g$$

$$= (n^2 - 2n + 1)g.g$$

$$= 1.g.g = g.g$$

$$= g \neq 0$$

$$so \ (n-1)g.(n-1)g = g.g$$

$$= g$$

$$= (n-1)g = -g$$

Implies $g + g = 0$

$$2g = 0$$

So that $n = 2$ and R is the ring of integers modulo 2.

Example 3.1.2.

Let R = $Z_2 \oplus Z_2$ with $S = (0,1), (1,0)$. Define the multiplication as follows:

$$a.b = f(x) = \begin{cases} a, & if\ b \in S \\ (0,0) & if\ b \notin S \end{cases}$$

We may verify that this multiplication satisfies the left distributive law as follows.

Suppose $a, b, c \in R$. Then:

a) If $b \in S\ and\ c \in S$, we have $a.(b + c) = (0,0) = a + a = a.b + a.c$

b) If $b \in S\ and\ c \notin S$, or if $b \notin S\ and\ c \in S$ we have $a.(b + c) = a$ $= a + (0,0) = a.b + a.c.$

c) If $b \notin S\ and\ c \notin S$ we have $a.(b + c) = (0,0) = (0,0) + (0,0)$ $= a.b + a.c.$

Thus R is a properly BZS ring since $(1,1).(1,1) = (0,0) and\ (0,1).(0,1) = (0,1)$

We can rewrite this example in terms of matrices.

Example 3.1.3.

Let R be the ring in example above put $x = (0,0), y = (1,1), e = (0,1)$, $f = (1,0)$. Now let S be the ring of 2×2 matrices over $\mathbb{Z}_2$ whose non-zero entries occur only in the last column. Thus ,

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} | 0,1 \in \mathbb{Z}_2 \right\}$$

The additive group of this ring is isomorphic to that of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Label this matrices as $\hat{x}, \hat{y}, \hat{e}, \hat{f}$ respectively. Then the map $g : R \longrightarrow S$ given by

$g(x) = \hat{x}, g(y) = \hat{y}, g(e) = \hat{e}, g(f) = \hat{f}$ is an isomorphism from R to S.

Lemma 3.1.4. If R is a properly BZS ring, then the additive group of R is isomorphic to a direct product of copies of $\mathbb{Z}_2$.

Proof:

By Lemma 2.2.4 and Lemma 2.2.7, we have that every non-zero element of R is of order 2; i.e., $(R, +,)$ is an elementary abelian group

Lemma 3.1.5. if a BZS ring R is commutative then R is not properly BZS.

Lemma 3.1.6. is R is a BZS ring then $N(R)$ is closed under subtraction.

Proof: if $x - y \notin N(R)$ *then* $0 \neq (x - y) \in Id(R)$

but then by lemma 2.2.5. $(x - y) + y = x$ is idempotent, contradiction!

Lemma 3.1.7. if R is BZS ring then $Nil(R)$ is closed under multiplication by any element of R.

Proof: if R is Boolean or if R consists only oof nilpotent elements then the result is trivial.

Otherwise, let e be a non-zero element $e \in Id(R)$ *and* $x \in N(R)$

by lemma 2.2.6 we have $(ex)^2 = 0$ *and* $(xe)^2 = 0$.

Hence $Id(R)\, N(R) \subseteq N(R)$ and $N(R)\, Id(R) \subseteq N(R)$.

Now let $x, y \in N(R)$ since lemma 3.1.6 implies that

$x + y \in N(R)$

$(x + y)^2 = 0$

$xy + yx = 0$ multiply on the right by xy to get

$(xy)^2 = 0$.

Lemma 3.1.8. If R is a BZS ring then $N(R)$ is an ideal.

Proof: this follows from Lemma 3.1.5 and Lemma 3.1.6 and the fact that

$0 \in N$.

Proposition 3.1.9. If R is a properly BZS ring and if $x, y \in N(R)$ then $xy = yx$.

Proof: since $N(R)$ is closed under multiplication we have

$0 = (x + y)^2 = xy + yx$. Thus $xy = -yx$. By lemma 2.2.6 we have

$-yx = yx$, and the result follows.

The ring given in example 3.1.2 and 3.1.3 shows that, in general idempotents of properly BZS rings need not commutative.

Proposition 3.1.10. If R is properly BZS ring and if $e, f \in Id(R)\{0\}$ then $e + f \in N$.

Proof: Let $e, f \in Id(R) \setminus \{0\}$. $if\ e = f\ then\ e + f = 0$.

Assume then that $e \neq f$.

Suppose that $e + f$ is a non-zero idempotent.

pick $0 \neq x \in N(R)$. Then $e + f + x$ is idempotent by Lemma 2.2.4 so

$e + f + x = (e + f + x)^2 = (e + f)^2 + ex + fx + xe + xf$

By Lemma 2.2.6 c) we have that

$x = ex + xe = fx + xf$

Hence $e + f + x = (e + f)^2 + x + x = e + f$

Which implies that $x = 0$, contradiction!

Thus $e + f$ is nilpotent.

Proposition 3.1.11. If R is BZS ring then Id(R) is closed under multiplication.

Proof: If R is Boolean or zero square, the result is trivial, so suppose R is properly BZS ring.

Let $e, f \in Id(R), e \neq f$. Note that

$(e + f)^2 = e + ef + fe + f$

By proposition 3.1.18 we have that $e+f$ is nilpotent. Thus

$e+ef+fe+f=0,$

So that $e+f=ef+fe.$

Multiply the last equation on the left by f and on the right by e and simplify to get

$fe=fefe.$

Proposition 3.1.13. If R is a BZS ring then one of the following holds.

$a)$ R is Boolean

$b)$ R is zero square

$c)$ R is an extension of a nil ring N whose elements are square zero by $\mathbb{Z}_2$.

Proof: This follows from proposition 3.1.12

3.2. Ideals of properly BZS rings

Proposition 3.2.1. If R is a properly BZS ring, then $N(R)$ is a maximal ideal of R of index 2. Further, $N(R)$ is the unique maximal ideal of R.

Proof: it follows from Lemma 2.2.5 and proposition 3.1.100 that the factor ring $R/N(R)$ has two distinct cosets:

$0+N(R)$ and $e+N(R)$ for some $e\in Id(R)\setminus\{0\}$ Since $N(R)$ has index 2 as an ideal in R.

If $I\nsubseteq N(R)$, then I contains a non-zero idempotent e.

Let $x\in N(R)$. Then $x=ex+xe\in I$ by Lemma 2.2.6 c).

Thus $N(R)\subsetneq I$.

Hence $I=R$ and the result follows.

Lemma 3.2.2. R is a properly BZS ring, then $|Id(R)\setminus\{0\}|=|N|$.

Proof: the result follows from proposition 3.2.1 and the fact that

$R=Id(R)\cup N(R)$ with $Id(R)\cap N(R)=\{0\}$.

Remark 3.2.3. In a Boolean ring every prime ideal must be maximal. Properly BZS share this special property.

Lemma 3.2.4. If R is a properly BZS ring, then for any $x \in N(R)$,

$(x)(x) = (0)\ where\ x$ is the principal two-sided ideal generated by x.

Proof: every element of (x) is a finite sum of terms of the form $x, ax, xb, or\ axb, for\ a, b \in R$.

It follows that every element of $(x)(x)$ is a sum of elements such that each summand either contains a factor of $x^2 = 0$ or contains a factor of the form xcx for some $c \in R$.

If $c \in Id(R)$, then Lemma 2.2.6, b) implies that $xcx = 0\ if\ c \in N(R)$ then proposition 3.1.9 implies that $xcx = cx^2 = 0$.

So each summad in an element of $(x)(x)$ is zero, and the result follows.

Theorem 3.2.5. If R is a properly BZS ring, then every prime ideal is a maximal ideal.

Proof: The result is trivial if R is the ring $\mathbb{Z}_2$, so we assume $|R| > 2$ (and hence $|N| \geq 2$).

If P is a prime ideal of R that is not maximal, P must be properly contained in the unique maximal ideal $Nl(R)$, thus there is some non-zero element $x \in N(R) \setminus P$.

From Lemma 3.2.4, we have $(0) = (x)(x) \subseteq P, whereas\ (x) \nsubseteq P$, contradiction!

Since they are always commutative, Boolean rings all have the property that every one-sided ideal is also a two sided ideal.

This property does not hold in general for BZS rings, as the following example shows.

Example 3.2.6. Let R be the ring from example 3.1.2 and let $I = \{(0,0), (0,1)\}$ then $(I, +)$ is clearly a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since $(0,0).r\ and\ (0,1).r$ are both in I

For every $r \in R, I$ is a right ideal of R.

However, I is not a left ideal of R since $(1,0).(0,1) = (1,0) \notin I$.

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الخلاصة

في هذا التقرير ندرس حلقة الصفرية التربيعية عديمة القوى (BZS) الحلقة R تسمى (BZS) اذا كانت جميع عناصرها اما عديمة القوى او ذو القوة 2 وقدمنا بعض الصفات على هذة الحلقة.

پوخته

ئێمه لەم توێژینەوەیە باسی Boolean-zero square ring (BZS) مان کرد ، دەڵێین R دەبێتە BZS ئەگەر هەموو دانەکانی ناو R یان Idempotent بێ یان Nilpotent of index 2 بی. وە بەشێک لە سیفاتەکانی ئەو رینگەمان ناساندووە.