Engineering Analysis

Lec.2

Fall course 2021-2022

3rd Year

Nyan Dawood

Second order Linear ODEs with Constant Coefficients

(Homogeneous)

We consider second-order homogeneous linear ODEs whose coefficients *a* and *b* are constant

$$y'' + ay' + by = 0. \quad \text{.......................(1)}$$

The equation can be solved by using ƛ operator where each

$$ƛ = \frac{d}{dx}, \text{ and } \quad ƛ^2 = \frac{d^2y}{dx^2}$$

Re write equation (1) substituting ƛ instead of derivative

$$(ƛ^2 + a\,ƛ + b)\,y = 0 \quad \text{..............(2)}$$

The proportional Integral solution is :

$$Yh = C_1 e^{ƛ_1 X} + C_2 e^{ƛ_2 X}$$

$$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant. The corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

(**Case I**) *Two real roots if* $a^2 - 4b > 0$,

(**Case II**) *A real double root if* $a^2 - 4b = 0$,

(**Case III**) *Complex conjugate roots if* $a^2 - 4b < 0$.

example

$Y'' - y = 0$ will be $(\lambda^2 - 1)y = 0$, Solve to find ƛ value

$(\lambda+1)(\lambda-1)=0$ or $\lambda = \pm 1$

$$y = c_1 e^{x} + c_2 e^{-x}.$$

If ($a^2 - 4b = 0$) this will result of just one root, $\lambda = -a/2$

This will make $y_1 = C_1 e^{\lambda x}$, and $y2 = C_2 \text{ x } e^{\lambda x}$

So the corresponding general solution will be :

$$y = (c_1 + c_2 x)e^{-ax/2}.$$

The complex conjugate root result :

$$ƛ1 = -\frac{a}{2} + iw \quad ; \quad ƛ2 = \frac{-a}{2} - iw$$

To find bases of real solution

$$y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x$$

And general solution :

$$y = e^{-ax/2} (A \cos \omega x + B \sin \omega x)$$

Tutorials : Solve 2nd order linear ODE

1) $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$

2) $y'' + y' + 0.25y = 0, \quad y(0) = 3.0, \quad y'(0) = -3.5.$

3) $y'' + 0.4y' + 9.04y = 0, \quad y(0) = 0, \quad y'(0) = 3.$

Second order Linear ODEs with Constant Coefficients

(non-Homogeneous)

Standard form : $y'' + a y' + b y = f(x)$

This equation solve by : $y = yh + yp$

$Yh = C_1 e^{\lambda_1 X} + C_2 e^{\lambda_2 X}$, let $u_1 = e^{\lambda_1 X}$; $u_2 = e^{\lambda_2 X}$

$Yp = u_1 v_1 + u_2 v_2$

$$v_1' = \frac{\begin{vmatrix} 0 & u_2 \\ f(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}}$$ then integrate to find v_1

$$v_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & f(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}}$$ then integrate to find v_2

Solve the following DE

- y''+ 3y'+2y $=e^{x}$

  Ans.: $C_1\ e^{-x} + C_2\ e^{-2x} + \frac{1}{6}\ e^{x}$

- y''+ y =tan x

  Ans.: C_1Cosx+ $C_2\ Sin\ x$ – Cos x ln | sec x + tan x|

Partial Differential Equations PDE

Partial differential equations are equations that involve partial derivatives of an unknown function depends on two or more independent variables.

$ODE:$ $\frac{d\,u(x)}{dx}$ =0 $\xrightarrow{\int}$ u(x)=C ;constant

Type equation here.

$PDE:$ $\frac{\partial\,u(x,y)}{\partial x}$ =0 $\xrightarrow{\int}$ u(x,y) =f(y) ; arbitrary function

Some important abbr. $\frac{\partial u}{\partial x}$ named u_x ; $\frac{\partial u}{\partial y}$ named u_y ; $\frac{\partial^2 u}{\partial x \partial y}$ named u_{xy}

$\frac{\partial^2 u}{\partial x^2}$ named u_{xx} ; $\frac{\partial^2 u}{\partial y^2}$ named u_{yy} ; *so on*

Second – order PDE   Standard form

$$A\, u_{xx} + B\, u_{xy} + C u_{yy} + D u_x + E\, u_y + F\, u = G$$

A, B, C, D, E , F, G { constant   or function of (x,y), or function of u }

Examples :

- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$
- $u_x + u_y = u^3$
- $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 1$
- $u_x\, u_{xx} + u_{xy} + 3u_{yy} + 5u_x + u_y + u = x^3$

Some special cases of 2nd order PDE

- Wave equation

$$u_{xx} = \frac{1}{C^2}\ u_{tt}$$

- Heat equation

$$u_{xx} = \frac{1}{K}\ u_t$$

- Laplace equation

$$u_{xx} + u_{yy} = 0$$

- Euler equation

$$A\, u_{xx} + B\, u_{xy} + C u_{yy} = 0$$

Euler Solution of P.D.E

Standard form of equation : A $u_{xx} + B\, u_{xy} + Cu_{yy}$=0 ……..(1)

Let: u_{xx}=1 , $u_{xy} = \lambda$, $u_{yy} = \lambda^2$

So equation (1) will be written as : A + B λ + C λ^2

If $\lambda_1 \neq \lambda_2$: u(x,y)=F(x + $\lambda_1 y) + G(x + \lambda_2$y)

If $\lambda_1 = \lambda_2$: u(x,y)=F(x + λy) + y G(x + λy)

or ,

Let: u_{xx}= λ^2 , $u_{xy} = \lambda$, u_{yy} =1

So equation (1) will be written as : A λ^2+ B λ + C

If $\lambda_1 \neq \lambda_2$: u(x,y)=F(y + $\lambda_1 x) + G(y + \lambda_2$x)

If $\lambda_1 = \lambda_2$: u(x,y)=F(y + λx) + x G(y + λx)

Find the solution of P.D.E

Ex1: $u_{xx} + 2\,u_{xy} + u_{yy}$=0

Sol: Let: u_{xx}=1 , $u_{xy} = \lambda$, $u_{yy} = \lambda^2$

λ^2+2 λ +1=0

λ_1= $\lambda_2 = -1$

General solution : u(x,y) =F(x -y) + y G(x -y)

Ex2: $u_{xx} + u_{yy}$=0

sol: λ^2+1=0 , $\lambda_1 = +i$, $\lambda_2 = -i$

General solution : u(x,y) =F(x +iy) + G(x -iy)

Ex3: $u_{xx} + 5\, u_{xy} + 6u_{yy}$=0

Ex4: $u_{xx} = \frac{1}{C^2} u_{tt}$ (wave equation)

Ex5: Find particular solution of PDE

$u_{xx} - 2\,u_{xy} + u_{yy}$=0 , if u(x,0)=$e^x$, $u(0,y) = y^3 + e^y$

Sol:

let u_{yy}=λ^2, u_{xy} =λ , u_{xx}=1

1 - 2$\lambda + \lambda^2$=0

(λ-1)(λ-1)=0 , λ1= λ2=1

u(x,y)=F(x+y) + yG(x+y)

if u(x,0)=$e^x = F(x) + 0$, F(x+y)=e^{x+y}

So; u(x,y)= e^{x+y}+ yG(x+y) , $u(0,y) = y^3 + e^y = e^y$+ yG(y)

G(y)= y^2 ; and G(x+y)= $(x + y)^2$

u(x,y)=e^{x+y}+y $(x + y)^2$