زانكۆى سەلاحەدين - هەولێر
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Square Idempotent Rings

Research Project

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Supervisor Certificate

This research project has been written under my supervision and has been submitted for the award of the degree of BSc. in (mathematics).

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Dedication

This project is dedicated to:

Allah Almighty, my Creator and my Master,

My great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life,

My homeland Kurdistan, the warmest womb,

The Salahadin University; my second magnificent home;

My great parents, who never stop giving of themselves in countless ways,

My beloved brothers and sisters;

To all my family, the symbol of love and giving,

My friends who encourage and support me,

All the people in my life who touch my heart.

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Abstract

The purpose of the present work is to introduce and study the concepts of square idempotent rings as a generalization of the concept of the idempotent ring and find the square idempotent elements in the rings of integer modulo n and study some of its properties.

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INTRODUCTION

An element x is said to be idempotent if $x * x = x$. Idempotent elements have an important role in a decomposition of rings. Within any ring including a unity element, at least there are two idempotent elements, namely 0 and 1. These particular idempotent elements are commonly referred to as the trivial idempotent. In **Invalid source specified.** the concept of m-idempotent element introduced as a generalization of an idempotent element. An element x of a ring R is m-idempotent if m is the least positive integer such that $x^m = x$. A ring is called Boolean if each elements are idempotent.

Many ring theoretic generalizations of Boolean rings. Boolean like rings was introduced by Foster (Foster., 1946). is a commutative ring with identity of characteristic 2 in which (1— a) a (1— b) b $=$ 0 holds for all elements a, b of the ring. .In recent times, there have been many studies on certain aspects of idempotent elements, for example see (Dereje , et al., 2022) (Venkateswarlu & Wasihun, 2020) and (Mellese, 2020)

In this paper, we introduce the concept of square idempotent elements. An element in R is called a $Square\ \ idempotent\ \ element$ (SIE) if a^4= a^2. A ring $(R, +, \ .)$ is called a Square idempotent ring (SIR) if R is of characteristic 2 and a^4= a^2 for each $a \in R$ for each $a \in R$.

The project consists three chapters. In chapter One, we give some necessary definition and theorems in a ring theory we needed in the project. The second chapter, consists of two section, in section one, we give some basic properties of square idempotent elements. The second section, we find the square idempotent element in Z_n, the ring of integer modulo n. The third chapter, we give some properties in ideals of SIR.

CHAPTER ONE

1.1. Background

Definition 1.1.1. (DAVID, 2004) a non empty set R is said to form a ring with respect to the binary operations addition (+) and multiplication(.) provided for arbitrary $a, b, c \in R$, the following properties hold:

1. $(a+b)+c = a+(b+c)$ (associative law addition)
2. $a+b = b+a$ (Commutative Law of addition)
3. $(a.b).c = a.(b.c)$ (associative law multiplicative)
4. $a(b+c) = a.b + a.c$ (distributive)

Example 1.1.2. $(R,+,.), (Q,+,.), (C,+,.)$ are rings

Definition 1.1.3. (DAVID, 2004)R is called a commutative ring if multiplication is commutative; and called with identity if R has a multiplicative identity element.

Definition 1.1.4. (Dummit, & Foote,, 2004) Let n be a fixed positive integer. Two integers $a\ and\ b$ are said to be ***congruent modulo n***, written $a \equiv b(mod\ n)$ if and only if $a - b = kn$ for some integer k or $(a-b)$ is divisible by n.

Theorem 1.1.5. (First isomorphism theorem)

If f is a homomorphism from the ring $(R,+,.)$ onto the ring $(R',+',.')$. Then

$$\left({}^{R}/_{\ker f}, +,'\right) \cong (R',+',.').$$

Proposition 1.1.6. In a ring with identity every proper ideal is contained in a maximal ideal

Proposition 1.1.7. Assume R is commutative. The ideal M is a maximal ideal if and only if the quotient ring R/ M is a field.

Proposition 1.1.8. Assume R is commutative. Then the ideal P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.

Corollary 1.1.9. Assume R is commutative. Every maximal ideal of R is a prime ideal.

Definition 1.1.10. (DAVID, 2004)An element a of a commutative ring R is called **nilpotent** if a^n= 0 for some positive integer n.

Definition 1.1.11. (DAVID, 2004)a left, right or two-sided ideal of a ring is said to be a **nil ideal** if each of its elements nilpotent.

Definition 1.1.12. (Dummit, & Foote,, 2004) An element a of a ring R is said to be **idempotent** if a^2=a.

Definition 1.1.13. (Dummit, & Foote,, 2004)**Boolean ring: :** is a commutative ring with unity if all elements of satisfying a^2=a

Definition 1.1.14 (Dereje , et al., 2022) **Boolean like ring:** is a commutative ring with unity is of characteristic 2 in which ab(1+a)(1+b)=0

Example 1.1.15. $(Z_2, +_2, \cdot_2)$

Remark. It is clear that every Boolean ring is a Boolean like ring but not conversely. We substantiate this in the following example.

Example 2.1.17. The ring $(H4, +, *)$ with $H4 = \{0, 1, p, q\}$ and + and * are defined by the following tables is a Boolean like ring, but not a Boolean ring

+	0	1	P	q
0	0	1	P	q
1	1	0	Q	p
P	p	q	0	1
Q	q	p	1	0

Table 1

*	0	1	P	Q
0	0	0	0	0
1	0	1	P	Q
P	0	p	0	P
Q	0	q	P	1

Table 2

Definition

Let R be a ring. We say that $a \in R$ is a quadratic residue if $a = x^2$, for some $x \in R$.

Definition 2.1.29. (Dummit, & Foote,, 2004)Let $\emptyset : R \to S$ be a ring homomorphism. The set $\{a \in R \mid \emptyset(a) == 0\}$ is called the kernel of $\emptyset$, denoted by Ker($\emptyset$).

Definition 2.2.15. (Dereje , et al., 2022)the ideal P of a SIR R is called **completely prime** if and only if $ab \in P$ implies $a \in P$ or $b \in P$.

CHAPTER TWO

2.1. Basic Properties in Square Idempotent Rings

In this section we give some properties in elements in square idempotent rings, we started by definition of SIR.

Definition 2.1.1. A ring $(R, +, \ .)$ is called a $Square\ \ idempotent\ \ ring$ (SIR) if R is of characteristic 2 and a^4= a^2 for each $a \in R$

Example 21.2. Let $U_2(Z_2)$ be the ring of 2×2 upper triangular matrices over with the usual addition and multiplication of matrices. Then $a^4 = a^2$ and $a + a = 0$, for all $a \in U_2(Z_2)$ Clearly is non-commutative ring but is Weak idempotent ring with unity.

Example 2.1.3. Let $R = Z_2$. Define " + " and "* " on $\overline{R} = R \times R$ by (a, b) + (c, d) = $(a + c, b + d)$ and (a, b) * (c, d) = (bc, bd) for $(a, b), (c, d) \in \overline{R}$. Then $(\overline{R}, +,*)$ is a Weak idempotent ring with unity $(a, 1)$, for any $a \in$ R, as a left unity. But R has no right unity and hence R has no unity.
Furthermore, the ring $\overline{R}$ is a non-commutative ring since (1,0) * $(a$, 1) = (0,0) ≠ (1,0) = $(a$, 1) * (1,0). Thus, $(\overline{R}, +,\cdot)$ is a non-commutative Weak idempotent ring and without unity.
Example 2.1.4. The quaternion ring Q over the field $\mathbb{Z}_2$ is a commutative ring with unity satisfies that $a^4 = a^2$ and $a + a = 0$ for all a ∈ Q. Hence Q is a commutative WIR with unity.

Lemma 2.1.5. Every idempotent element is a square idempotent element.
But the convers is not true .

Example 2.1.6.($Z_4, +_4, ._4$).

Lemma 2.1.7. Let R be a SIR with unity. Then $a \in R$ is a unit if and only if $a^2 = 1$

Proof. Let a be a unit with inverse a^{-1}. Since

$a^2(1 + a^2) = a^2 + a^4 = a^2 + a^2 = 0;$

then by multiplying by a^{-2}on the left we get $1+a^2 = 0$ that implies the conclusion.

Lemma 2.1.8. Let R be a SIR with unity. Then $a \in R$ is a unit if and only if
1 + a is nilpotent.

Proof. $(1 + a)^2 = 1 + a^2 = 0$ iff $a^2 = 1$

Lemma 2.1.9. Every non-zero, non-unit in a SIR R with unity is a zero-divisor.

Proof: Let $0 \neq a \in R$ be a non-unit. Since $a^4 = a^2$ implies $a(a^3 + a) = 0$.

If $(a^3 + a) \neq 0$, then a is a zero divisor.

If $(a^3 + a) \neq 0$, that is $a(a^2 + 1) = 0$. . Since a is non-unit, $a^2 + 1 \neq 0$.

Hence, a is a zero divisor in R.

Proposition 2.1.10. The set of all unit elements of a SIR with unity is precisely $\{1+n\colon n\in N\}$.

Proof. Let R be a SIR and a be a unit element of R. Then $(1 + a)^2 = 1 + a + a + a^2 = 0$, as $a^2 + 1 = 0$. Hence, $1 + a$ is nilpotent and $a = 1 + (1 + a)$. On the other hand, for any nilpotent element n in R, $(1 + n)^2 = 1 + n + n + n^2 = 1$.

Thus, $1 + n$ is a unit element in R.

Lemma 2.1.11. Let R be a SIR. Then for all $a \in R$

(1) a^n= a, a^2 or a^3 for any positive integer n.

(2) If 0≠ a is a nilpotent element, then a^3= 0.

(3) a = a^2+ (a^2+ a), where a^2 is idempotent and a^2+ a is nilpotent.

Proof. If 0 = $a^n = a^3$*; then* $a^2 = a^4 = aa^3 = a0$ = 0.

3. (a^2+ a)^2 = $a^4 + a^2 = a^2 + a^2$= 0.

Remark 2.1.12: Let R be a SIR. We denote the set of all idempotent elements of R by $Id(R)$ and the set of all nilpotent elements of R by $Nil(R)$.

Proposition 2.1.13: Let R be a commutative square idempotent ring with unity**.** For any two elements a and b of R , the following are satisfied.

1. $Id(a + b) = Id(a) + Id(b)$ *and* $Nil(a + b) = Nil(a) + Nil(b)$
2. $Id(ab) = Id(a)Id(b)$ and

$$Nil(ab) = Id(a)Nil(b) + Nil(a)Id(b) + Nil(a)Nil(b)$$

3. $Id(ab) = 0$ and $Nil(ab) = ab, if\ b$ is nilpotent.

2.2. Square idempotent elements in Z_n

In this section, we find the square idempotent elements in the ring of integer modulo n.

Proposition 2.2.1: In Z_p, p is prime. The only nontrivial square idempotent element is $p - 1$.

Example 2.2.2: In z_{11} , the only SIE is 10 since $10^4 = 1 = 10^2$

Proposition 2.2.3: In Z_{2p}, p is prime. The nontrivial square idempotent elements are $p - 1, p, p + 1$ *and* $2p - 1$.

Example 2.2.4: In Z_{14}, the nontrivial SIE are $6, 7, 8$ and 13.
Since $6^4 = 8 = 6^2$, $7^4 = 7 = 7^2$, $8^4 = 8 = 8^2$, $13^4 = 1 = 13^2$.

Examp2.2.5: Each element of Z_4 is square idempotent but $a + a = 0$ is not true for all $a \in Z_4$ since $3 + 3 = 2 \neq 0$. Thus, Z_4 is not a SIR.

Proposition 2.2.6: In Z_{pq} is prime. The nontrivial square idempotent elements are $q - 1, q, q + 1, 2q - 1, 2q, 2q + 1$ *and* $pq - 1$

Example 2.2.7: In Z_{15}, the nontrivial SIE are 4,5,6,9,10,11*and* 14.
Since $4^4 = 1 = 4^2$, $5^4 = 10 = 5^2$, $6^4 = 6 = 6^2$, $9^4 = 6 = 9^4$, $10^4 = 10 = 10^2$, $11^4 = 1 = 11^2$, $14^4 = 1 = 14^2$

Proposition 2.2.8: In Z_{pqr} is prime. The nontrivial square idempotent element are $q - 1, q, q + 1$

Example 2.2.9: In Z_{30}, the nontrivial SID are 4,5,6,9,10,11,14,15,16,19,20,21,24,25,26,29

Since

$4^4 = 16 = 4^2$, $5^4 = 25 = 5^2$, $6^4 = 6 = 6^2$, $9^4 = 21 = 9^2$, $10^4 = 10 = 10^2$, $11^4 = 1 = 11^2$, $14^4 = 16 = 14^2$, $15^4 = 15 = 15^2$, $16^4 = 16 = 16^2$, $19^4 = 1 = 19^2$, $20^4 = 10 = 20^2$, $21^4 = 21 = 21^2$, $24^4 = 6 = 24^2$, $25^4 = 25 = 25^2$, $26^4 = 16 = 26^2$, $29^4 = 1 = 29^2$.

Remark 2.2.10:.

The ring of integers modulo n is not a SIR

Chapter Three

1.3. Ideal and quotient in square Idempotent rings.

Lemma 3.1.1. Let R be a square idempotent ring. If R is a non-commutative, then the set of all idempotent elements $\mathrm{Id}(\mathrm{R})$ need not be a subring of R.

Remark 3.1.2. Let R be a square idempotent ring Commutativity is sufficient condition for $\mathrm{Id}(\mathrm{R})$ to be a subring of R and Nil(R) to be an ideal of R.

Proposition 3.1.3. If R is a local ring with unity, then the only idempotents are 0 and 1.

Proof. Let R be a local ring and $R = A \cup M$, where A is the set of all unit elements in R and M is the maximal ideal of R and for every $x \in R$ such that $x^2 = x$, either $x \in A$ or $x \in M$.

If $x \in A$, then $x = 1$. Otherwise, $x \in M$ and hence $1 - x \in A$ and $1 - x$ is an idempotent. So $x - 1 = 1$ implies that $x = 0$. .

Hence 0 and 1 are the only idempotents of the ring

Theorem 3.1.4. If R is a SIR with unity such that $I(R) = \{0,1\}$, then every proper ideal of R is a nil ideal.

Proof: For every $x \neq 0,1$, Since x^2 is idempotent, then either $x^2 = 0$ or $x^2 = 1$. Thus x is nilpotent or a unit. Let M be proper ideal of R. If $x \in M$ then x≠1. Suppose x≠0. Then x is nilpotent or a unit. But M does not contain a unit element. Thus x is nilpotent and hence $M \subseteq N$. Hence, M is nil.

Theorem 3.1.5. Let R be SIR which is a local ring with unity. Then we have:

1. The set $Nil(R)$ of all nilpotent elements of R is the unique maximal ideal of R.
2. R is a commutative ring.

Proof. (1) Let M be the unique maximal ideal. By Lemma and by Theorem we have that M is a nil ideal. Let a be a nilpotent element such that $a \in M$. Then there exist $r,s \neq 0$ such that 1 = *ras*. This implies that *r* = *rras* and *s* = *rass*. Hence, r and s are units $(i.e., \ r_2 \ = \ s_2 \ = \ 1)$. By 1 = *ras* we derive *rs* = *rrass* = a. Thus *rsrs* = 0 and, multiplying by $s^{-1}r^{-1}$ on the left, we get

a = *rs* = 0. Contradiction. We conclude that $M \ = \ N(R)$.

(2) The product of units is commutative: *ab* = *ba* iff *a* = *abb* = *bab* iff 1 = *aa* = *abab*. This last condition is true because *ab* and *ba* are units. The product of nilpotents is commutative. If *a*, *b* are nilpotent, then 1 + *a* and 1 + *b* are units. Then, 1+*b*+*a*+*ab* = (1+*a*)(1+*b*) = (1+*b*)(1+*a*) = 1+*a*+*b*+*ba*. By subtracting 1+*a*+*b* we get *ab* = *ba*. The product of a unit and a nilpotent is commutative. If *a* is nilpotent and b is a unit, then we have *b* + *ab* = (1 + *a*)*b* = *b*(1 + *a*) = *b* + *ba*. Then subtracting b we get ab = ba.

Theorem 3.1.6. Every non-commutative square idempotent ring R with unity is not local.

Proof. Let R be a non-commutative ring with unity. Suppose R is a local ring. Then by Theorem 3, N is an ideal of R and for all $a, b \in N$ and $ab = ba$.

Let $c, d \in R$, where $c = c_B + c_N$ and $d = d_B +$ d_N. Then $cd = (c_B + c_N)(d_B + d_N) = c_B d_B + c_B d_N +$

$cNdB$ + $cNdN$= $dBcB$ + $dBcN$ + $dNcB$ + $dNcN$ = dc since cB and dB are in the center of the ring (either 0 or 1). Thus R is commutative and it is a contradiction. Hence R is not local.

Theorem 3.1.7. Let I is an ideal of a square idempotent ring R R , then R/I is a Square idempotent ring.

Proof: It is obvious that R/I is a ring .

Let $a + I \in R/I$. Then $(a + I) + (a + I)$=$a + a + I = 0 + I = I$

and $(a + I)^4 = a^4 + 1 = a^2 + 1 = (a + 1)^2$

Hence, R/I is a SIR.

Note 3.1.8. For a ring , if R/I Square idempotent rings, then R need not be Square idempotent ring.

Example 3.1.9: $R = Z_4$, the set of all integers modulo 4, $I = \{0,2\}$and . Then R/I and I are Square idempotent rings but R is not a SIR as the characteristic of R is 4, not 2.

Lemma 3.1.10. If R is a commutative SIR, then the map $f: R \to R$, define by $f\ (x)$ =x^2, is an endomorphism of R.

Proof: It is obvious that f is well defined. Let $a, b \in R$. Then $f(a + b) = (a + b)^2$= $a^2 + b^2 = f(a) + f(b)$ and $f(ab)$= $(ab)^2$=a^2b^2= $f(a)\,f(b)$.

Hence, f is an endomorphism of R.

Theorem 3.1.11. Every completely prime ideal of a SIR with unity is maximal ideal.

Proof: Assume P is a completely prime ideal of R and J is an ideal of R such that $P \subsetneq J \subset R$. Let $a \not\subset P$and $a \in J$. Then $a^2 \not\subset P$ because of P is a completely prime ideal of R. Now a^4=a^2 implies $a^2(a^2+1) = 0 \in P$. Then $a^2+1 \in P$ because of P is a completely prime ideal of R. Thus, $a^2+1 \in J$.Since $a^2 \in J$, $a^2+a^2+1 \in J$ which implies $1 \in J$. Thus, $J = R$. Hence, P is a maximal ideal of R.

Theorem 5. Let R be a commutative square idempotent ring. Then Id(R) is isomorphic to $R/Nil(R)$.

Proof. Put $Nil(R) = N$Define $f : Id(R) \to R/N$ by $f(a) = a + N$.

Clearly f is a well-defined ring homomorphism

Suppose for $a, b \in Id(R)$, $f(a) = f(b)$.

Then $a + N = b + N \Rightarrow a + b \in N \Rightarrow (a + b)^2 = 0 \Rightarrow a^2 + ab + ba + b^2 = 0$

$\Rightarrow a + b = 0 \Rightarrow a = b$.

Thus, f is monomorphism. For a + N $\in R/N$,

$a + N$ = Id(a) + Nil(a) + N = Id(a) + N = f(Id(a)), where i d (a)∈Id(R).

Thus f is an epimorphism and hence it is an isomorphism.

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پوختە

ئامانجی ئەم کارە ناساندن ولێکۆلینەوە لەچەمکەکانی square idempotent rings وەک گشتاندنێکە بۆ چەمکی idempotent ring دۆزینەوەی square idempotent elements لە rings of integer modulo n لێکۆلینەوە لە هەندێک تایبەتمەندیەکانی.