Solve the following problems in Group Theory

Q1/ Suppose $(H,*)$ and $(K,*)$ are normal subgroups of the group $(G,*)$ with $H \cap K = [e\}$.
Show that $h * k = k * h$ $for\ all\ h \in H\ and\ k \in K.$

Q2/ In the commutative group $(G,*)$, let the set H consist of all elements of finite order.
Prove that

a) $(H,*)$ is normal subgroup of $(G,*)$, called the torsion subgroup .
b) The quotient group $(G/H,\ \otimes)$ is torsion –free; that is, none of its elements other than the identity are of finite order.

Q3/ Show that a group $(G,*)$ is commutative if and only if $[G,G] = \{e\}$.

Q4/ Let $(H,*)$ be a subgroup of $(G,*)$. If $x^2 \in\ H\ for\ all\ x\ \in\ G$, prove that

(a) $(H,*)$ is a normal subgroup of $(G,*)$.
(b) The quotient $(G/H,\ \otimes)$ is commutative.

Q5/ In the following, determine whether the function f is a homomorphism.

a) $f(a) = -a,\ f\colon(\mathbb{R},+) \longrightarrow (\mathbb{R},+)$.
b) $f(a) = |a|,\ f\colon(\mathbb{R}-\{0\},.) \longrightarrow (\mathbb{R}^{+},.)$.
c) $f(a) = a/q\ ,(q\ a\ fiext\ integer)\ \ f\colon(Z,+) \longrightarrow (Q,+)$

Q6/ Consider two groups $(Z,+)$ and $(G,.)$ $with$ $G = \{-1,1,-i,i\}$ where $i^2 = -1$. Show that
the mapping defined by $f(n) = i^n, for\ n \in Z$ is a homomrphism from $(Z,+)$ onto $(G,*)$
and determine $ker\ f$.

Q7/ Prove that if the quotient group $(G/cen(G),\ \otimes)$ is cyclic , then $(G,*)$ is a commutative group.

Q8/ Show that two groups $(R^{\#},+)and\ (R^{\#}-\{0\},.)$are not isomorphic.

Q9/ Let the set $G = Z \times Z$ and binary operation * on G given by the rule
$(a,b)*(c,d) = (a+c,b+d)$. Then

a) Show that $(G,*)$ is a commutative group.
b) Show that the mapping $f\colon G \longrightarrow Z$ defined by $f[(a,b)] = a$ is a homomorphism .
c) Determine the kernel of f.
d) If $H = \{(a,a) \mid a\ \in Z\}$, prove that $(H,*)$ is a subgroup of $(G,*)$.

Q10/ Let $(H,*)$ be a proper subgroup of $(G,*)$ such that $for\ all\ x,y\ \in\ G-H, xy\ \in\ H.$ Prove
that $(H,*)$ is a normal subgroup of $(G,*)$.

Q11/ Let $(G,*)$ be a group and $H\ and\ N$ are proper normal subgroups of $(G,*)$. Suppose
$G\ = H \cup N$ and $H \cap N = \{e\}$. Prove that $(G,*)$ is commutative.

Q13/ Let $H = \{e, (1\ 4)(2\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$, where e is the identity permutation. Determine whether or not H is a normal subgroup of S_4.

Q14/ Let $(H,*)$ and $(K,*)$ be subgroups of a group $(G,*)$ such that $(H,)$ is a normal subgroup of

$(G,*)$. Prove that $H \cap K$ is a normal subgroup of K.

Q15/ Let $(G,*)$ be a finite commutative group. Let $n \in Z$ be such that n and |G| are relatively prime. Show that the function $f : G \longrightarrow G$ defined by $f(a) = a^n \ for\ all\ a \in G$ is an isomorphism of $(G,*)$ onto $(G,*)$.

Q16/ Determine whether the indicated function f is a homomorphism from the first group into the second group. If f is a homomorphism, determine its kernel.

a) $f(a) = a^2,\ (\mathbb{R}^+,.), (\mathbb{R}^+,.)\ for\ all\ a \in \mathbb{R}^+$.
b) $f(a) = |a|,\ (\mathbb{R} - \{0\},.), (\mathbb{R}^+,.)\ for\ all\ a \in \mathbb{R} - \{0\}$.
c) $f([a]) = [5a], (Z_8,\ +_8), (Z_8,\ +_8)$

Q17/ Show that $(Q, +)$ is not cyclic.

Q18/ Give an example of a group $(G,*)$ and a subgroup $(H,*)$ of $(G,*)$ such that $aH = bH$, but

$Ha \neq Hb$ for some $a, b \in G$.

Q19/ Let G be a noncyclic group of order p^2, p a prime integer. Show that the order of each non-identity element is p.

Q20/ Let $G = \{a, b, c, d\}$ be a group. Complete the following Cayley table for this group.

*	a	b	c	d
a				
b				
c			b	
d		b		

Q22/ Prove or disprove

1- Let $G = (a)$ be a cyclic group of order 30. Then $[G : (a^5)] = 5$.
2- Every proper subgroup of a group of order p^2 (p aprime) is cyclic.
3- A subgroup $(H,*)$ of a group $(G,*)$ is a normal subgroup if and only if every right coset of H is also a left coset.
4- If $A, B\ and\ C$ are normal subgroups of a group G, then $A(B \cap C)$ is a normal subgroup of G.
5- Every commutative subgroup of a group G is a normal subgroup of G.

6- If G is a group of order $2p$, p an odd prime, then either G is commutative or G contains a normal subgroup of order p.
7- If every element of a group G is of finite order, then G is a finite group.
8- The group $(Z, +)$ is isomorphic to $(Q, +)$.
9- If f and g are two epimorphisms of a group G onto a group H such that $Ker\ f = Ker\ g$, then $f = g$.
10- $(Z \times Z, +)$ is a cyclic group.

Solve the following problems in Ring Theory

Q1/ In a ring $(Z,\oplus,\odot)$, where $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$, $for\ all\ a, b \in Z$. Find zero element and identity element.

Q2/ Let R denote the set of all functions $f: R^{\#} \longrightarrow R^{\#}$. The sum $f + g$ and the product $f.g$ of

two function $f, g \in R$ are defined by
$(f+g)(x) = f(x) + g(x)$, $(f.g)(x) = f(x).g(x), x \in R^{\#}$.
Show that $(R, +, .)$ is the commutative ring.

Q3/ Let $(R, +, .)$ be an arbitrary ring. In R define a new binary operation * by
$a * b = a.b + b.a\ for\ all\ a, b \in R$. Show that $(R, +, *)$ is a commutative ring.

Q4/ Show that the multiplicative identity in a ring with unity R is unique.

Q5/ Suppose that R is a ring with unity and that $a \in R$ is a unit of R. Show that the

Multiplicative inverse of a is unique.

Q6/ Let (3Z , +) be an abelian group under usual addition where $3Z = \{3n \mid n \in Z\}$. Show that $(3Z, +, \odot)$ is a commutative ring with identity 3, where $a \odot b = \frac{ab}{3}$, $for\ all\ a, b \in 3Z$.

Q7/ Let $(R, +, .)$ be a ring which has the property that $a^2 = a\ \ for\ every\ a \in R$. Prove that $(R, +, .)$ is a commutative ring. [Hint: First show $a + a = 0, for\ any\ a \in R$].

Q8/ Prove that a ring R is commutative if and only if
$$a^2 - b^2 = (a+b)a - b), for\ all\ a, b \in R.$$

Q9/ Prove that a ring R is commutative if and only if
$$(a + b)^2 = a^2 + 2ab + b^2, for\ all\ a, b \in R.$$

Q10/ Let R be the set of all ordered pairs of nonzero real numbers. Determine whether $(R, +, .)$ is
a commutative ring with identity.
(a) $(a,b) + (c,d) = (ac, bc + d), \quad (a,b).(c,d) = (ac, bd)$
(b) $(a,b) + (c,d) = (a + c, b + d), \quad (a,b).(c,d) = (ac, ad + bc)$.

Q11/ Find all units in the rings

1- $(Z_9, +_9, \times_9)$. 2- $Z \times Z$ 3- $Z_3 \times Z_3$ 4- $Z_4 \times Z_6$.

Q12/ Is Z_2 a subring of Z_6? Is $3Z_9$ a subring of Z_9?

Q13/ Give an example of a division ring which is not a field.

Q14/ Prove that $T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \ a, b, c \in \mathbb{R} \right\}$ is a subring of $M_2(\mathbb{R})$.

Q15/ In $(Z_{12}, +_{12}, \times_{12})$, find (i) $(2)^2 +_{12} (9)^{-2}$.

Q16/ Suppose that a and b belong to a commutative ring and ab is a zero-divisor. Show that either a or b is a zero-divisor.

Q17/ Complete the operation tables for the ring $R = \{a, b, c, d\}$:

+	a	b	c	d
	a	b	c	d
B	b	a	d	c
C	c	d	a	b
D	d	c	b	a

.	a	b	c	d	
a	a	a	a	a	
b	a	b			
c	a			a	
d	a	b	c		

Is R a commutative ring? Does it have a unity? What is its characteristic?
Hint. $c.b = (b + d).b$; $c.c = c.(b + d)$; etc.

Q18 Let R and S be commutative rings. Prove or disprove the following statements.
(a) An element $(a,b) \in R \times S$ is nilpotent if and only if a nilpotent in R and b is nilpotent in S.

(b) An element $(a,b) \in R \times S$ is a zero divisor if and only if a is a zero divisor in R and b is a zero divisor in S.

Q19/ Show that $Q[\sqrt{2}] = \{a + b\sqrt{2} \in R \mid a, b \in Q\}$ is a subfield of the field R.

Q20/ Find a subring of $\mathbb{Z} \times \mathbb{Z}Z$ that is not an ideal.

Q21/ If A and B are ideals of a ring, show that the sum of A and B, $A + B = \{a + b: a \in A, b \in B\}$ is an ideal.

Q22/ Let I be an ideal of a commutative ring R. Define the annihilator of I to be the set

$ann(I) = \{r \in R \mid ra = 0 for\ all\ a \in I\}$. Prove that $ann(I)$ is an ideal of R.
In the ring Z_{20}, $I = <4>$ is an ideal. Find $ann(I)$.

Q23/ Let $(I, +, .)$ be an ideal of the $(R, +, .)$. Define $C(I)$ to be the set

$C(I) = \{r \in R \mid ra - a.r \in I, for\ all\ a \in R\}$. Prove that $C(I)$ is a subring of R.

Q24/ Which of the following rings are fields? Are integral domain? Why?
i) $(Z, +, .)$ ii) $(Z_5, +_5, ._5)$ iii) $(Z_{25}, +_{25}, ._{25})$.

Given that $(I, +, .)$ is an ideal of the ring $(R, +, .)$, answer the following equations:

Q25/ Whenever $(R, +, .)$ is commutative with identity, then so is the quotient ring $(R/I, +, .)$.

Q26/If $(R, +, .)$ is a principal ideal ring , then so is the quotient ring $(R/I, +, .)$.

Q27/The ring $(R/I, +, .)$ may have divisor of zero, even though $(R, +, .)$ does not have any.

Q28/ In the ring Z_{24}, show that $I = \{[0], [8], [16]\}$ is an ideal. Find all elements of the quotient ring Z_{24}/I.

Q29/ Let R be an ideal of a ring R. Prove that the quotient ring R/I is a commutative ring if and only if $ab - ba \in I\ for\ all\ a, b \in R$.

Q30/ Find all prime ideals and all maximal ideals of $Z_2 \times Z_4$.

Q31/ Find a prime ideal of Z x Z that is not maximal.

Q32/ Find a nontrivial proper ideal of Z x Z that is not prime.

Q33/ Find a subring of the ring Z x Z that is not an ideal of Z x Z.

Q34/ Describe the quotient rings in $Z/4Z$, $Z_{12}/(3)$, $2Z/8Z$ and $Z \times Z/2Z \times \{0\}$.

Q35/ Consider the equation $x^2 - 5x + 6 = 0$. Find all solutions of this equation in Z_7, Z_8 and Z_{12} .

Q36/ Find all units, zero-divisors, and nilpotent elements in the rings Z_{24} $Z \times Z$, $Z_3 \times Z_3$, $Z \times Q$and $Z_4 \times Z_6$.

Q37/ Prove or disprove

1- If $f:(R,+,.) \longrightarrow (R',+',.')$ is a homomorphism and $(I,+,.)$ is an ideal of $(R,+,.)$, then $(f(I),+',.')$ is an ideal of $(R',+',.')$.
2- If a ring $(R,+,.)$ have divisor of zero, then so is $(R/I,+,.)$.
3- Every subring is an ideal of the ring $(R,+,.)$.
4- Every maximal ideal of commutative ring with identity is prime.
5- Every primary ideal is a prime.
6- If a ring $(R/I,+,.)$ have divisor of zero, then so is $(R,+,.)$.
7- The cancellation law holds in any ring.
8- Every field is an integral domain.
9- Let $(R,+,.)$ be a ring. Then $(cent\ (R),+,.)$ is a subring of $(R,+,.)$
10- If $(R,+,.)$ is an integral domain, then so is $(R/I,+,.)$.
11- If $(I,+,.)$ is an ideal of the ring $(R,+,.)$ containing the identity element, then $I = R$.
12- Every integral domain is a field.
13- Every ring has a multiplicative identity.
14- Every maximal ideal of a ring R is a prime ideal.
15- Every prime ideal of a ring R is a prime ideal.
16- If $(R,+,.)$ is an integral domain, then so is $(R \times R,+,.)$.
17- The characteristic of an integral domain $(R,+,.)$ is either zero or a prime.
18- Any ring without identity, then so is subring.
19- Any ring with identity, then so is subring.
20- If $(R,+,.)$ is a ring such that $R \neq \{0_R\}$, then the element 0_R and 1_R are distinct.
21- The union of two ideals is ideal
22- The intersection of two ideals is ideal.
23- Every finite integral domain is field.
24- Every division ring is field.
25- Every field is division ring.
26- Every maximal ideal of commutative ring with identity is a prime ideal.
27- If $(R,+,.)$ is commutative with identity, then so is the quotient ring $(R/I,+,.)$.
28- the multiplicative identity in a ring with unity R is unique.